Appendix 2B: Cascade phenomenology and spectral analysis

One of the properties of turbulence to which we appeal to justify the cascade model developed in the following chapters is that the dynamical interactions are strongest between structures whose sizes are nearly the same. This means that for the energy flux to pass from a large to a small eddy/structure it must pass through numerous intermediate steps: large structures don't spontaneously break up into numerous small ones but instead pass energy flux from one scale to another in a cascade-like manner. The development below is close to Rose and Sulem (1978) and shows simply that on condition that $1 < \beta < 3$, the main contribution to the dynamically significant v_n across structures of size l_n is from wavenumbers in the octave near wavenumber $1/l_n$.

Following Section 2.4, consider the dynamically significant velocity gradient. We will express v_n in terms of $E(k)$:

$$v_n{}^2 = \left\langle |\Delta v(\underline{l}_n)|^2 \right\rangle = \left\langle |v(\underline{r}) - v(\underline{r} + \underline{l}_n)|^2 \right\rangle \quad (2.114)$$

$$= 2\left\{ \langle v^2 \rangle - \langle v(\underline{r}) v(\underline{r} + \underline{l}_n) \rangle \right\} \quad (2.115)$$

The first term, $\langle v^2 \rangle = u(0)$ is the total energy, $\int_0^\infty dp\, E(p)$ (we will not worry about constant factors such as π etc.). The second term is just the trace of the velocity correlation tensor $u(l_n) = u_{ii}(l_n)$. Now:

$$u(\underline{l}_n) = \int d\underline{p}\, e^{i\underline{p}\cdot\underline{l}_n}\, \tilde{u}(\underline{p}) \quad (2.116)$$

but in our case (isotropic turbulence) $l_n = |\underline{l}_n|$ and $\underline{p}\cdot\underline{l}_n = p l_n \cos\theta$, where θ is the angle between $\underline{p}$ and l_n. Hence:

$$u(l_n) = \int_0^\infty dp\, E(p) \int_\Omega d^{d-1}\Omega e^{ipl_n\cos\theta} \quad (2.117)$$

where Ω is the (solid) angle in Fourier space. In spherical polar coordinates (θ, ϕ) $(d = 3)$, we have $\delta^{\delta-1}\Omega = \cos\theta\, d\theta\, d\phi$. Then we have:

$$v_n{}^2 = \langle |\underline{\Delta} v(l_n)|^2 \rangle = \int_0^\infty dp\, E(p)\left(1 - \int_\Omega d^{d-1}\Omega e^{ipl_n\cos\theta}\right)$$
$$= \int_0^\infty dp\, E(p) \int_\Omega d^{d-1}\Omega\left(1 - e^{ipl_n\cos\theta}\right) \quad (2.118)$$

where we have used the fact that the normalization has been defined so that:

$$\int_\Omega d^{d-1}\Omega = 1 \quad (2.119)$$

To estimate this integral in Eqn. (2.118), we use $k_n = \frac{2\pi}{l_n}$ and divide the range of integration into three parts:

(I) $0 \le p \le \dfrac{k_n}{\sqrt{2}}$ (low frequency);

(II) $\dfrac{k_n}{\sqrt{2}} \le p \le \sqrt{2}$ (medium frequency);

(III) $\sqrt{2}k_n \le p < \infty$ (high frequency).

We will now consider each case, starting with the limiting cases I, III.

Term (I)

pl_n is small, i.e. $pl_n \to 0$ and discarding all imaginary parts in first order term of p (since we know a priori that the integral must be real) we are left (ignoring constant factors) with second-order terms $O((pl_n)^2)$:

$$\int_0^{k_n/\sqrt{2}} dp\,(pl_n)^2 E(p) \sim l_n{}^2 \int_0^{k_n/\sqrt{2}} dp\,p^2\,E(p) \sim l_n^2\langle|\underline{\omega}|^2\rangle \tag{2.120}$$

where we have used $E_\omega(p) = p^2E(p)$. This is the large-eddy contribution to $v_n{}^2$. This result can be understood physically in the following way. The effect of large-scale vorticity is to produce a nearly constant velocity gradient across the eddy, and the velocity difference will be approximated by $l_n\omega(p)$ (since $\underline{\omega} = \nabla \times \underline{v}$ and we are interested in a "typical" gradient), hence the mean squared difference will be:

$$l_n{}^2\langle|\underline{\omega}|^2\rangle = l_n{}^2 \int_0^{k_n/\sqrt{2}} dp\,p^2\,E(p) \tag{2.121}$$

Term (III)

This is the small-eddy contribution. $pl_n \to \infty$:

$$\int_\Omega d^{d-1}\Omega(1 - e^{ipl_n\cos\theta}) \to 1 \tag{2.122}$$

since the exponential will oscillate very rapidly and will yield zero on average. So the contribution to $v_n{}^2$ due to small structures is

$$\int_{\sqrt{2}k_n}^{\infty} dp\,E(p) \tag{2.123}$$

where the contributing wavenumbers are greater than$\sqrt{2}k_n$. The physical interpretation is that the small-scale eddies cause the boundary of l_n-scale eddy to execute a highly convoluted random walk. In the mean, the effect is diffusive. The diffusion constant depends on the mean square velocity of all the contributing eddies, which is:

$$\int_{\sqrt{2}k_n}^{\infty} dp\,E(p) \tag{2.124}$$

Term (II)

Take

$$E_n \equiv \int_{k_n/\sqrt{2}}^{\sqrt{2}\,k_n} dp\,E(p) \int_\Omega d^{d-1}\Omega = (1 - e^{ipl_n\cos\theta}) \tag{2.125}$$

as the definition of energy in band n. We will investigate under which conditions this term is the main contribution to $v_n{}^2$. When it dominates terms I and II, the energy spectrum is termed "local," since most of the contribution to the dynamically significant quantity $v_n{}^2$ is due to structures with neighbouring wavenumbers; otherwise, it is "nonlocal." The final expression for $v_n{}^2\big((\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})\big)$ is:

$$v_n{}^2 \approx l_n{}^2 \int_0^{k_n/\sqrt{2}} dp\,p^2\,E(p) + E_n + \int_{\sqrt{2}k_n}^{\infty} dp\,E(p) \tag{2.126}$$

Due to the scaling, the dominant behaviour of the spectrum will be a power law. We now consider how the value of the scaling exponent affects the relative value of various terms. Considering $E(p) \sim p^{-\beta}$ (ignoring constant factors) then (I) becomes:

$$l_n{}^2\int_0^{k_n/\sqrt{2}} dp\,p^2\,p^{-\beta} = l_n^2\,p^{3-\beta}\Big|_0^{k_n/\sqrt{2}} \begin{cases} \sim l_n^2k_n^3 & \text{for} \quad \beta < 3 \\ \to \infty & \text{for} \quad \beta > 3 \end{cases} \tag{2.127}$$

When $\beta \geq 3$ then the term diverges – this low-frequency divergence is called an "infrared catastrophe" and indicates that the spectrum is dominated by low frequencies – it will be nonlocal.

Term (III) becomes:

$$\int_{\sqrt{2}k_n}^{\infty} dp\,p^{-\beta} \sim p^{1-\beta}\Big|_0^{\infty} \quad \begin{cases} \sim k_n{}^{1-\beta} & \text{for} \quad \beta > 1 \\ \to \infty & \text{for} \quad \beta < 1 \end{cases} \tag{2.128}$$

Hence if $\beta < 1$ the term diverges, we have an "ultraviolet catastrophe," and again the spectrum is nonlocal, this time due to dominance of the higher frequencies.

We can now conclude that if $1 < \beta < 3$, all the terms are dominated by the contributions from wavenumbers near $k_n = \frac{2\pi}{l_n}$, and hence the spectrum will be local. Now as long as $\beta > 1$, term III is negligible and the sum of terms I and II can be approximated by:

$$v_n{}^2 \approx l_n{}^2\int_0^{k_n} dp\,p^2\,E(p); \quad \beta > 1 \tag{2.129}$$

(we are interested in an order-of-magnitude estimate only; the angular integration will give a constant correction to the above of order unity).

When viscosity is negligible, the only way to define a quantity with dimensions of time is as follows:

$$\tau_n = \frac{l_n}{v_n} \sim \left(\int_0^{k_n} dp\, p^2 E(p) \right)^{-1/2} \qquad (2.130)$$

which is an estimate of the eddy turnover time.