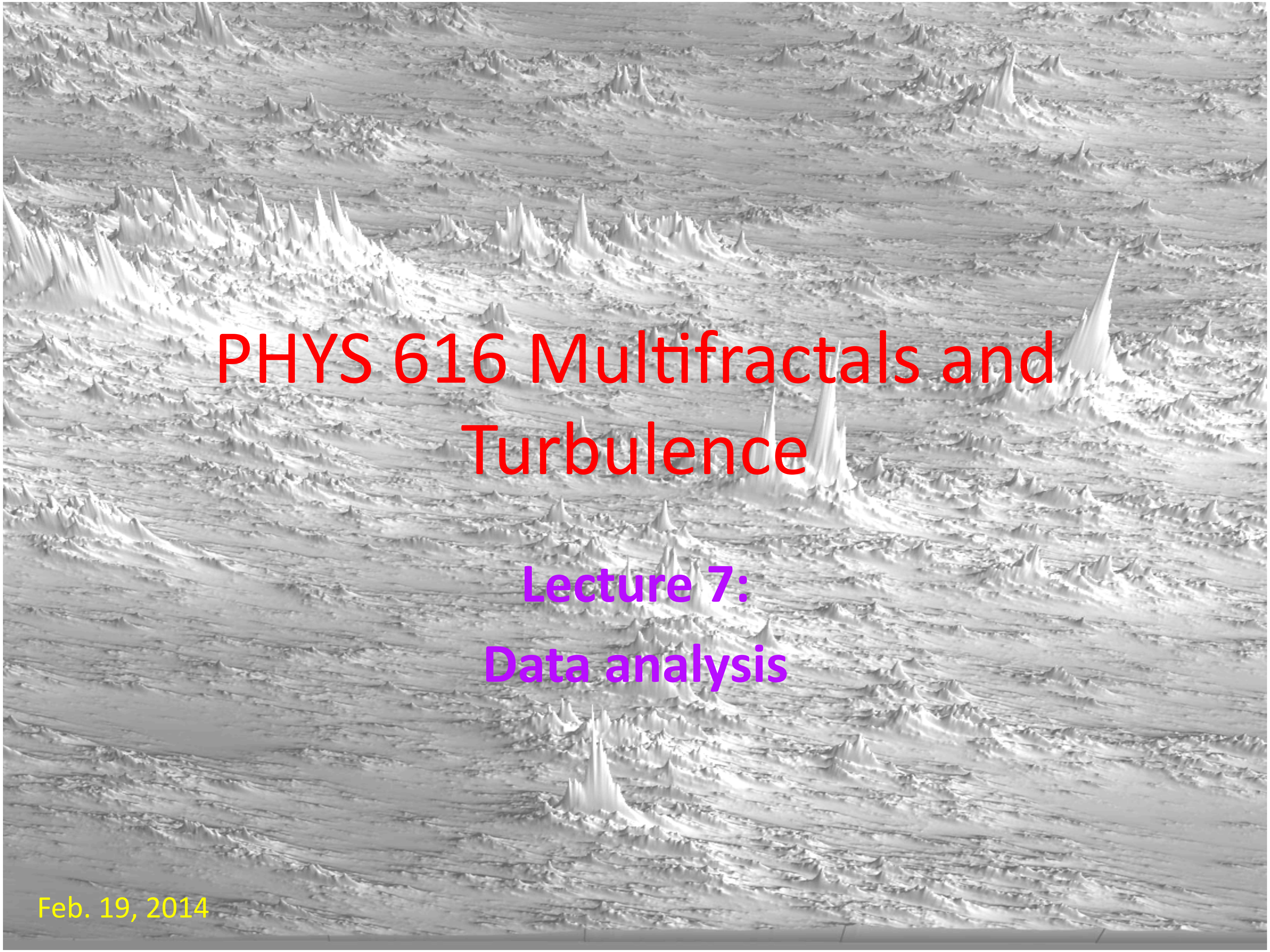

PHYS 616 Multifractals and Turbulence
Lecture 7:
Data analysis
Feb. 19, 2014

Data analysis

The space-time variability of natural systems, can often be broken up into various "scaling ranges" over which the fluctuations vary in a power law manner with respect to scale. Over these ranges, the fluctuations follow

$$\Delta v = \varphi_{\Delta x} \Delta x^{H}$$

The flux at resolution Δx

Taking the q^{th} power and ensemble averaging, we see that the statistical characterization of the fluctuations in terms of generalized structure function:

$$S_q(\Delta x) = \left\langle \Delta v(\Delta x)^q \right\rangle = \left\langle \varphi_{\Delta x}^q \right\rangle \Delta x^{qH} \approx \Delta x^{\xi(q)}; \quad \left\langle \varphi_{\Delta x}^q \right\rangle = \left(\frac{L}{\Delta x} \right)^{K(q)}; \quad \xi(q) = qH - K(q)$$

Hence, we seek H, K(q)

With universality: $K(q) = \frac{C_1}{\alpha - 1}\left(q^{\alpha} - q\right)$ i.e. we seek H, C_1, α

Empirical analysis: Estimating fluxes from the fluctuations

Explicit multiplicative cascades and we examined some of their consequences, notably that the general statistics of the cascades can be specified by their statistical moments via the simple multifractal cascade equation:

$$\left\langle \varphi_\lambda^q \right\rangle = \lambda^{K(q)}$$

The empirical determination of the outer scale is fairly straightforward. Consider L_{eff}, the "effective outer scale" where the cascade begins, and use the symbol

$\lambda' = L_{eff}/L$

for the (unknown scale ratio from the beginning of the cascade and the resolution of the flux φ. We will instead use the symbol λ as the ratio of a convenient reference scale to the resolution scale.

$$M_q = \left\langle \varphi_\lambda'^{q} \right\rangle; \quad \varphi_\lambda' = \frac{\varphi_\lambda}{\left\langle \varphi_1 \right\rangle}$$

obey the generic multiscaling relation

$$M_q = \lambda'^{K(q)} = \left(\frac{L_{eff}}{\Delta x}\right)^{K(q)} = \left(\frac{L}{\Delta x}\right)^{K(q)}\left(\frac{L_{eff}}{L}\right)^{K(q)} = \left(\frac{\lambda}{\lambda_{eff}}\right)^{K(q)}; \qquad \lambda' = \frac{L_{eff}}{\Delta x} = \frac{\lambda}{\lambda_{eff}}; \quad \lambda = \frac{L_e}{\Delta x}; \quad \lambda_{eff} = \frac{L_e}{L_{eff}}$$

Scaling range flux estimates

If atmospheric dynamics are controlled by scale invariant turbulent cascades of various (scale by scale) conserved fluxes φ then in a scaling regime, the (absolute) fluctuations $\Delta I(\Delta x)$ in an observable I (e.g. wind, temperature or radiance) over a distance Δx are related to the turbulent fluxes by a relation of the form:

$$\Delta v(\Delta x) \approx \varphi \Delta x^H$$

This is a generalization of the Kolmogorov law for velocity fluctuations (the latter has H = 1/3 and $\varphi = \varepsilon^\eta, \eta = 1/3$ where ε is the energy flux to smaller scales). Without knowing η nor H - nor even the physical nature of the flux - we can use this to estimate the normalized (nondimensional) flux φ' at the smallest resolution ($\Delta x = \lambda$) of our data:

$$\frac{\Delta v}{\langle \Delta v \rangle} = \frac{\varphi \Delta x^H}{\langle \varphi \Delta x^H \rangle} = \frac{\varphi}{\langle \varphi \rangle} = \varphi'$$

Note that if the fluxes are realizations of pure multiplicative cascades then the normalized η powers are also pure multiplicative cascades, so that $\varphi' = \varphi/<\varphi> = 1$ is a normalized cascade ($<\varphi>$ is the ensemble mean large scale flux, i.e. the climatological value, it is independent of scale, hence there is no need for a subscript).

The fluctuation, Δv (Δx) can be estimated in various ways; in 1-D a convenient method (which works for the common situation where $0 \leq H \leq 1$) is to use absolute differences: $\Delta v(\Delta x) = |v(x + \Delta x) - v(x)|$

Dissipation scale flux estimates

For data at resolutions high enough for viscous dissipation to be important, the scaling law can no longer be used to estimate the fluxes. In the atmosphere these scales are typically millimetric and such data is rarely encountered. However in reanalyses, the finest resolutions are regularized using artificial "hyper - viscosities", so that their interpretation must be different. To see this, consider the example of the energy flux, recalling that at the dissipation scale the viscous term is dominant:

$$\varepsilon \approx \nu \underline{v} \cdot \nabla^2 \underline{v}$$

At dissipation scales, viscous term dominates

where ν is the viscosity, $\underline{v}$ the velocity. Standard manipulations give:

$$\varepsilon \approx \nu \sum_{i,j=1}^{3} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 \approx \nu \left(\frac{\Delta v}{\Delta x} \right)^2$$

(the *i*, *j* index the velocity components). Therefore if Δx is in the dissipation range (e.g. the finest resolution of the model) then:

$$\Delta v \approx \left(\frac{\varepsilon}{\nu} \right)^{1/2} \Delta x$$

Note different exponent

Hyperviscosity

Since the meteorological models and reanalyses actually use hyper-viscosities w ith hyper-viscous coefficient ν* and a Laplacian taken to the power *h* (typically *h* =3 or 4), we have:

$$\varepsilon \approx \nu^{*} \underline{v} \cdot \nabla^{2h} \underline{v}$$

which leads to the estimate

$$\Delta v \approx \left(\frac{\varepsilon}{\nu^{*}} \right)^{1/2} \Delta x^{h}$$

In all cases, we therefore have (independently of *h*):

$$\varphi' = \frac{\Delta v}{\langle \Delta v \rangle} = \frac{\varepsilon^{1/2}}{\langle \varepsilon^{1/2} \rangle}$$

Exponent independent of h

η Powers (General)

We see that this is the same the dissipation scale estimate with $\varphi=\varepsilon^{1/3}$, the only difference is that for the wind field, the exponent η = 1/2 holds in the dissipation range rather than η = 1/3 which holds in the scaling regime. If we introduce $K_\eta(q)$ which is the scaling exponent for the normalized η flux φ' then :

$$\varphi' = \frac{\varphi}{\langle\varphi\rangle} = \frac{\varepsilon^\eta}{\langle\varepsilon^\eta\rangle}$$

hence

$$\langle\varphi'^q\rangle = \left\langle\left(\frac{\varepsilon^\eta}{\langle\varepsilon^\eta\rangle}\right)^q\right\rangle = \frac{\lambda^{K(q\eta)}}{\lambda^{qK(\eta)}}$$

Defining $K_\eta(q)$ $\langle\varepsilon_\lambda^q\rangle = \lambda'^{K_\eta(q)}$

This shows that: $K_\eta(q) = K_\varepsilon(q\eta) - qK_\varepsilon(\eta)$

which for universal multifractals yields

$$K_\eta(q) = \eta^\alpha K_1(q)$$

(note: $K_1(q) = K(q)$), i.e. in obvious notation:

$$C_{1,\eta} = \eta^\alpha C_{1,1}$$

Comparing the dissipation estimate (η = 1/2) and the scaling range estimate (η = 1/3), we have:

$$C_{1,diss} = \left(\frac{3}{2}\right)^\alpha C_{1,scaling}$$

For wind, α=1.8 hence: $\left(\frac{3}{2}\right)^\alpha \approx 2.07$

Passive scalars

The extension of this discussion to passive scalars is also relevant and shows that the interpretation of the empirically/numerically estimated fluxes in terms of classical theoretical fluxes can be nontrivial. Denoting by ρ the density of the passive scalar, and χ its variance flux, the dissipation range formula is:

$$\chi \approx \rho\kappa\nabla^2\rho$$

recall: $\chi = -\frac{\partial \rho^2}{\partial t}$

Molecular diffusivity

κ is the molecular diffusivity, hence:

$$\Delta\rho \approx (\chi/\kappa)^{1/2}\Delta x$$

Dissipation range

Whereas the corresponding formula in the scaling range is

$$\Delta\rho \approx \chi^{1/2}\varepsilon^{-1/6}\Delta x^{1/3}$$

the Corrsin-Obukhov law

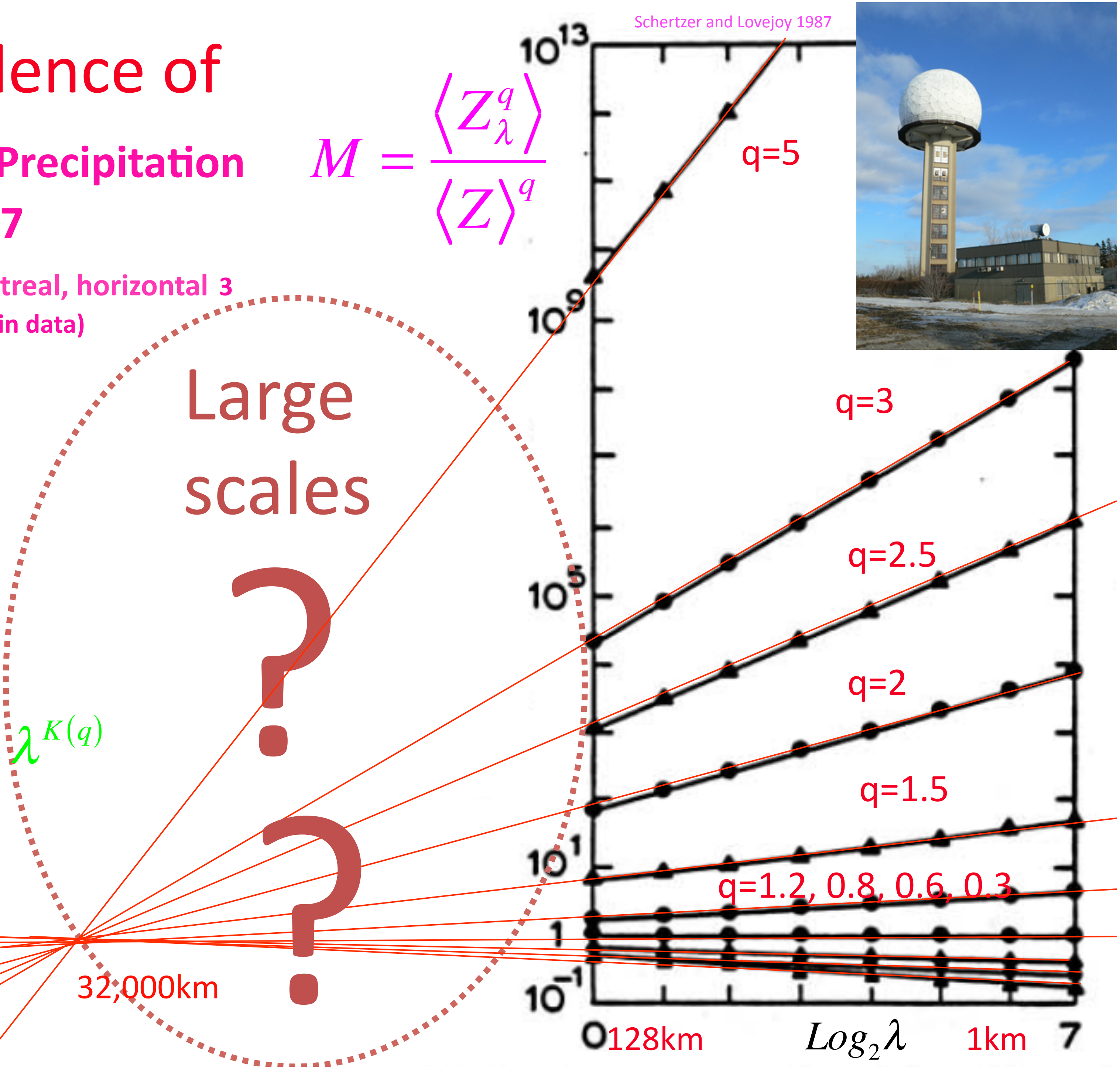
The combined effective flux $\phi \approx \chi^{1/2}\varepsilon^{-1/6}$

Two (presumably statistically *dependent*) cascade quantities rather than just one

Early evidence of cascades: **Precipitation 1987**

(70 Radar Scans, Montreal, horizontal 3 weeks of rain data)

$$M = \frac{\langle Z_\lambda^q \rangle}{\langle Z \rangle^q}$$

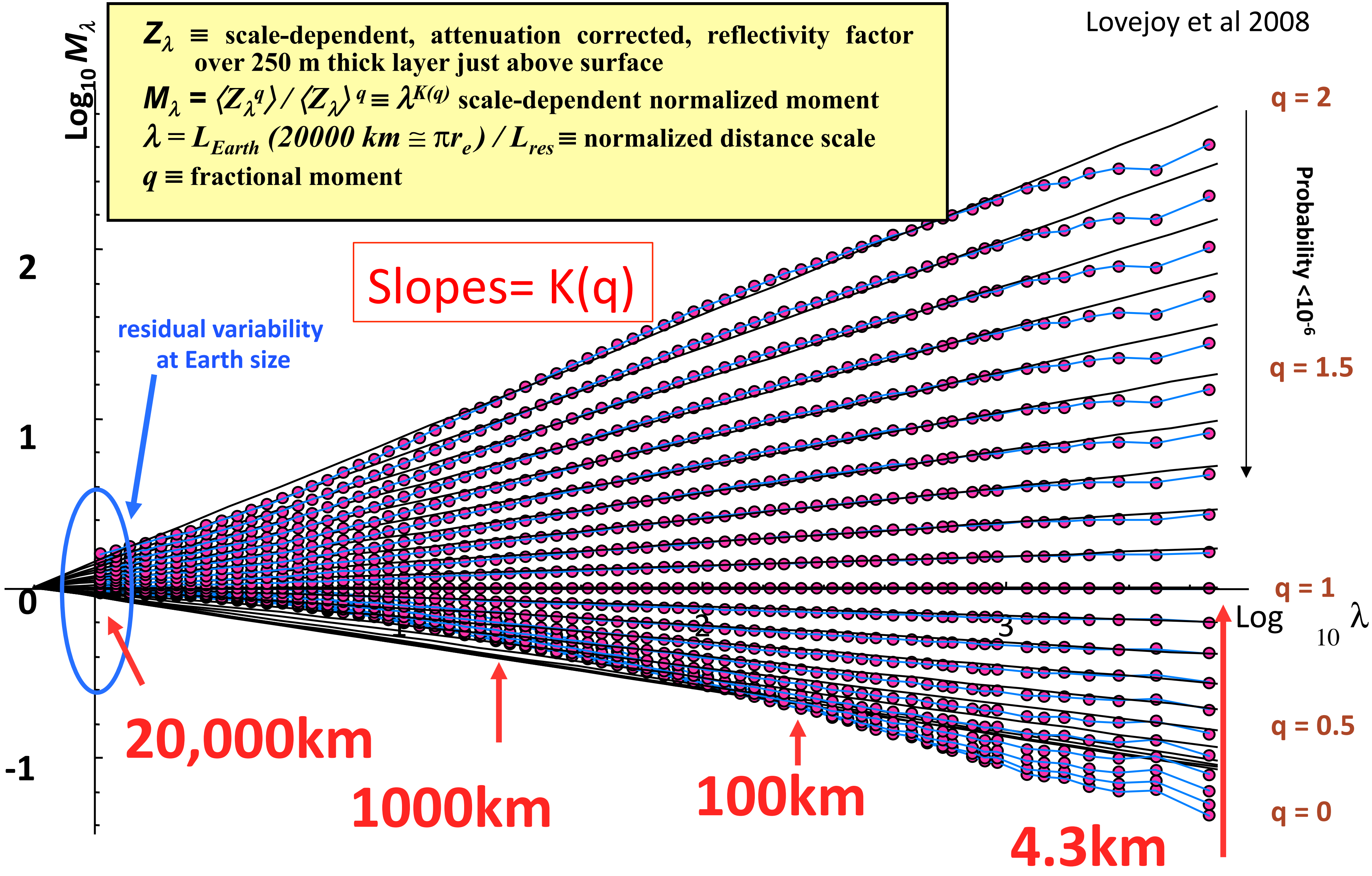


Cascade prediction:

$$\langle Z_\lambda^q \rangle / \langle Z_1 \rangle^q = \lambda^{K(q)}$$

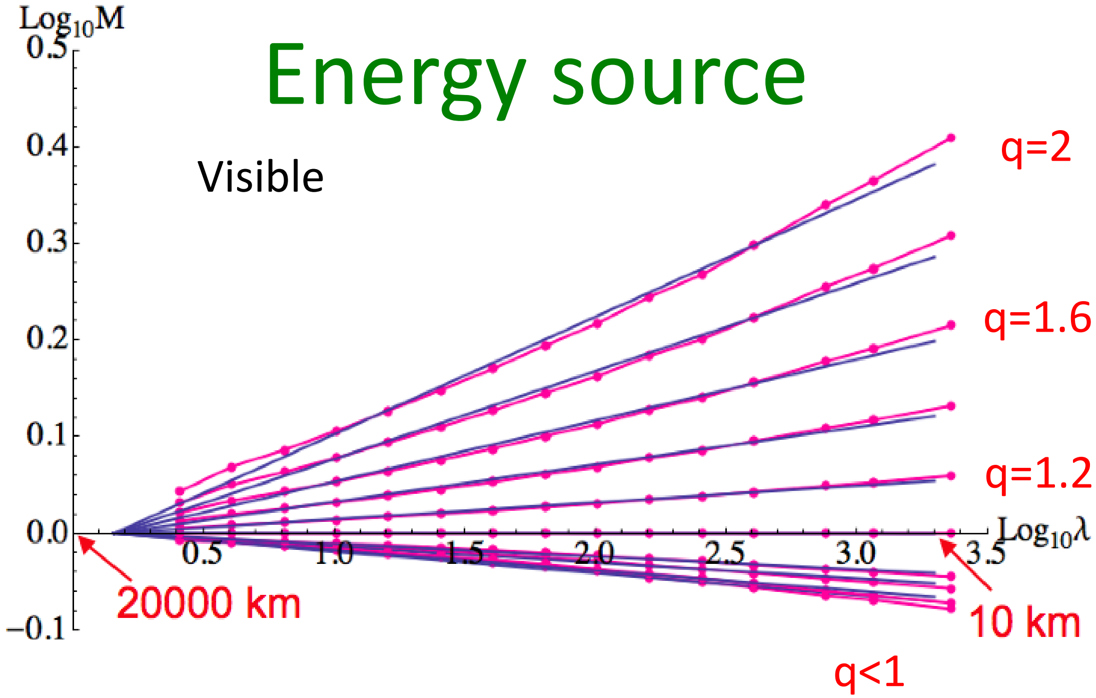
$$\lambda = L_{eff} / L_{res}$$

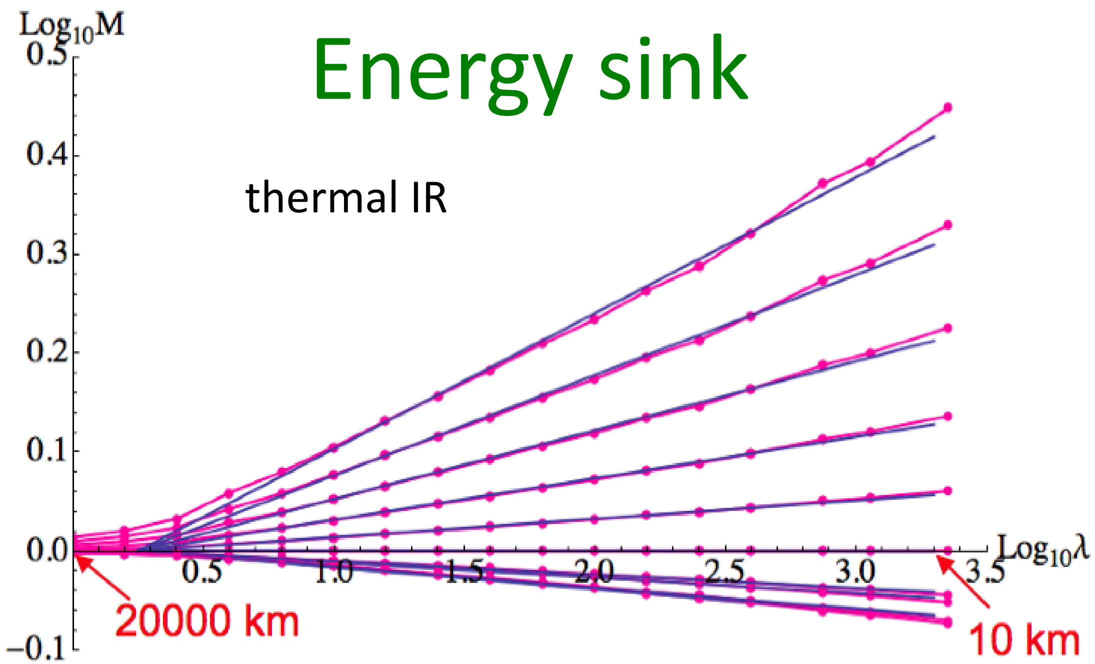
Scale-dependent TRMM PR Attenuation Corrected Reflectivity Factor [Z_λ] (1176 consecutive orbits -- ~70 days)



TRMM satellite data, ≈1000 orbits

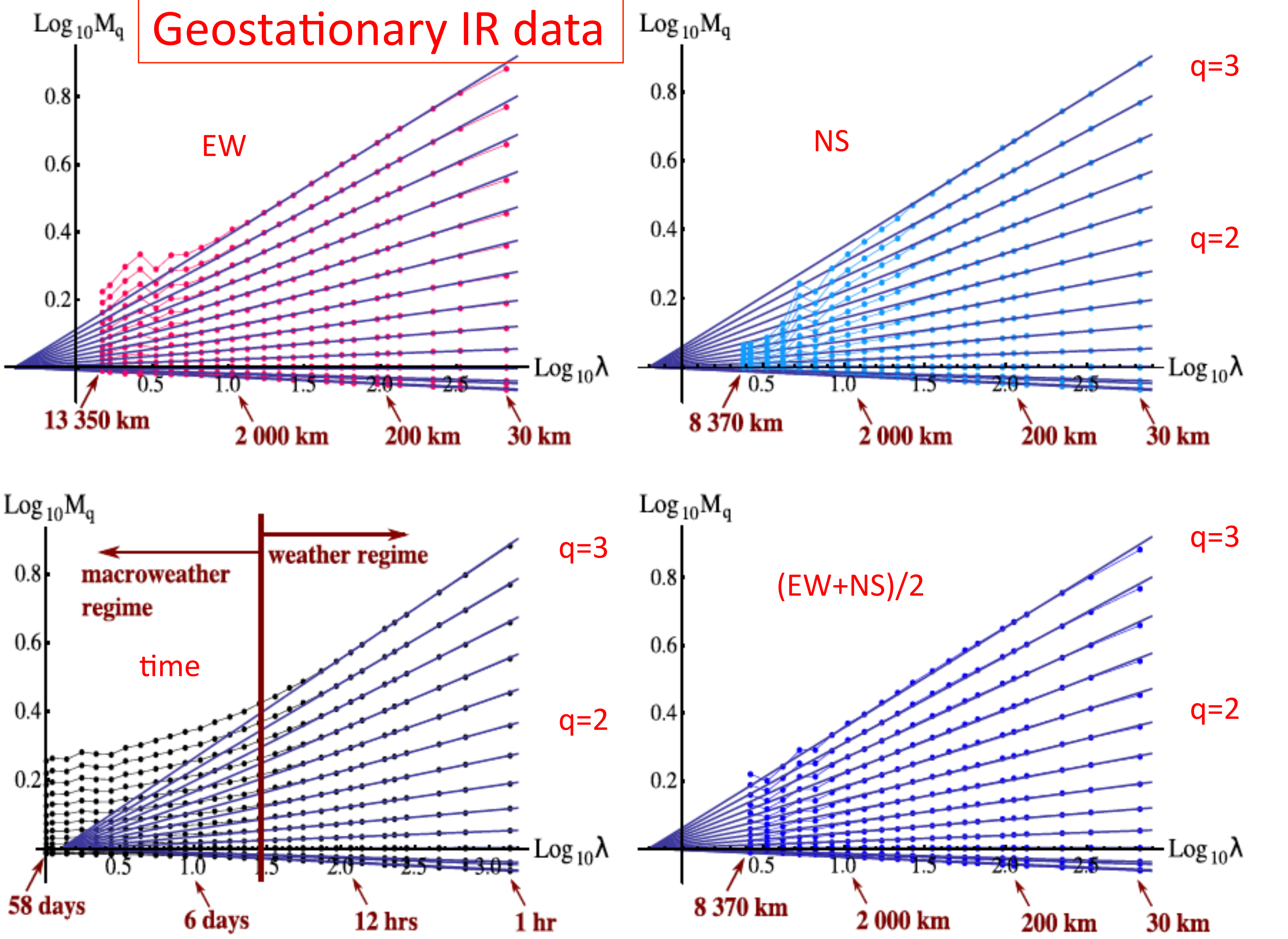
Energy budget



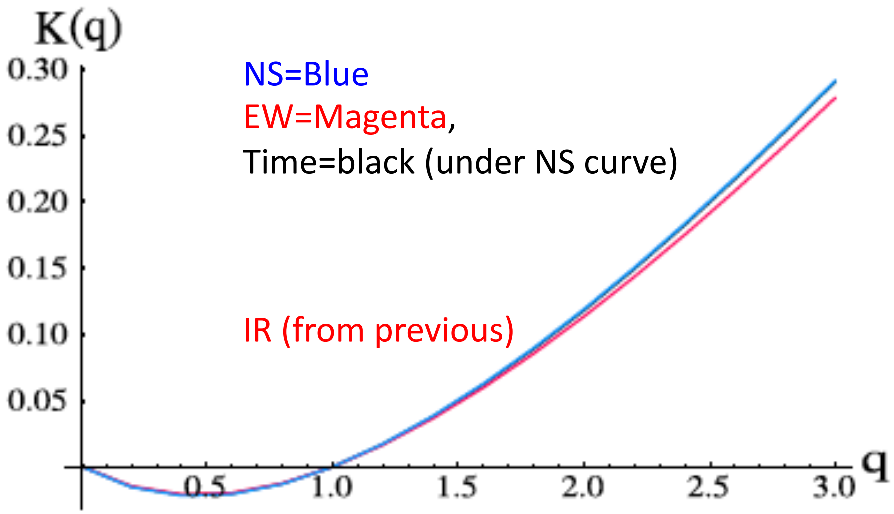


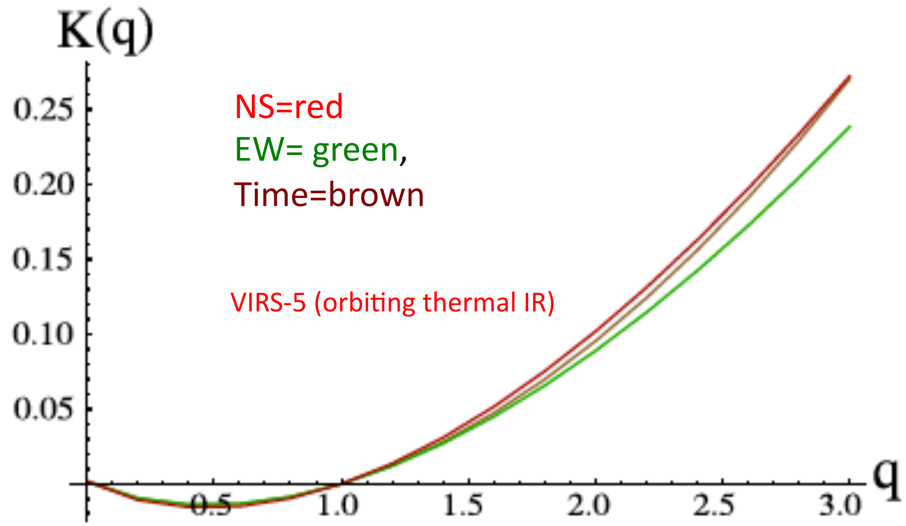
Lovejoy et al 2009

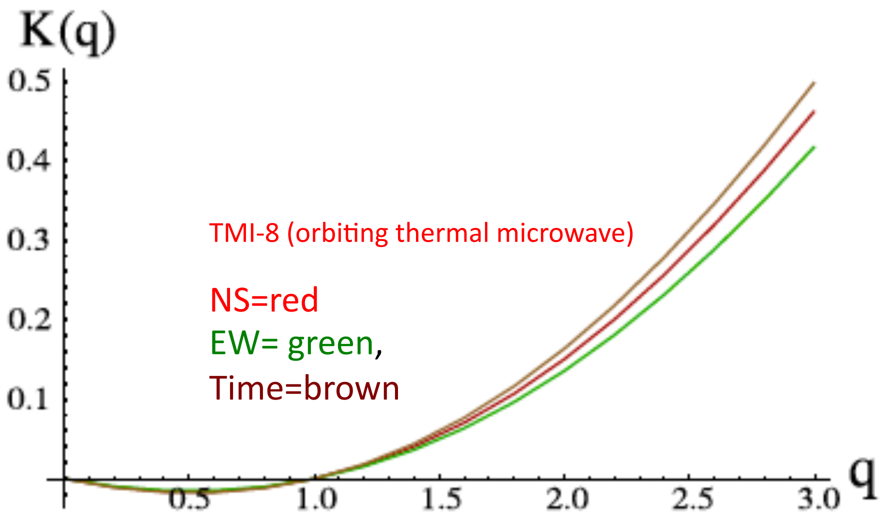
Geostationary IR data

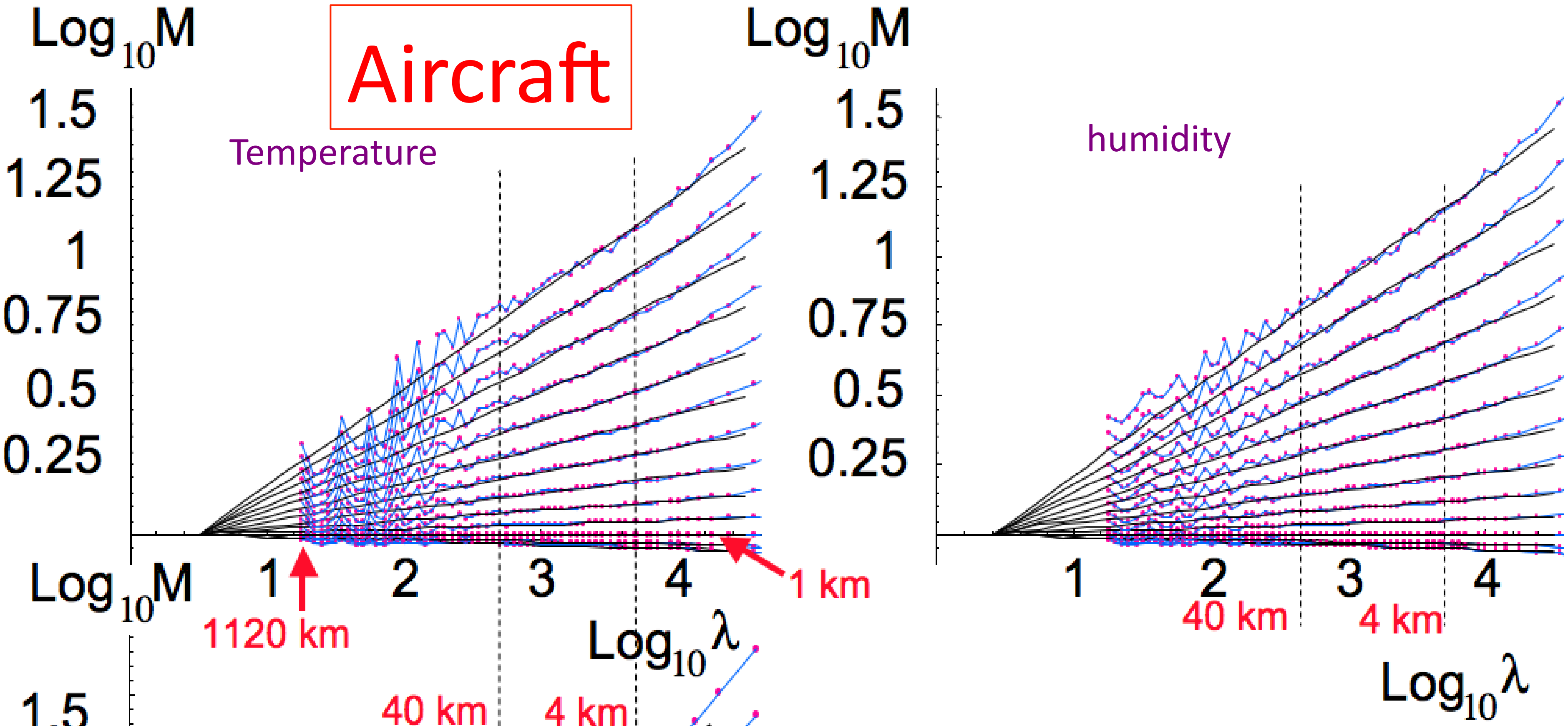


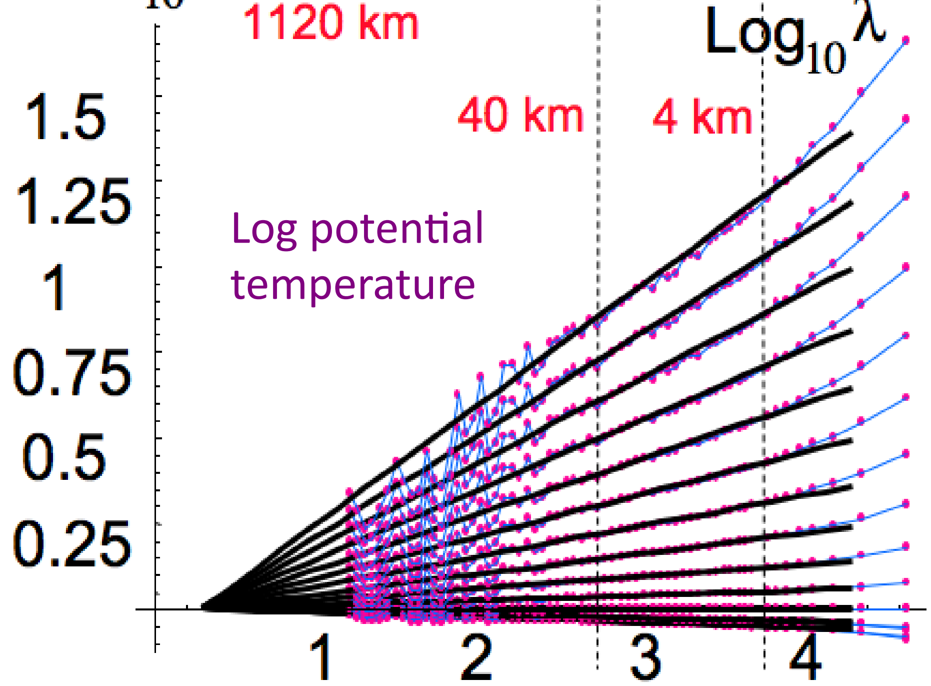
K(q) satellite data







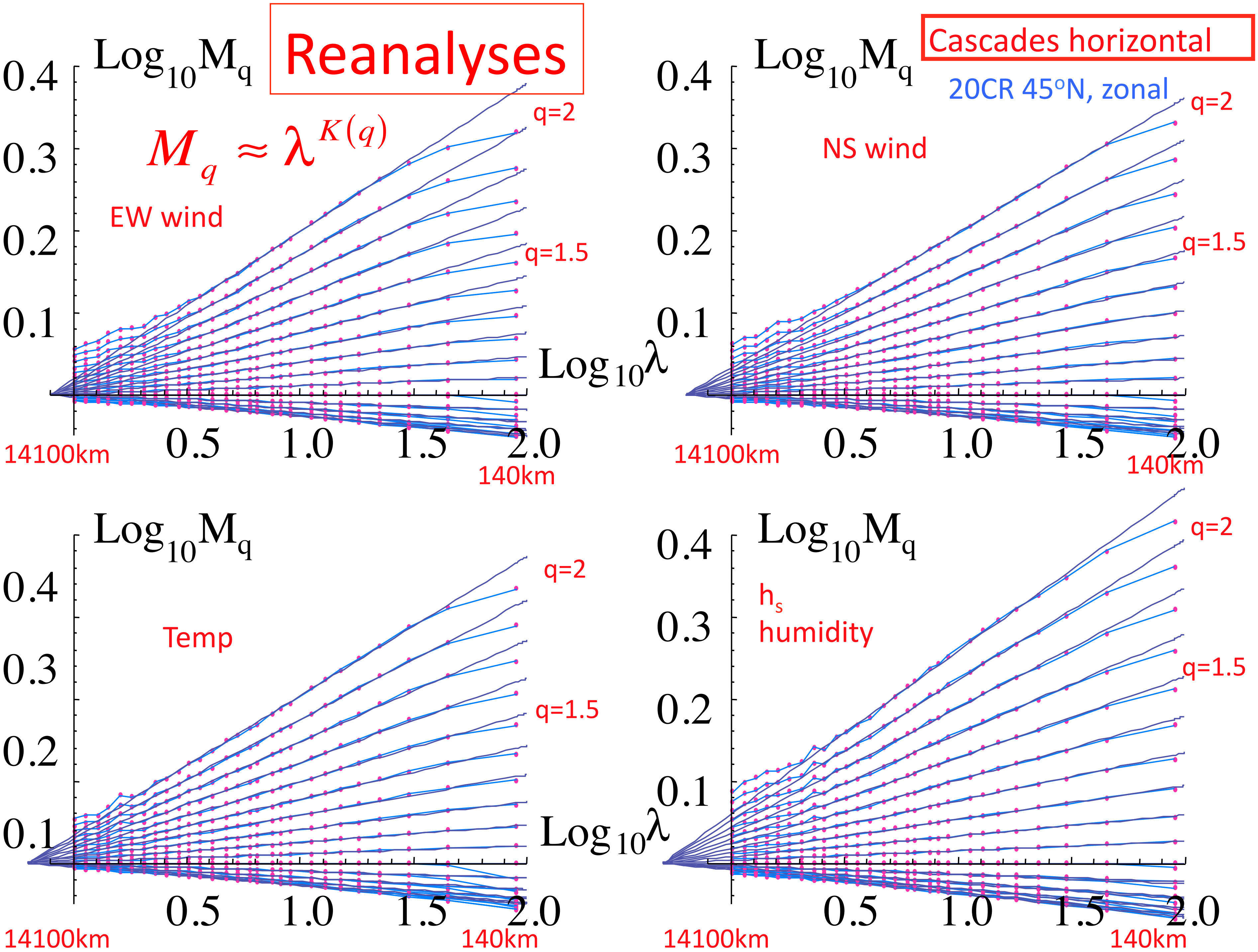




Horizontal cascades from 24 aircraft legs

(11-13km)

Fields that are relatively unaffected by the trajectories

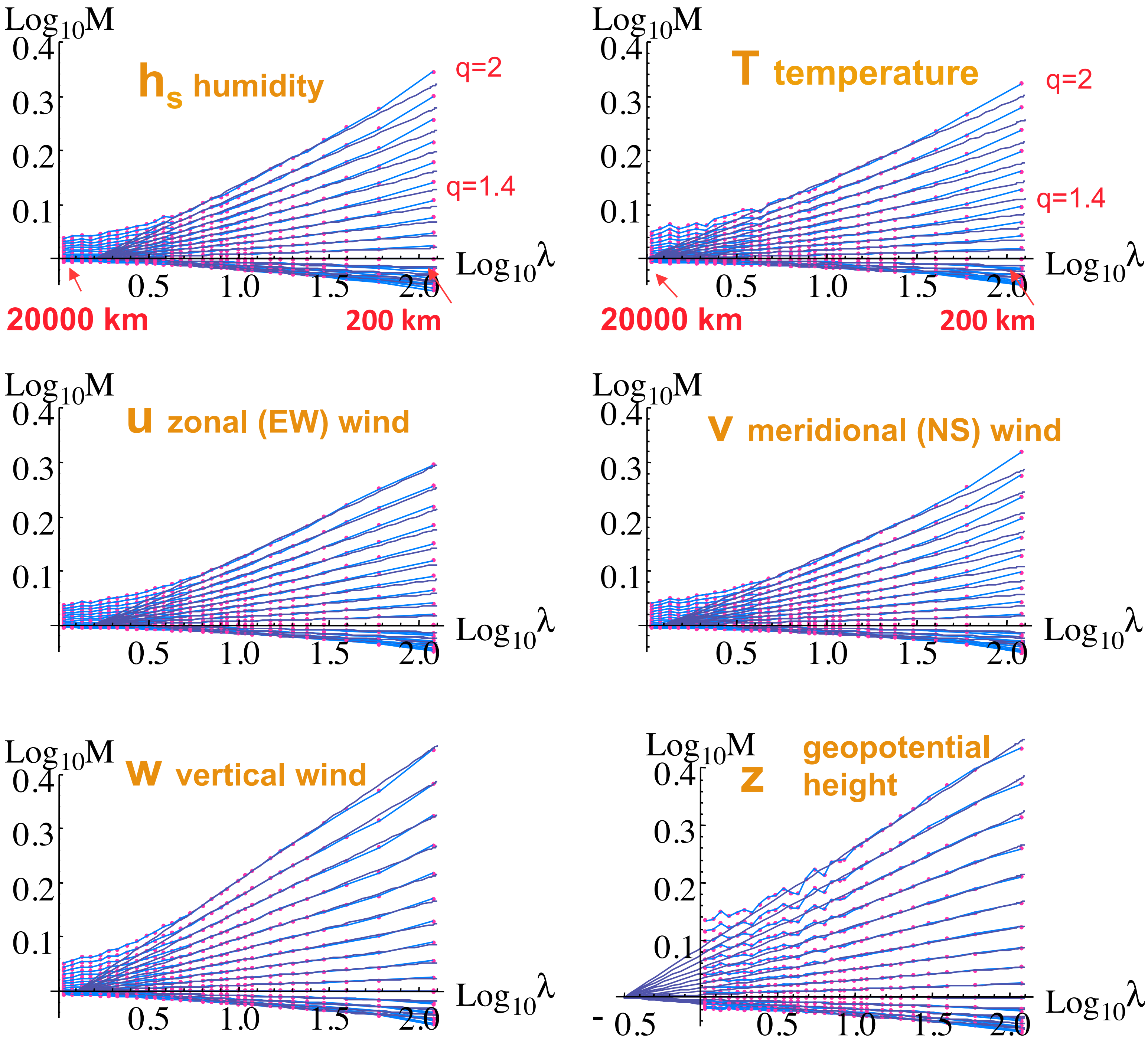

Reanalyses
Cascades horizontal
20CR 45°N, zonal
$M_q \approx \lambda^{K(q)}$
$\mathrm{Log}_{10}M_q$
$\mathrm{Log}_{10}\lambda$
EW wind
NS wind
Temp
h_s humidity
q=2
q=1.5
0.4
0.3
0.2
0.1
0.5
1.0
1.5
2.0
14100km
140km

$M = \langle \phi_\lambda^q \rangle / \langle \phi \rangle^q$

ECMWF Reanalysis

East-West

(2006, 0Z, 700 mb)

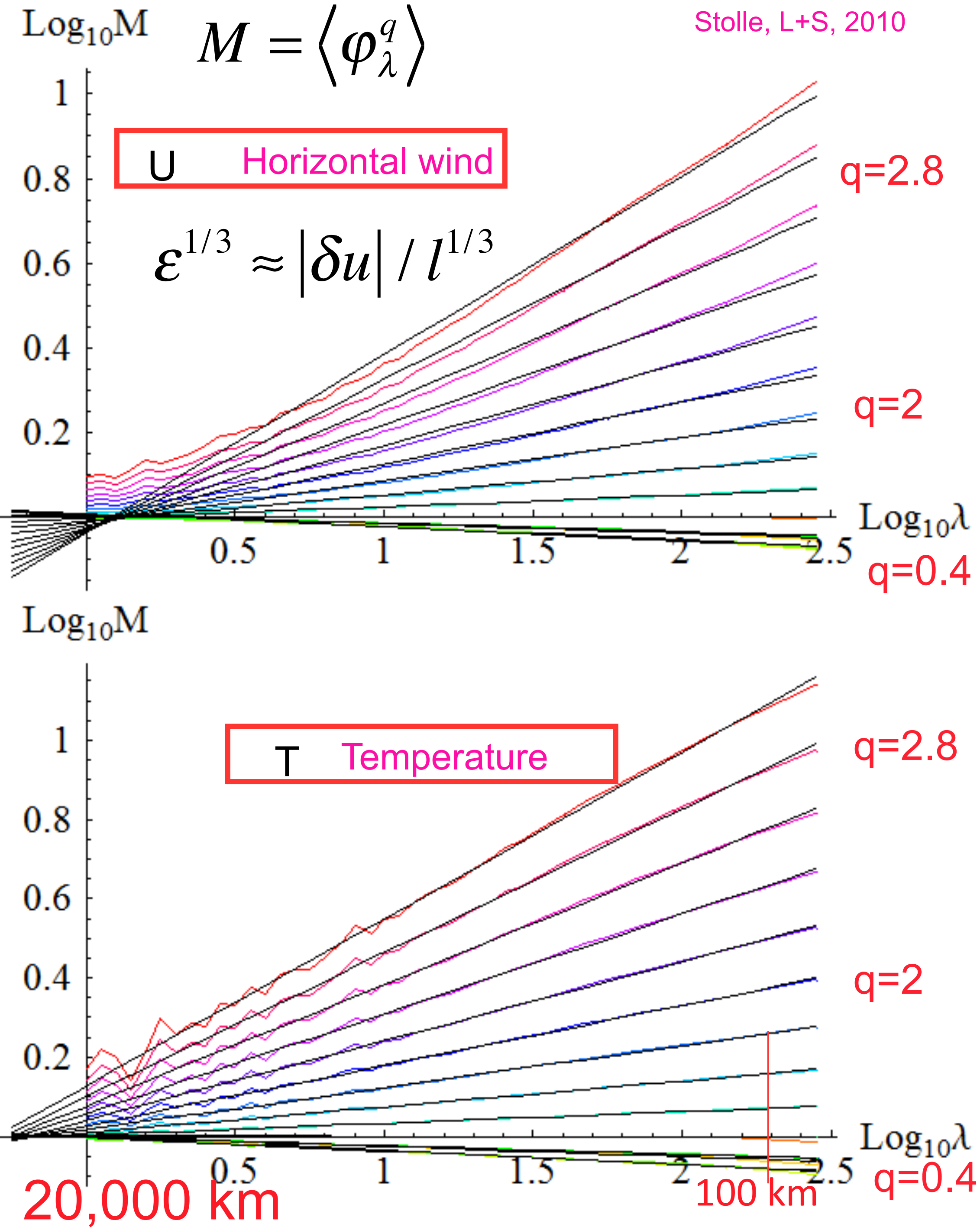


Weather forecasting models

Global GEMS Model 00h

Analysis of four months
U,T at 1000 mb

(48 h forecasts are almost the same)



Results on reanalyses

	h_s	T	u	v	w	Z
β	1.90	2.40	2.40	2.40	0.40	3.35
$C_{1,uni}$	0.102 ± 0.009	0.077 ± 0.005	0.084 ± 0.006	0.087 ± 0.012	0.121 ± 0.007	0.088 ± 0.006
C_1	0.101 ± 0.009	0.072 ± 0.005	0.082 ± 0.007	0.085 ± 0.013	0.115 ± 0.008	0.083 ± 0.005
α	1.77 ± 0.06	1.90 ± 0.006	1.85 ± 0.012	1.85 ± 0.011	1.92 ± 0.009	1.90 ± 0.012
H	0.54	0.77	0.77	0.78	0.14	1.26
$L_{eff,EW}$ (km)	13 000	20 000	13 000	16 000	16 000	63 000
$L_{eff,NS}$ (km)	6300	16 000	8000	10 000	13 000	40 000
$T_{eff,time}$ (days)	46	58	29	29	37	290
δ (%)	0.32 ± 0.04	0.35 ± 0.02	0.31 ± 0.09	0.28 ± 0.10	0.33 ± 0.10	0.52 ± 0.30

Model and reanalysis exponents

		ECMWF interim	ERA40	20CR	GEM	GFS	aircraft
u	α	1.86	1.93	1.87	1.68	1.80	1.94
	C_1	0.081	0.096	0.089	0.104	0.082	0.088
	L_{eff}	12 700	12 000	11 200	11 000	9000	25 000
T	α	1.89	2.11	1.85	1.94	2.00	1.78
	C_1	0.074	0.094	0.088	0.077	0.080	0.107
	L_{eff}	20 000	14 500	11 200	8300	8600	5000
h	α	1.70	1.75	1.73	1.60	1.74	1.81
	C_1	0.095	0.094	0.077	0.100	0.091	0.083
	L_{eff}	12 700	11 000	35 000	11 800	9000	10 000

Table 4.2b A comparison of the 1000 mb fields. The triplets (GEM) represent the parameter estimates for integrations of $t = 0, 48, 144$ hours, and the pairs (GFS) for $t = 0, 144$ hours.

	C_1			α			L_{eff} (km)			δ (%)		
T (GEM)	0.125	0.115	0.112	1.64	1.68	1.69	25 700	20 500	25 700	0.27	0.26	0.80
T (GFS)	0.142	0.138		1.72	1.71		27 900	26 000		0.59	0.60	
u (GEM)	0.121	0.122	0.123	1.68	1.62	1.61	11 000	11 000	12 300	0.32	0.36	1.24
u (GFS)	0.114	0.107		1.80	1.84		12 300	11 200		0.54	0.64	
h (GEM)	0.109	0.106	0.112	1.81	1.80	1.77	15 900	13 800	14 100	0.51	0.49	1.51
h (GFS)	0.128	0.128		1.86	1.81		21 700	20 900		0.46	0.46	

Aircraft estimates

	T	Logθ	h	v_{long}	v_{trans}	p	z
H	0.50 ± 0.01	0.51 ± 0.01	0.51 ± 0.01	0.46	0.37	0.36	0.43
C_1	0.052 ± 0.012	0.052 ± 0.010	0.040 ± 0.012	0.033	0.046	0.031	0.068
α	*1.78*	1.82	1.81	2.10	2.10	2.2	2.15
L_{eff} (km)	5000	10 000	10 000	10^5	25 000	1600	50
δ (%)	0.5	2.0	0.5	0.4	0.8	0.5	2.6

Horizontal spatial Scaling exponents

		C_1	α	H	β	L_{eff}
State variables	u, v	0.09	1.9	1/3, (0.77)	1.6, (2.4)	(14 000)
	w	(0.12)	(1.9)	(−0.14)	(0.4)	(15 000)
	T	0.11, (0.08)	1.8	0.50, (0.77)	1.9, (2.4)	5000 (19 000)
	h	0.09	1.8	0.51	1.9	10 000
	z	(0.09)	(1.9)	(1.26)	(3.3)	(60 000)
Precipitation	R	0.4	1.5	0.00	0.2	32 000
Passive scalars	Aerosol concentration	0.08	1.8	0.33	1.6	25 000
Radiances	Infrared	0.08	1.5	0.3	1.5	15 000
	Visible	0.08	1.5	0.2	1.5	10 000
	Passive microwave	0.1–0.26	1.5	0.25–0.5	1.3–1.6	5000–15 000
Topography	Altitude	0.12	1.8	0.7	2.1	20 000
Sea surface temperature	SST (see Table 8.2)	0.12	1.9	0.50	1.8	16 000

Satellite Scaling exponents

Channel	Wavelength	Resolution (km)	δ (%) line[a]	δ (%) uni[b]	α	C_1	H	L_{eff} (km)
VIRS 1	0.630 µm	8.8	0.60	0.71	1.35	0.077	0.19	9800
VIRS 2	1.60 µm	8.8	0.83	1.37	1.41	0.079	0.21	5000
VIRS 3	3.75 µm	22.	1.10	1.58	1.99	0.065	0.27	17 800
VIRS 4	10.8 µm	8.8	0.48	0.53	1.56	0.081	0.26	12 600
VIRS 5	12. 0 µm	8.8	0.47	0.81	1.63	0.084	0.33	15 800
AVHRR 14 vis[c]	0.58–0.68 µm	2.2	_	_	1.92	0.075	0.32	18 700
AVHRR 14 IR[c]	11.5–12.5 µm	2.2	_	_	1.91	0.079	0.36	25 200
MTSAT[d]	10.8 µm	30	_	_	1.5	0.74	0.31	40 000
Photography	0.3–0.7 µm	0.5–5 m	_	_	1.77	0.061	0.61	_

[a] This is the residual with respect to pure power-law scaling.
[b] This is the residual with respect to universal multifractal scaling with $\alpha = 1.5$, C_1 _ 0.08, only the outer scale is fit to each channel.
[c] These were from 153 visible, 214 IR scenes each 280 × 280 km over Oklahoma, from Lovejoy *et al.* (2001), Lovejoy and Schertzer (2006).
[d] This is the average of the north–south and east–west parameters; see Table 4.7.

Table 4.8b The characteristics of the five (TRMM) TMI channels and the Precipitation Radar reflectivity (not rain rate), from Lovejoy *et al.* (2009a). All used vertical polarization. The H estimates are based on structure functions.

Channel	Wavelength	Resolution (km)	δ (%) line[2]	δ (%) uni[3]	α	C_1	H	L_{eff} (km)
TMI1	3.0 cm (10.6 GHz)	111.4	1.40	1.55	1.35	0.255	0.50	15 900
TMI 3	1.58 cm (19.35 GHz)	55.6	1.71	1.93	1.76	0.193	0.331	6900
TMI 5	1.43 cm (22.24 GHz)	27.8	1.62	1.82	1.93	0.157	0.453	5000
TMI 6	8.1 mm (37 GHz)	27.8	1.73	1.95	1.76	0.15	0.377	4400
TMI 8	3.51 mm (85.5 GHz)	13.9	1.40	1.70	1.90	0.102	0.238	6300
TRMM Z	2.2 cm (13.2 GHz)	4.3	6.0*	4.6*	1.50	0.63	0.00	32 000

Observables: additive and multiplicative processes

$$\Delta I = \varphi \Delta x^{H}$$

Linear scaling part, additive

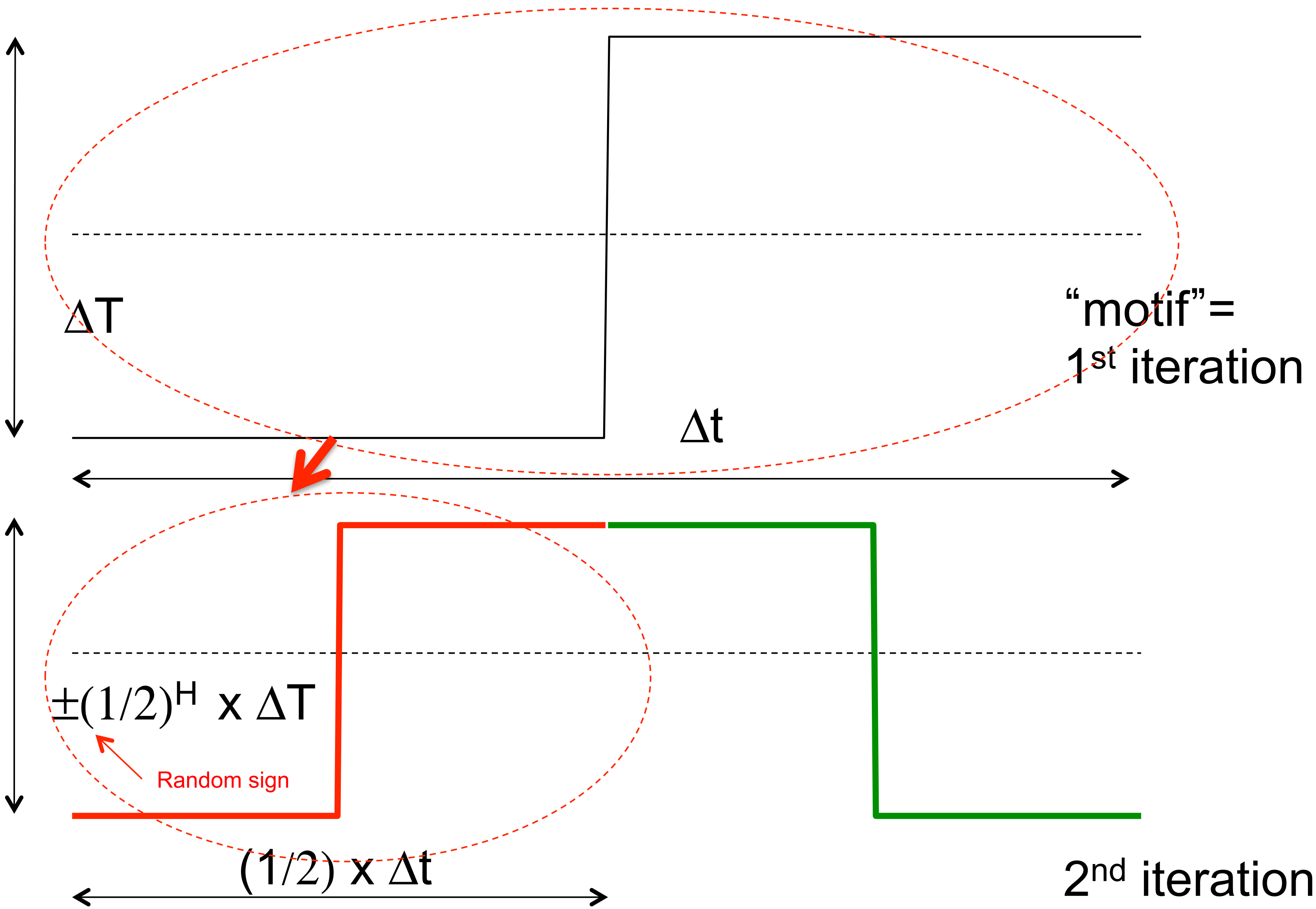
Observable: multiplicative and additive

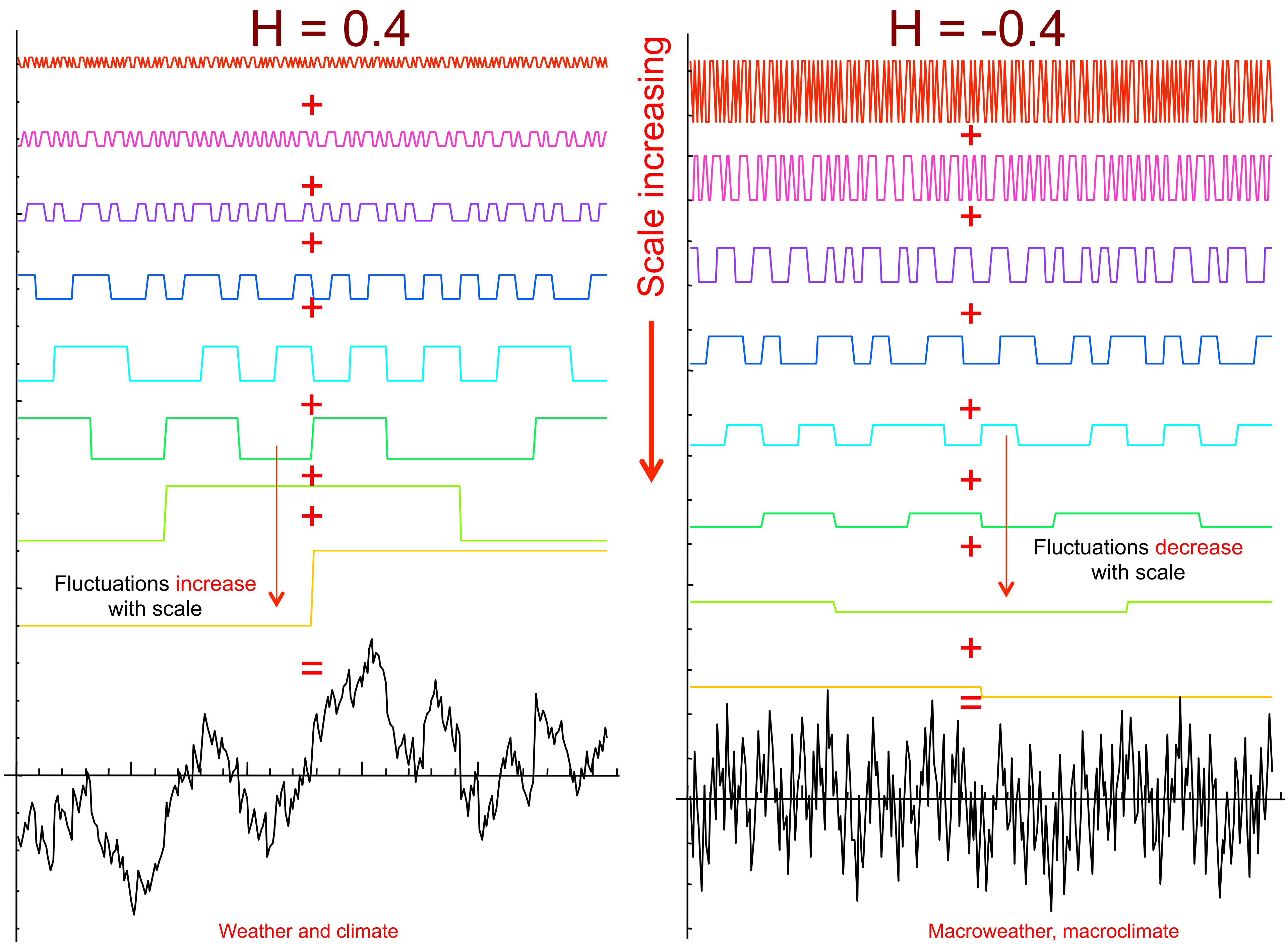
Multiplicative cascade

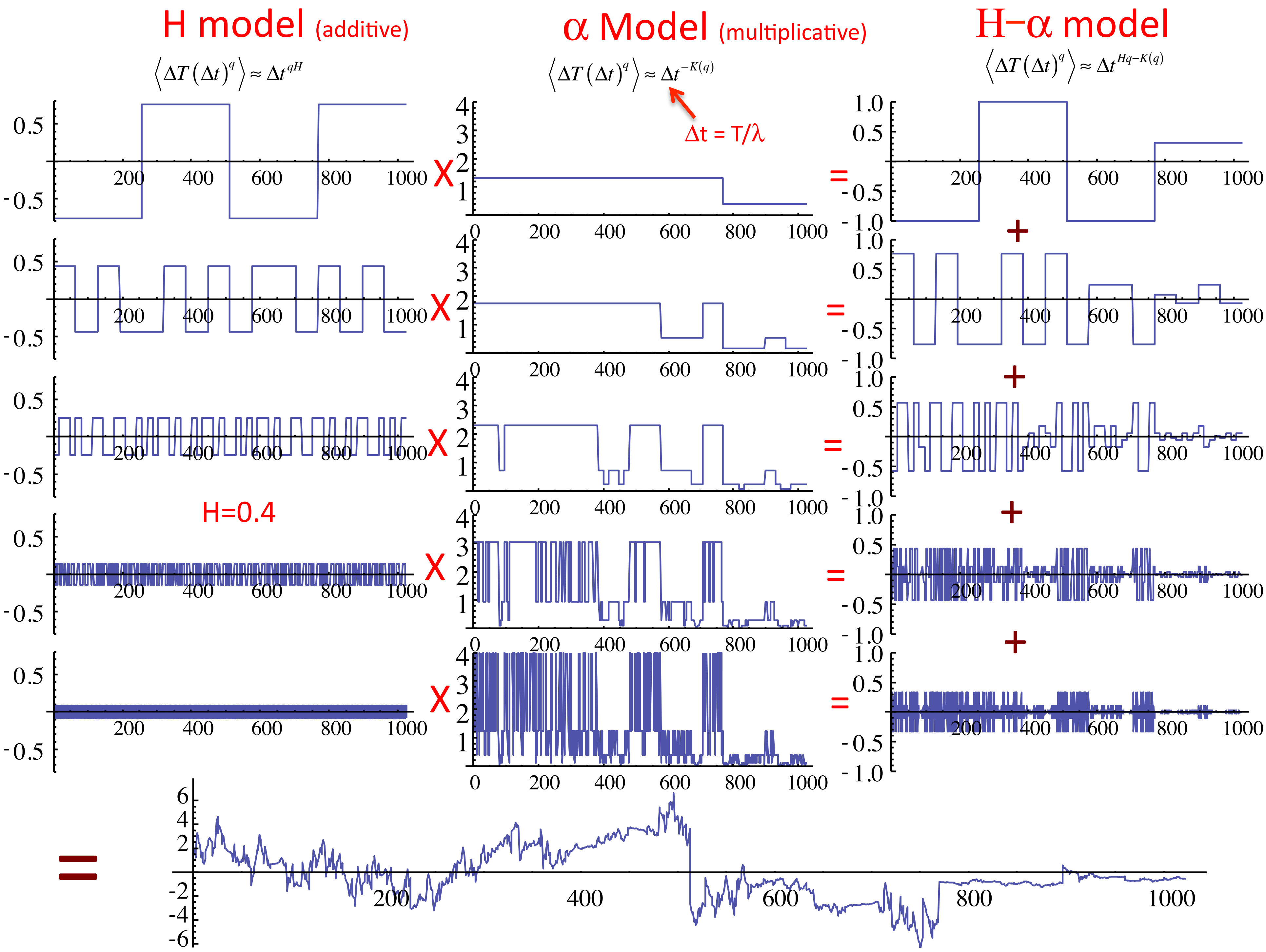
The fractal H model

(Lovejoy 2013)

(fractal dimenension= 2-H)




H = 0.4
H = -0.4
Scale increasing
Fluctuations increase with scale
Fluctuations decrease with scale
Weather and climate
Macroweather, macroclimate



H model (additive)
$\langle \Delta T(\Delta t)^q \rangle \approx \Delta t^{qH}$
α Model (multiplicative)
$\langle \Delta T(\Delta t)^q \rangle \approx \Delta t^{-K(q)}$
Δt = T/λ
H−α model
$\langle \Delta T(\Delta t)^q \rangle \approx \Delta t^{Hq-K(q)}$
H=0.4
X
=
+

Spectral analysis

Basic equation

$$\Delta f = \varphi \Delta x^{H}$$

$$R(\Delta x) = \langle f(x) f(x-\Delta x) \rangle$$ autocorrelation

$$S(\Delta x) = \left\langle \left(f(x) - f(x-\Delta x) \right)^2 \right\rangle$$ Structure function

$$S(\Delta x) = 2\left(R(0) - R(\Delta x) \right)$$ Relation between them

Hence we can use spectral analysis

$$E(k) = \left\langle \left| \widetilde{f(k)} \right|^2 \right\rangle; \quad \widetilde{f(k)} = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

$$R(\Delta x) = \int_{-\infty}^{\infty} E(k) e^{ik\Delta x} dk$$ Wiener-Khintchine theoreom

$$R(0) = \int_{0}^{\infty} E(k) dk$$

$$S(\Delta x) = 2\int_{0}^{\infty} \left(1 - e^{ik\Delta x} \right) E(k) dk$$ Real space-
Fourier space relation

Tauberian theorem

If the spectrum is of power law form:

$$E(k) \approx k^{-\beta}$$

then put

$$k' = \lambda k$$

so that:

$$E(\lambda k) = \lambda^{-\beta} E(k)$$

$$S(\Delta x/\lambda) = 2\int_{-\infty}^{\infty}\left(1-e^{ik'\Delta x/\lambda}\right)\lambda^{-\beta}E(k'/\lambda)\lambda^{-1+\beta}dk' = \lambda^{-1+\beta}S(\Delta x)$$

so that we conclude:

$$S(\Delta x) \approx \Delta x^{2H}; \quad H = (\beta - 1)/2$$

(a similar result holds for $R(\Delta x)$).

We conclude:

POWER LAWS <->F.T. POWER LAWS

Note this is valid for $1<\beta<3$ ($0\leq H\leq 1$) for $S(\Delta x)$, $1>\beta>-1$; for $R(\Delta x)$ ($-1/2<H<0$) (see later)

Practical spectral analysis

The aim is to make the most accurate spectrum

$$E(k) = \left\langle \left| \widetilde{f(k)} \right|^2 \right\rangle; \quad \widetilde{f(k)} = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

With finite data, we obtain finite spectrum over a length n:

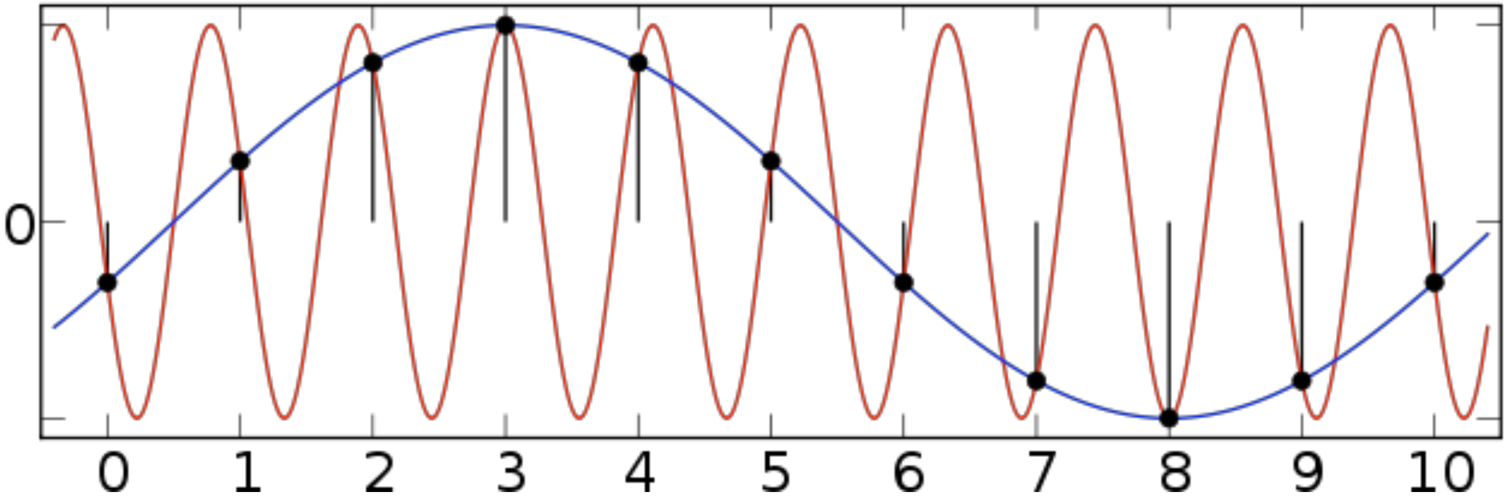
Discrete transform →

$$\left(\widetilde{f(k)} \right)_{disc} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(j) e^{2\pi i (k-1)(j-1)/n}$$

We want: $\left(\widetilde{f(k)} \right)_{disc} \approx \widetilde{f(k)}$

There is a both high wavenumber and low wavenumber bound (k=1/n, 1 respectively).

High wavenumbers: Aliasing



Black points, frequency ω=1

Red sinusoid, frequency ω_s =0.9

blue sinusoid, frequency ω_s =0.1

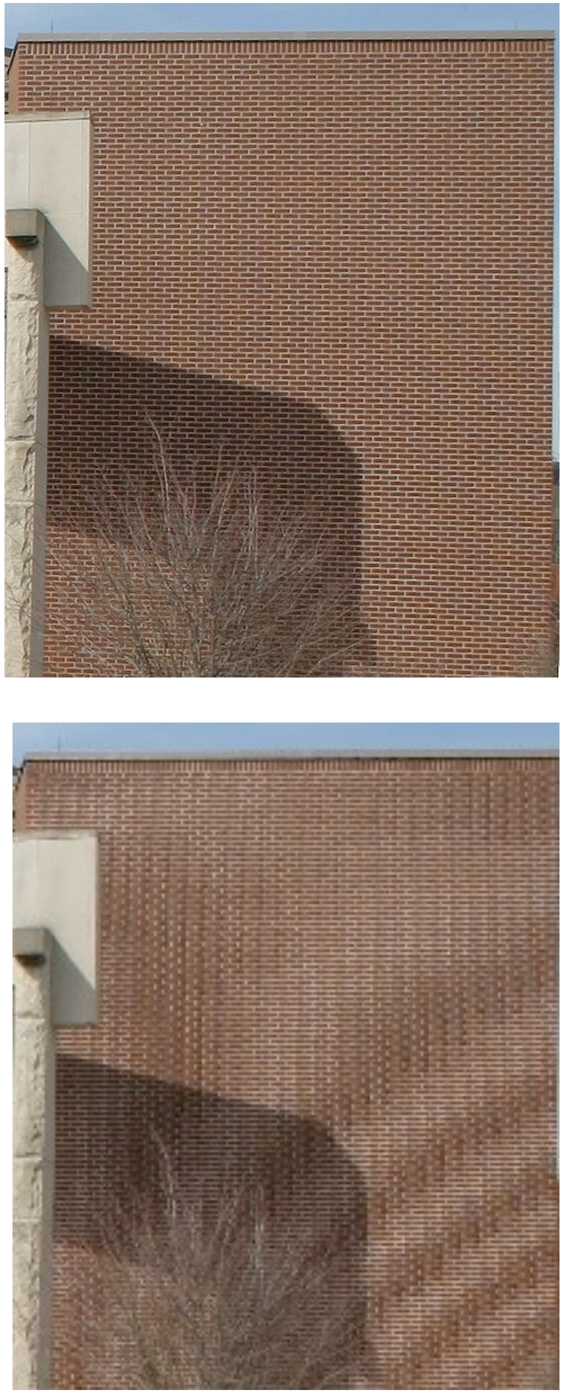
In general, when a sinusoid of frequency ω is sampled with frequency ω_s the resulting samples are indistinguishable from those of another sinusoid of frequency $|\omega - N\omega_s|$ for any integer N (here, take N=1: $|1-0.9| = 0.1$

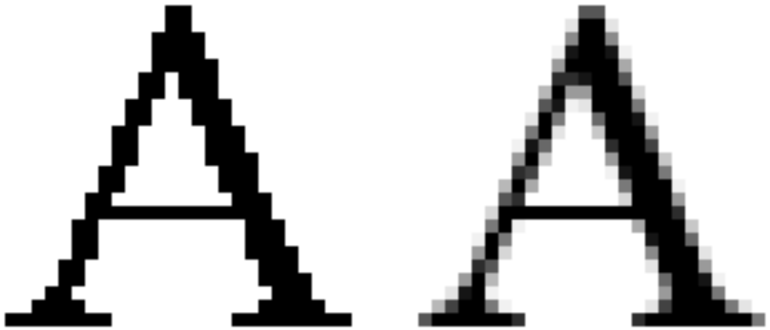
Two different sinusoids that fit the same set of samples (points at frequency ω_s= 0.9, 0.1). The black points are at a slightly higher frequency (1). If we try to reproduce the sample with Fourier analysis, we will generally have contributions from more than one frequency. To avoid aliasing it is sufficient that the original signal has no frequencies above the Nyquist frequency = $\omega_s/2$.

Analogue data at a fixed resolution Δt (by definition: it takes two “pixels” to define a sinusoid, i.e. 2Δt) has no frequencies >1/ (2Δt) so that the criterion is satisfied if the sampling is at intervals Δt. Digital data with interval Δt will be aliased if it is sampled at intervals <Δt.

Examples of Aliasing

Example of spatial aliasing is the Moiré pattern one can observe in a poorly pixelized image of a brick wall





Aliasing example of the A letter in Times New Roman. Left: aliased image, right: anti-aliased image.

Windowing, low wavenumbers/frequencies

Problem:

The low wavenumbers will be essentially the same as:

(Ignore high wavenumber issues)

$$\left(\widetilde{f(k)}\right)_{disc} = \int_0^n f(x)e^{ikx}\,dx = \int_{-\infty}^{\infty} B_n(x) f(x) e^{ikx}\,dx$$

where

$$B_n(x) = \begin{cases} 1; & 0 \le x \le n \\ 0; & otherwise \end{cases} \qquad \widetilde{B_n(k)} = \frac{2}{k} e^{-ikn/2} \sin\frac{kn}{2}$$

Hence: $\left(\widetilde{f(k)}\right)_{disc} = \widetilde{B_n(k)} * \widetilde{f(k)}$

The $\widetilde{B_n(k)}$ Thus "smears" the spectrum out so that we don't well resolve the spectral amplitudes very well. This is called "spectral leakage" because the variance at one wavenumber gets spread to neighbouring wavenumbers

Solution:

we "premultiply the function f(x) by an appropriate "windowing function" W(x)

$$\widetilde{f(k)} \approx \left(\widetilde{f(k)}\right)_{window} = \int_0^n f(x) W(x) e^{-ikx}\,dx$$

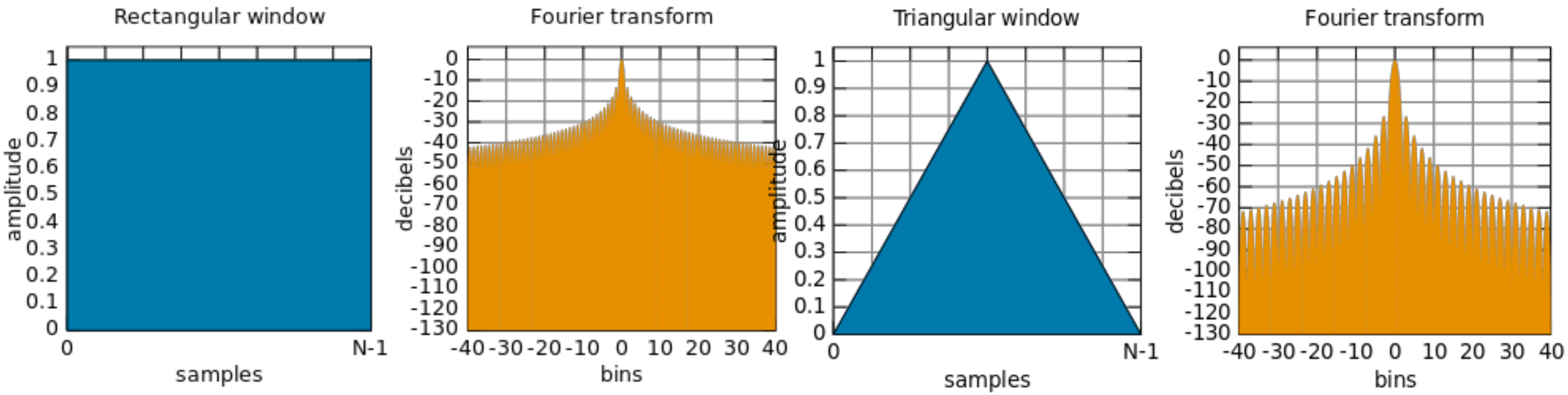
Many "windows" are possible, it doesn't make much difference which is used. Try

Hann window

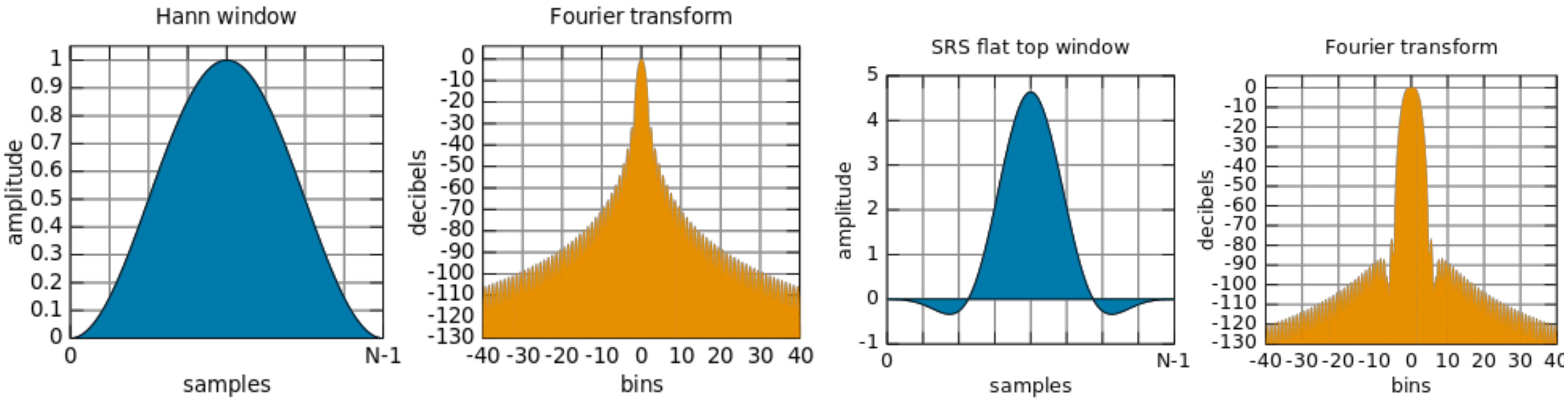
$$W(x) = \begin{cases} \frac{1}{2}\left(1 - \cos\left(\frac{2\pi x}{n}\right)\right) & 0 \le x \le n \\ 0 & otherwise \end{cases}$$

This is narrower in Fourier space, hence less "leakage"

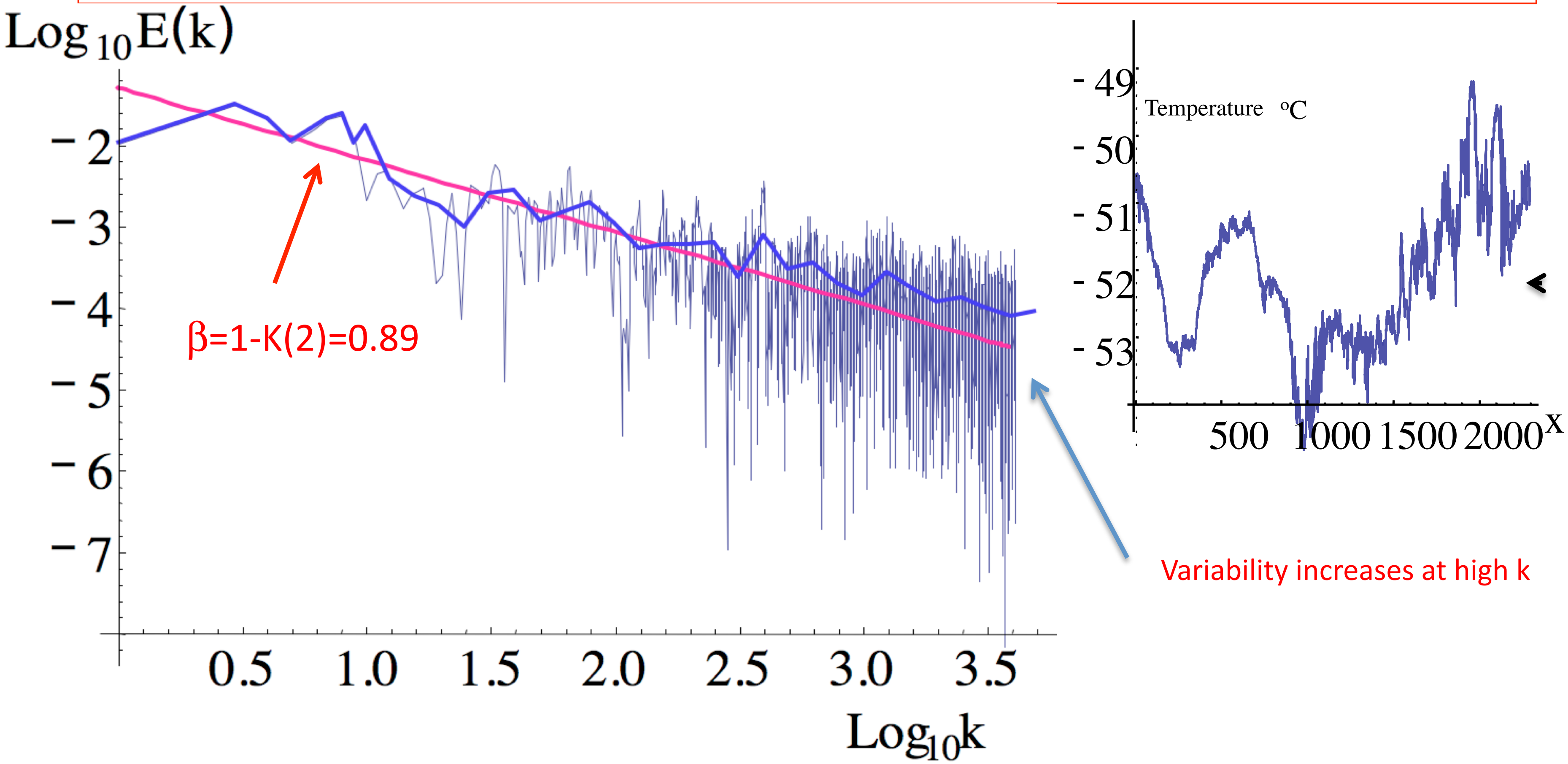
A few of the windows in the literature….



W(x)=B_n(x); the “box car” window



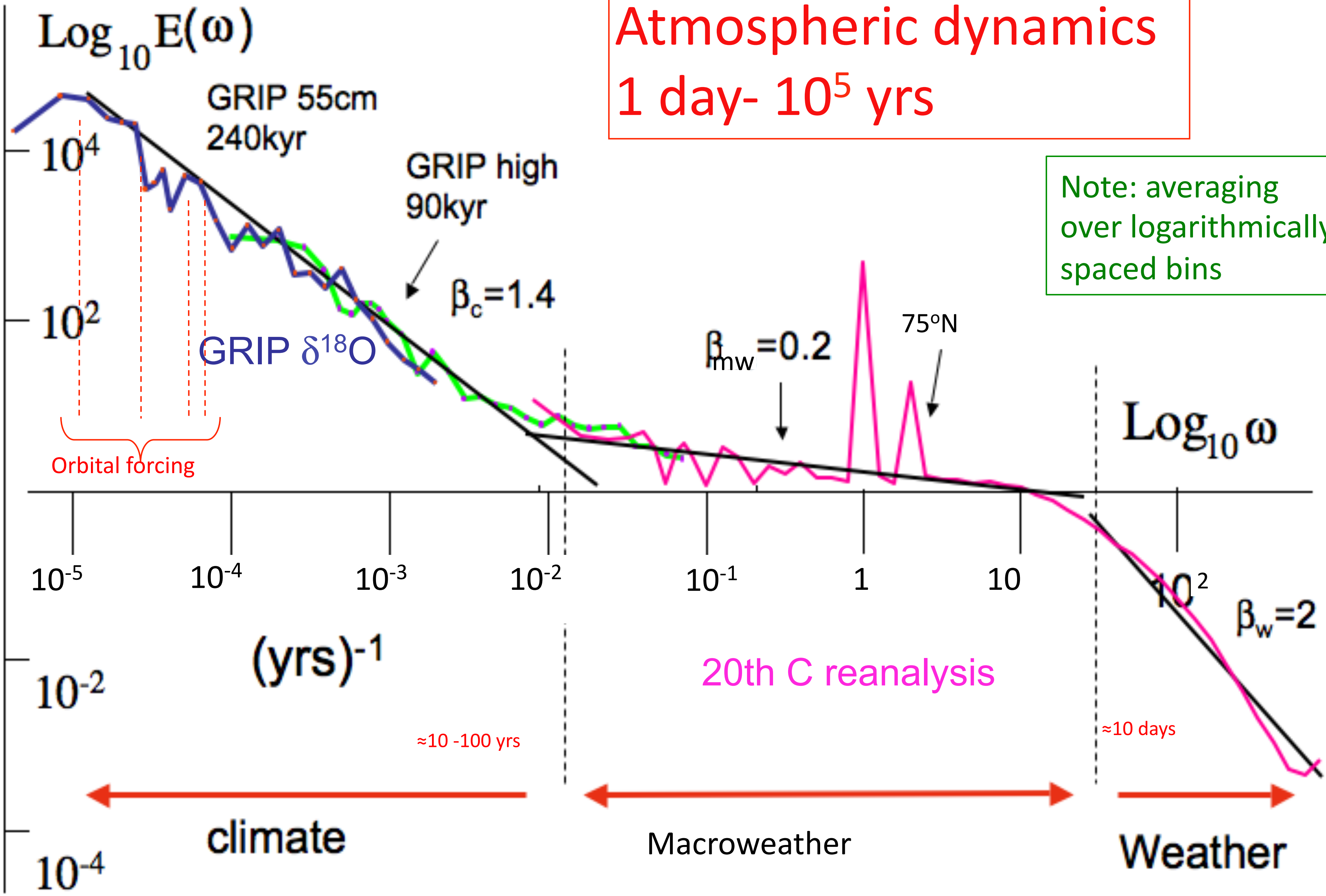
Ex: Spectrum of temperature flux



This is the spectrum (thin line) of the fluxes from the aircraft transect shown in at right with its average over logarithmically spaced bins (thick line) along with a reference line with slope -0.89 ($K(2)$ = 0.11, the value for $C_1 = K'(1) = 0.06$, $\alpha = K''(1)/K'(1) = 1.8$).

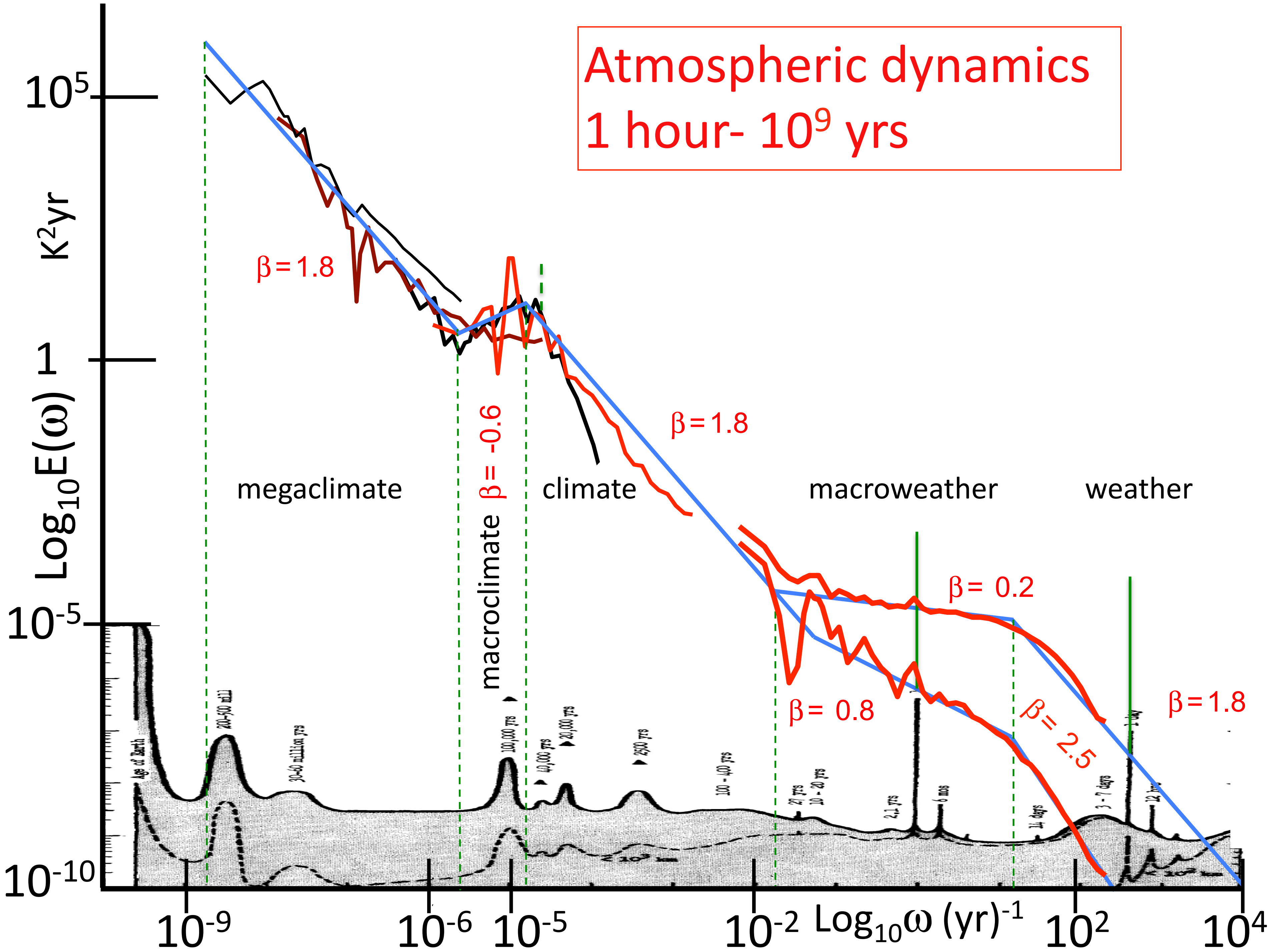
Atmospheric dynamics
1 day- 10^5 yrs

Note: averaging over logarithmically spaced bins



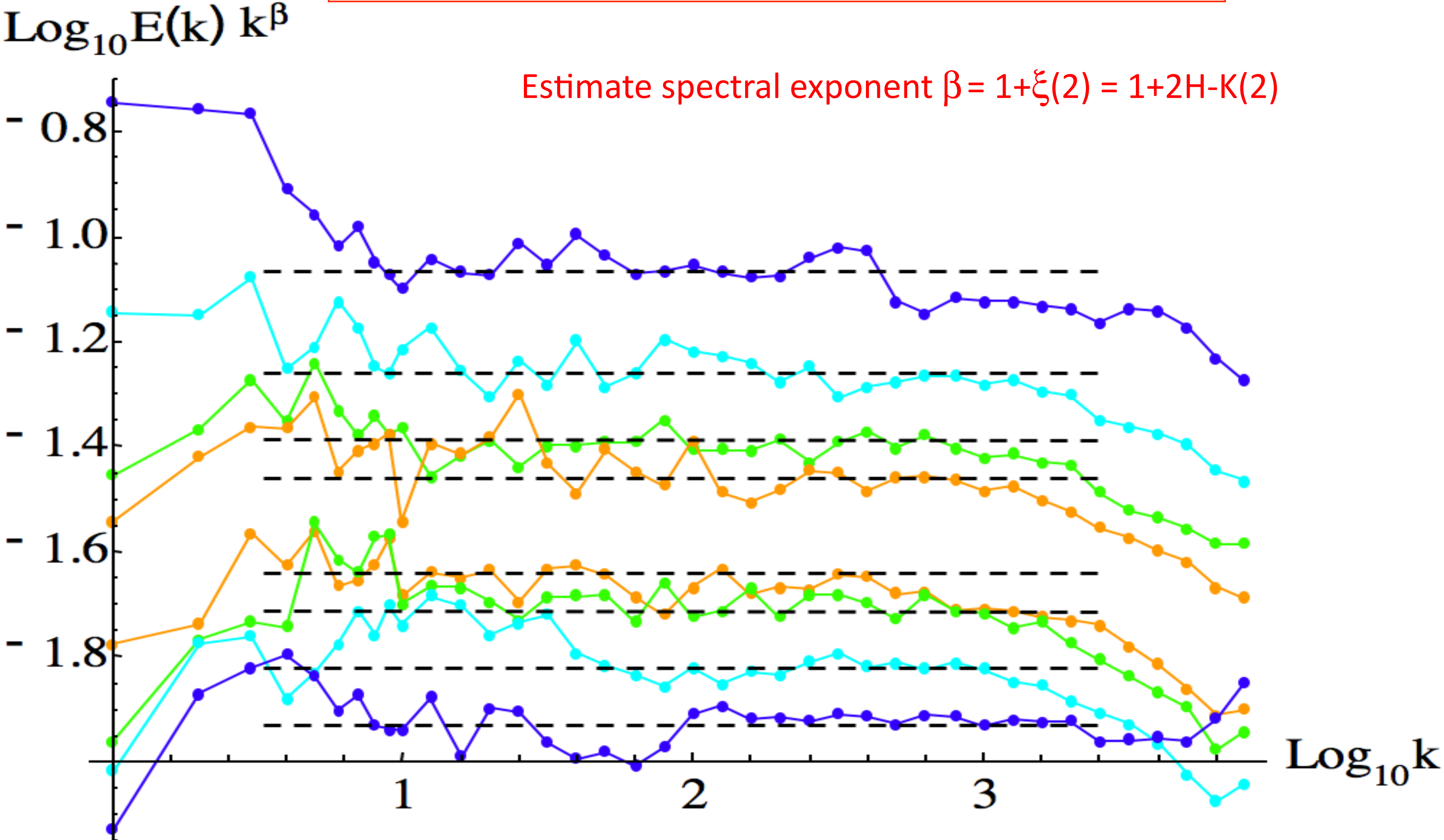
Two data sources only GRIP, 20CR

Lovejoy and Schertzer 2011



Atmospheric dynamics
1 hour- 10^9 yrs
K²yr
Log10E(ω)
10^5
1
10^-5
10^-10
β=1.8
β = -0.6
macroclimate
megaclimate
climate
β=1.8
macroweather
weather
β= 0.2
β= 0.8
β= 2.5
β=1.8
10^-9
10^-6
10^-5
10^-2
Log10ω (yr)^-1
10^2
10^4

Practical spectral analysis



The compensated spectra for an ensemble of 50 realizations, 2^{14} each, a = 1.8, C_1 = 0.1, intermittency correction = $K(2)$ = 0.18, with H increasing from top to bottom from -7/10 to 7/10. The dashed horizontal line is the theoretical behaviour indicated over the range used to estimate the exponent (i.e. the highest and lowest factor of $10^{0.5}$ in wavenumber has been dropped). Each curve was offset in the vertical for clarity.

Analysis using fluctuations

$$\Delta f = \varphi \Delta x^{H}$$

↑

Need definition of fluctuation

Classical: fluctuation = difference: $\Delta f(\Delta x) = f(x)-f(x+\Delta x)$

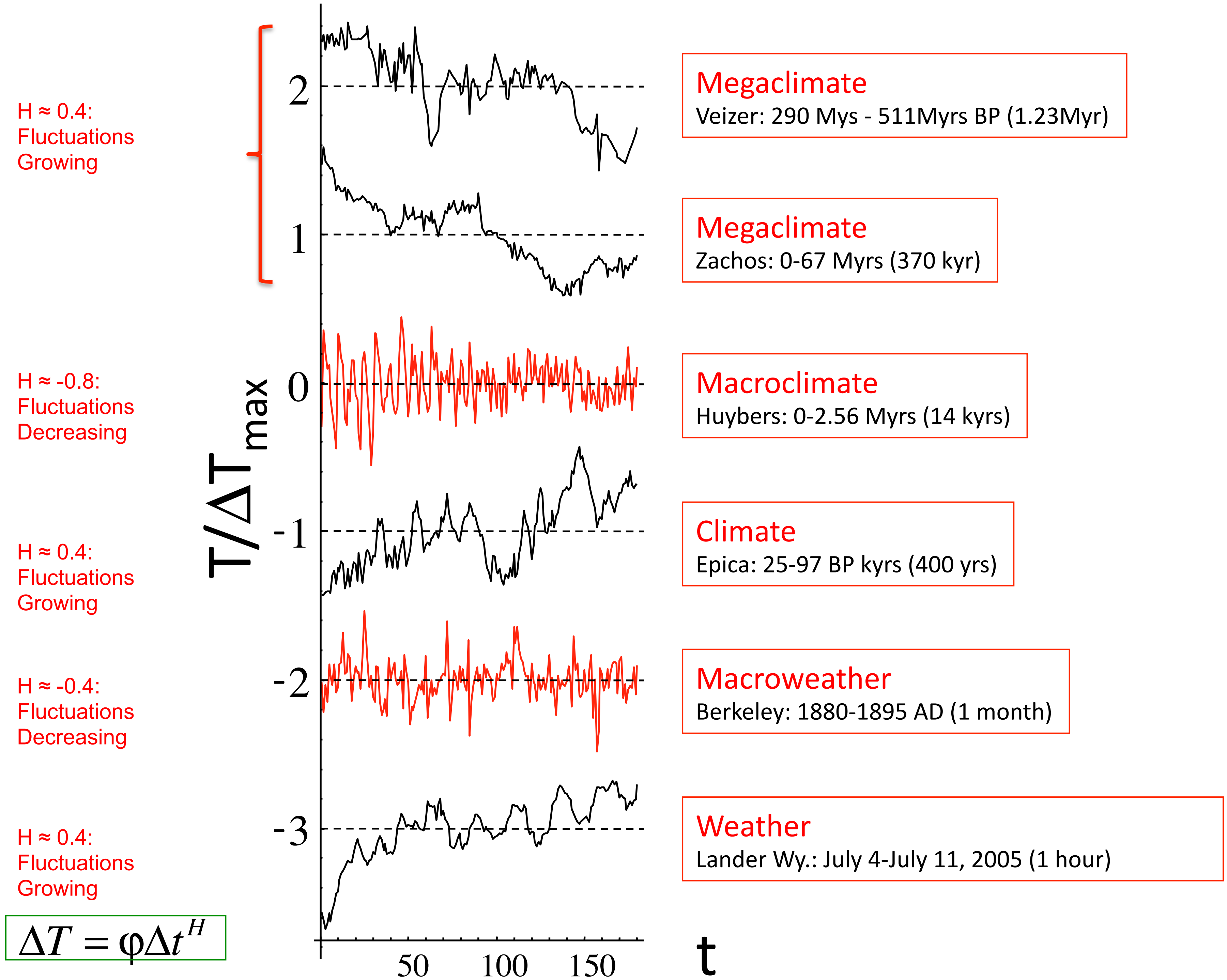
$$S_q(\Delta x) = \left\langle \Delta f(\Delta x)^q \right\rangle = \left\langle \varphi_{\Delta x}^q \right\rangle \Delta x^{qH} \approx \Delta x^{\xi(q)}; \quad \left\langle \varphi_{\Delta x}^q \right\rangle = \left(\frac{L}{\Delta x} \right)^{K(q)}; \quad \xi(q) = qH - K(q)$$

With universality:

$$K(q) = \frac{C_1}{\alpha - 1}\left(q^{\alpha} - q\right)$$

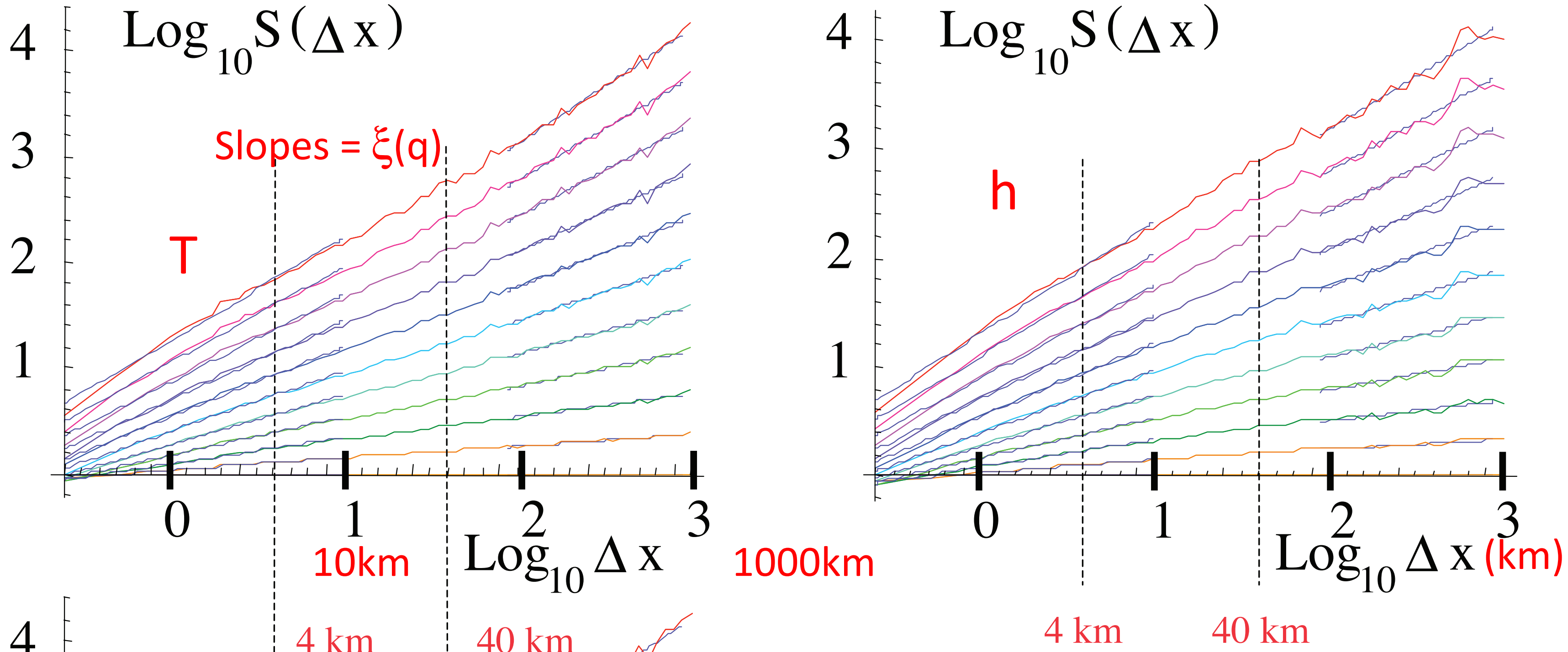
i.e. we seek H, C_1, α

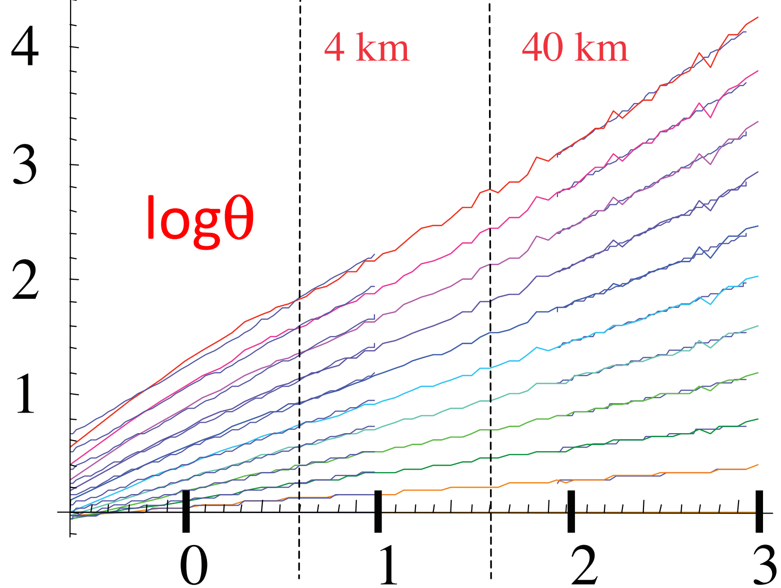
$$\xi(q) = qH - K(q) = qH - \frac{C_1}{\alpha - 1}\left(q^{\alpha} - q\right)$$



H ≈ 0.4: Fluctuations Growing
H ≈ -0.8: Fluctuations Decreasing
H ≈ 0.4: Fluctuations Growing
H ≈ -0.4: Fluctuations Decreasing
H ≈ 0.4: Fluctuations Growing
$\Delta T = \varphi \Delta t^{H}$
$T/\Delta T_{max}$
2
1
0
-1
-2
-3
50
100
150
t
Megaclimate
Veizer: 290 Mys - 511Myrs BP (1.23Myr)
Megaclimate
Zachos: 0-67 Myrs (370 kyr)
Macroclimate
Huybers: 0-2.56 Myrs (14 kyrs)
Climate
Epica: 25-97 BP kyrs (400 yrs)
Macroweather
Berkeley: 1880-1895 AD (1 month)
Weather
Lander Wy.: July 4-July 11, 2005 (1 hour)

Aircraft structure function estimates



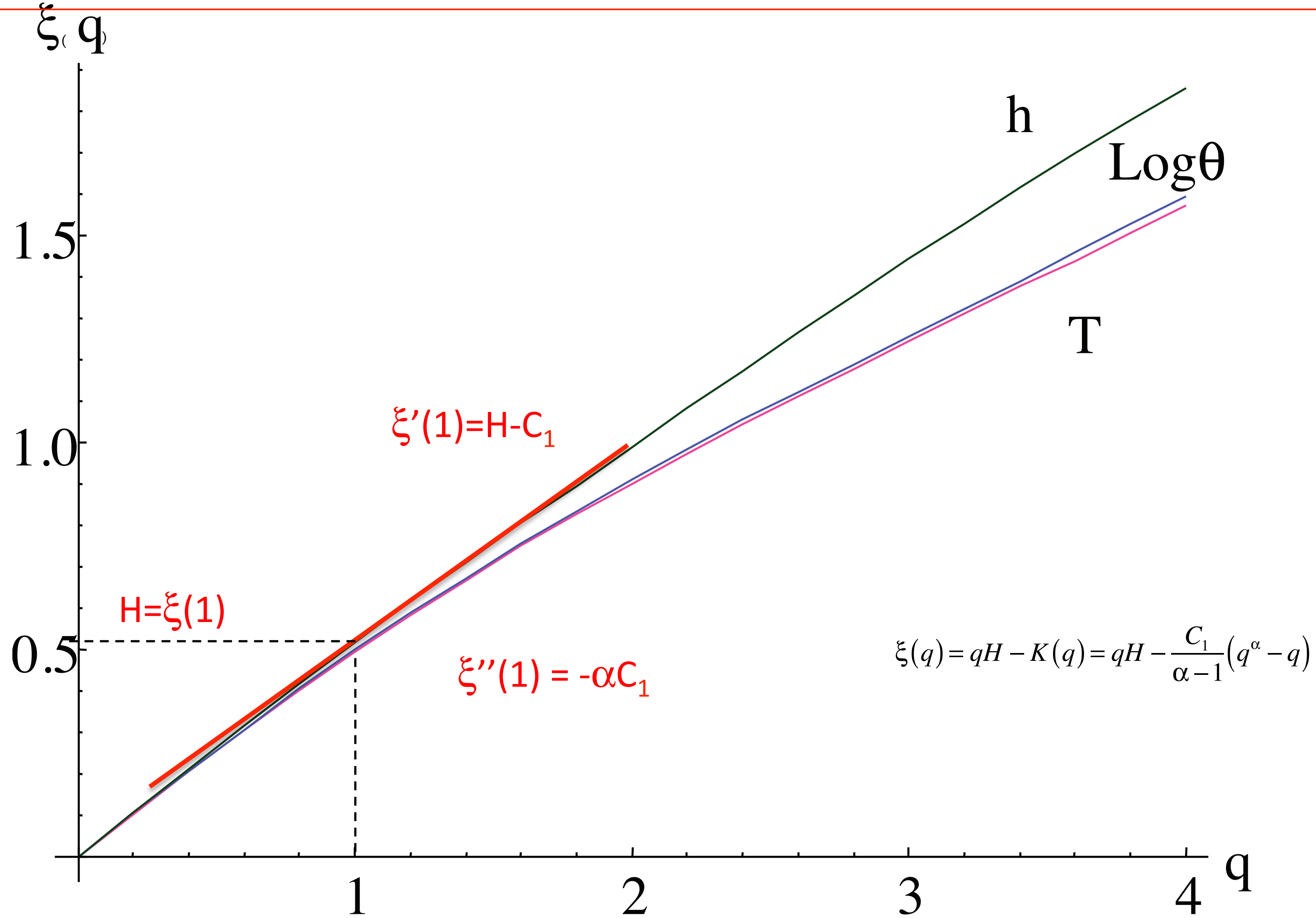


Fluctuations as differences

Temperature (Upper left), humidity (upper right), log potential temperature (lower left)

The structure functions of order q = 0.2, 0.4, ..., 1.9, 2.0 are shown (from bottom to top). All have been nondimensionalized by dividing by the absolute mean first difference at the finest scale (280 *m*)

$\xi(q)$

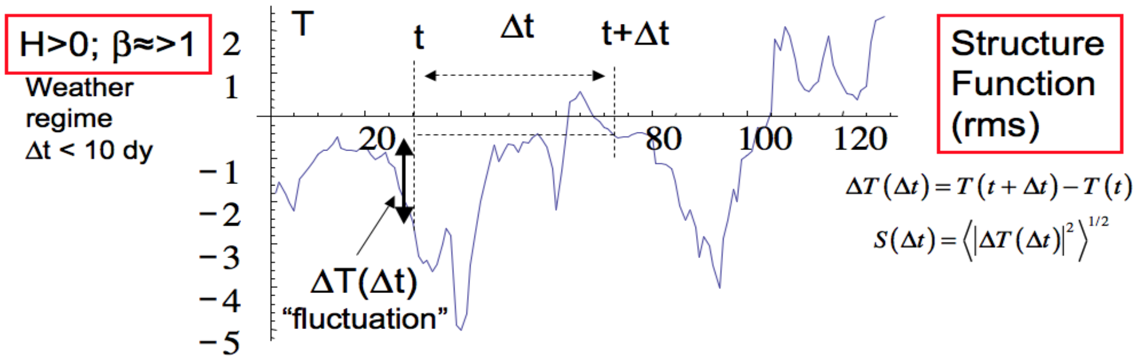


The structure function exponents for *T*, logθ, *h* from the aircraft data analysed in the previous slide. The exponents were estimated by fitting the structure functions over the "optimal" range 4 – 40 *km*.

Differences, tendencies

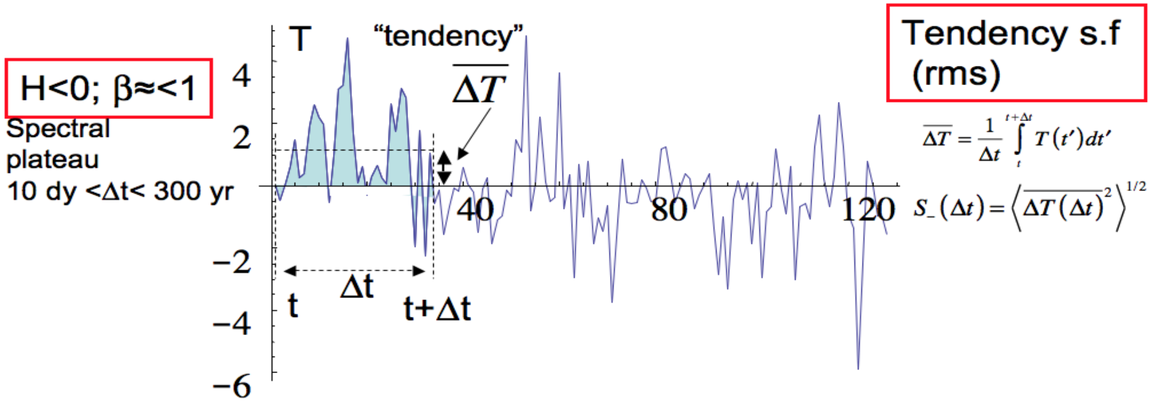
Fluctuations defined as a difference: $(\Delta v)_{diff} = v(x+\Delta x/2) - v(x-\Delta x/2)$

Tendencies $\overline{\Delta v}(\Delta x) = \frac{1}{\Delta x}\int_x^{x+\Delta x} v'(x')dx'; \quad v'(x) = v(x) - \overline{v(x)}$



$\Delta T(\Delta t) = T(t+\Delta t) - T(t)$

$S(\Delta t) = \left\langle |\Delta T(\Delta t)|^2 \right\rangle^{1/2}$



$\overline{\Delta T} = \frac{1}{\Delta t}\int_t^{t+\Delta t} T(t')dt'$

$S_{-}(\Delta t) = \left\langle \overline{\Delta T(\Delta t)}^2 \right\rangle^{1/2}$

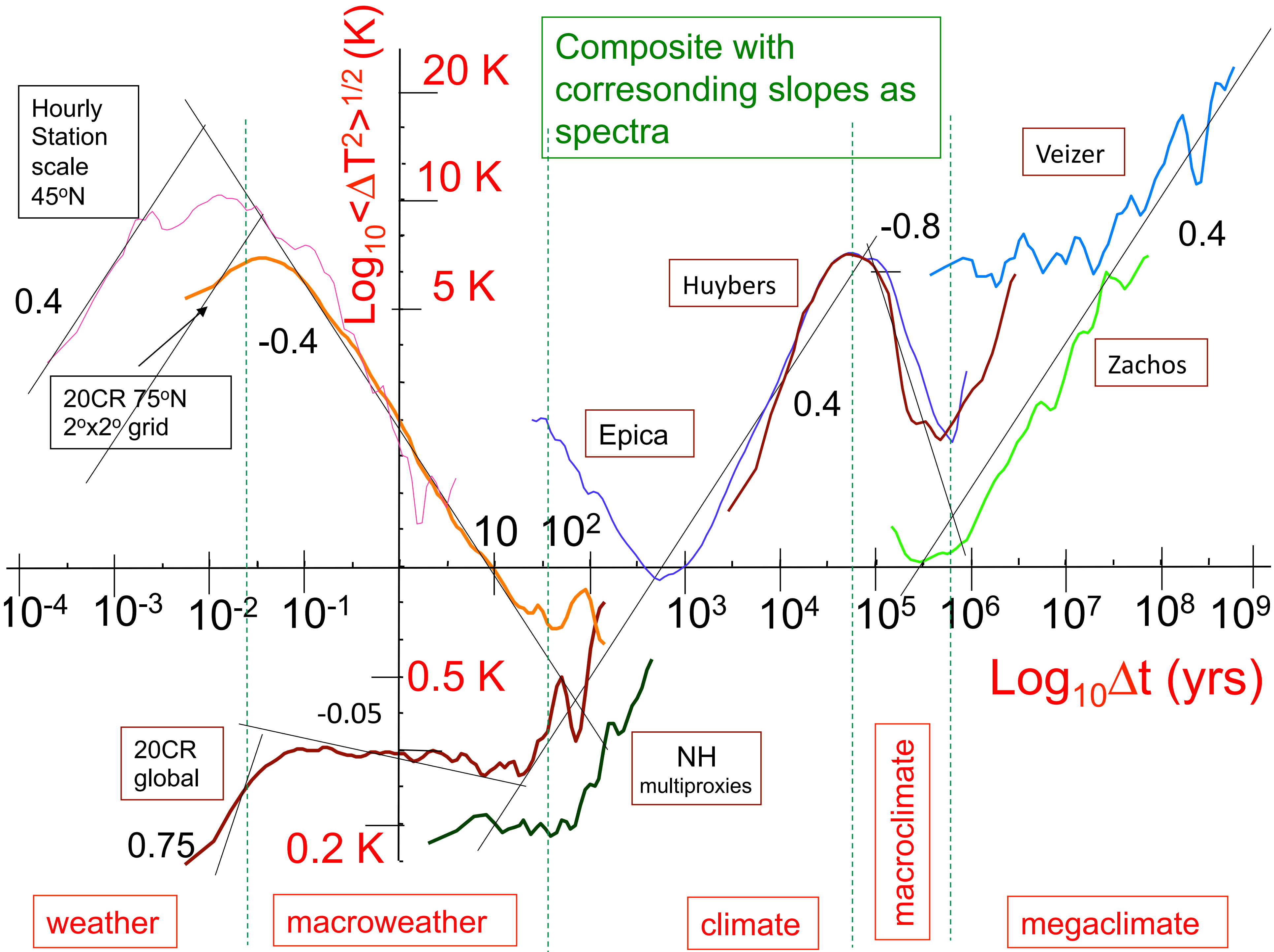
Difference, Tendency, Haar fluctuations

Differences: The difference in temperature between t and $t+\Delta t$

Tendency: The average of the temperature (with overall mean removed) between t and $t+\Delta t$

Haar: The difference between the average of the temperature from t and $t+\Delta t/2$ and from $t+\Delta t/2$ and $t+\Delta t$

Relations: When $1 > H > 0$: Haar ≈ difference
When $0 > H > -1$: Haar ≈ tendency



Composite with corresonding slopes as spectra
Hourly Station scale 45oN
20CR 75oN 2ox2o grid
20CR global
Epica
Huybers
Veizer
Zachos
NH multiproxies
0.4
-0.4
-0.05
0.75
-0.8
0.4
0.4
20 K
10 K
5 K
0.5 K
0.2 K
$Log_{10}<\Delta T^2>^{1/2}$ (K)
10^{-4} 10^{-3} 10^{-2} 10^{-1} 10 10^2 10^3 10^4 10^5 10^6 10^7 10^8 10^9
$Log_{10}\Delta t$ (yrs)
weather
macroweather
climate
macroclimate
megaclimate

Defining fluctuations using wavelets (1)

We have seen that data analyses constantly rely on defining fluctuations at a given scale and location; the simplest definition of fluctuation at position x, scale Δx, being $\Delta v(x, \Delta x) = v(x+\Delta x) - v(x)$. Note that since we typically assume that the statistics of the fluctuations are independent of position, we previously suppressed the x argument. We have already mentioned that other definitions of fluctuation are possible and are occasionally necessary, let us examine this a bit more closely.

Consider the statistically translationally invariant process $v(x)$ in 1-D: the statistics are thus independent of x and this implies that the Fourier components are "δ correlated":

$$\left\langle \tilde{v}(k)\tilde{v}(k')\right\rangle = \delta_{k+k'}\left\langle \left|\tilde{v}(k)\right|^2\right\rangle; \quad \tilde{v}(k) = \int e^{-ikx} v(x)\,dx$$

If it is also scaling then the spectrum $E(k)$ is a power law: $E(k) \approx \left\langle \left|\tilde{v}(k)\right|^2\right\rangle \approx k^{-\beta}$ (where here and below, we ignore constant terms such as factors of 2π etc.). In terms of its Fourier components, the fluctuation is thus:

$$\Delta v(x,\Delta x) = v(x+\Delta x) - v(x) = \int e^{ikx}\tilde{v}(k)\left(e^{ik\Delta x} - 1\right)dk$$

Defining fluctuations using wavelets (2)

so that the F.T. of $\Delta v(x,\Delta x)$ is $\tilde{v}(k)\left(e^{ik\Delta x}-1\right)$. We first consider the statistics of quasi-Gaussian processes for which $C_1 = 0$, $\xi(q) = Hq$. Exploiting the statistical translational invariance, we drop the x dependence and relate the second order structure function to the spectrum:

$$\left\langle \left|\Delta v(\Delta x)\right|^2 \right\rangle = 4\int e^{ik\Delta x}\left\langle \left|\tilde{v}(k)^2\right|\right\rangle \sin^2\left(\frac{k\Delta x}{2}\right)dk \approx \int e^{ik\Delta x}k^{-\beta}\sin^2\left(\frac{k\Delta x}{2}\right)dk$$

As long as the integral on the right converges, then the usual Tauberian argument shows that:

$$\left\langle \left|\Delta v(\Delta x)\right|^2 \right\rangle \propto \Delta x^{\xi(2)} \approx \int e^{ik\Delta xx}k^{-\beta}\sin^2\left(\frac{k\Delta x}{2}\right)dk \propto \Delta x^{\beta-1}$$

so that $\beta = \xi(2)+1 = 2H+1$ ($C_1 = 0$ here). However, for large k, the integrand $\approx k^{-\beta}$ which has a large wavenumber divergence whenever $\beta<1$. However, since for small k, $\sin^2(k\Delta x/2)\propto k^2$, there will be a low wavenumber divergence only when $\beta>3$.

Conclusion: $\left\langle \left|\Delta v(\Delta x)\right|^2 \right\rangle$ will be dominated by wavenumbers $k \approx 1/\Delta x$ only if: $1<\beta<3$, or equivalently, $0\leq H\leq 1$

Defining fluctuations using wavelets (3)

In the divergent cases, although real world (finite) data will not diverge, the structure functions will no longer characterize the fluctuations, but will depend spuriously on the either the highest or lowest wavenumbers present in the data. In the quasi gaussian case, or when C_1 is small, we have $\xi(2) \approx 2H$ and we conclude that the using first order differences to define the fluctuations leads to second order structure functions being meaningful in the sense that they adequately characterize the fluctuations whenever $1<\beta<3$, i.e. $0<H<1$.

Since $0<H<1$ is the usual range of geophysical H values, and the difference fluctuations are very simple, they are commonly used. However, we can see that there are limitations; in order to extend the range of H values, it suffices to define fluctuations using finite differences of different orders. To see how this works, consider using second (centred) differences:

$$\Delta v(x,\Delta x) = v(x) - \frac{1}{2}\left(v(x+\Delta x) + v(x-\Delta x)\right) = \int e^{ikx}\tilde{v}(k)\left[1 - \frac{1}{2}\left(e^{ik\Delta x/2} + e^{-ik\Delta x/2}\right)\right]dk$$

$$= 2\int e^{ikx}\tilde{v}(k)\sin\left(\frac{k\Delta x}{4}\right)^2 dk$$

repeating the above arguments we can see that the relation $\beta = \xi(2)+1$ holds now for $1<\beta<5$, or (with the same approximation) $0<H<2$.

Defining fluctuations using wavelets (4)

More generally, going beyond Gaussian processes we can consider intermiitent FIF processes, which have $\tilde{v}(k) \approx \tilde{\varepsilon}(k)|k|^{-H}$, we see that the F.T. of $\Delta v(x,\Delta x)$ is $\left(e^{ik\Delta x}-1\right)|k|^{-H}\tilde{\varepsilon}(k)$. This implies that for low wavenumbers, $\tilde{v}(k) \approx |k|^{1-H}\tilde{\varepsilon}(k)$ ($k<<1/\Delta x$) whereas at high wavenumbers, $\tilde{v}(k) \approx |k|^{-H}\tilde{\varepsilon}(k)$ ($k>>1/\Delta x$) hence since the second characteristic function of $\varepsilon(x)$ has logarithmic divergences with scale for both large and small scales ($\log\left\langle \varepsilon_\lambda^q \right\rangle = K(q)\log\lambda$), we see that for $0<H<1$, that the fluctuations $\Delta v(\Delta x)$ are dominated by wavenumbers $k \approx 1/\Delta x$, so that for this range of H, fluctuations defined as differences capture the variability of Δx sized structures, not structures either much smaller or much larger than Δx.

More generally, since the F.T. of the n^{th} derivative $d^n v/dx^n$ is and the finite derivative is the same for small k but "cut-off" at large k, we find that n^{th} order fluctuations are dominated by structures with $k \approx 1/\Delta x$ as long as $0 < H < n$. This means that $\Delta v(\Delta x)$ does indeed reflect the Δx scale fluctuations.

Finally, summing is the inverse of a finite difference (integration the inverse of differentiation), hence we can n =-1 and extend the range to -1<H<0 by summing.

Definition of wavelets

In wavelet analysis, one defines fluctuations with the help of a basic "mother wavelet" $\Psi(x)$ and performs the convolution:

$$\Delta v(x,\Delta x) = \frac{1}{\Delta x}\int v(x')\Psi\left(\frac{x'-x}{\Delta x}\right)dx'$$

↑

Needed to convert to "fluctuations"

where we have kept the notation Δv to indicate "fluctuation". The basic "admissibility" condition on $\Psi(x)$ (so that it is a valid wavelet) is that it has zero mean.

Some simple wavelets

Fluctuations defined as a difference:

$$(\Delta v)_{diff} = v(x+\Delta x/2) - v(x-\Delta x/2)$$

$$\Psi(x) = \delta(x-1/2) - \delta(x+1/2)$$

"poor man's" wavelet

Fourier: $\sin(k/2)$

Check: $\int v(x')\delta\left(\frac{x'-x}{\Delta x}-\frac{1}{2}\right)dx' = \int v\left(\left(x''+\frac{1}{2}\right)\Delta x + x\right)\delta(x'')\Delta x dx'' = \Delta x v\left(x+\frac{\Delta x}{2}\right)$ $\quad x'' = \frac{x'-x}{\Delta x}-\frac{1}{2}$

and same for second δ function

Fluctuations defined as second differences:

$$(\Delta v)_{2nd} = \frac{1}{2}\left(v(x+\Delta x/2) + v(x-\Delta x/2)\right) - v(x)$$

$$\Psi(x) = \frac{1}{2}\left(\delta\left(x+\frac{1}{2}\right) + \delta\left(x-\frac{1}{2}\right)\right) - \delta(x)$$

the second finite difference wavelet

Fourier: $\sin^2(k/4)$

Tendency Fluctuation

$$\overline{\Delta v}(\Delta x) = \frac{1}{\Delta x} \int_{x}^{x+\Delta x} v'(x')\,dx'; \quad v'(x) = v(x) - \overline{v(x)}$$

In terms of wavelets, this can be seen to be equivalent to using the wavelet:

$$\Psi(x) = I_{[-1/2,1/2]}(x) - \frac{I_{[-L/2,L/2]}(x)}{L}; \quad L >> 1$$

where *I* is the indicator function:

$$I_{[a,b]}(x) = \begin{matrix} 1 & a \le x \le b \\ 0 & otherwise \end{matrix}$$

Haar fluctuations and wavelets

$$\Delta v_{Haar}(x,\Delta x)=\frac{2}{\Delta x}\left[\int_{x}^{x+\Delta x/2} v(x')dx' - \int_{x-\Delta x/2}^{x} v(x')dx'\right]$$ -1<*H*<1

where we have added the extra $1/\Delta x$ factor so that the scaling is the same as for the poor man's fluctuation, i.e. $\xi_{Haar}(q) = \xi(q)$ for processes with $0<H<1$.

Haar wavelet

$$\Psi(x)= \begin{cases} 1/2; & 0\le x<1/2 \\ -1/2; & -1/2\le x<0 \\ 0; & otherwise \end{cases}$$

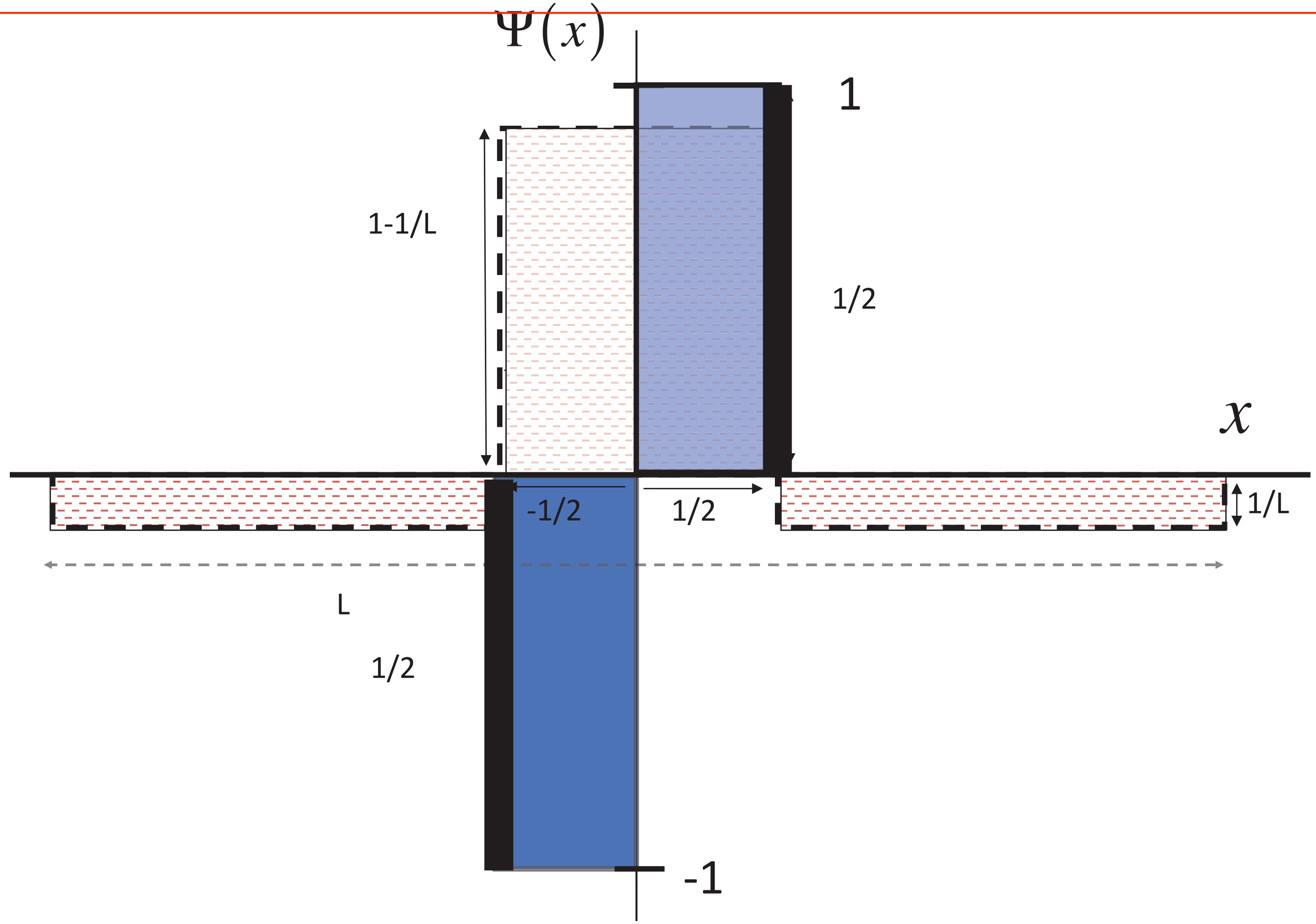
Fourier: $2ik^{-1}\sin^2(k/4)$

Quadratic Haar:

$$\Psi(x)= \begin{cases} -1/3; & 1/3<x\le 1 \\ 2/3; & -1/3\le x\le 1/3 \\ -1/3; & -1\le x<-1/3 \\ 0; & otherwise \end{cases}$$

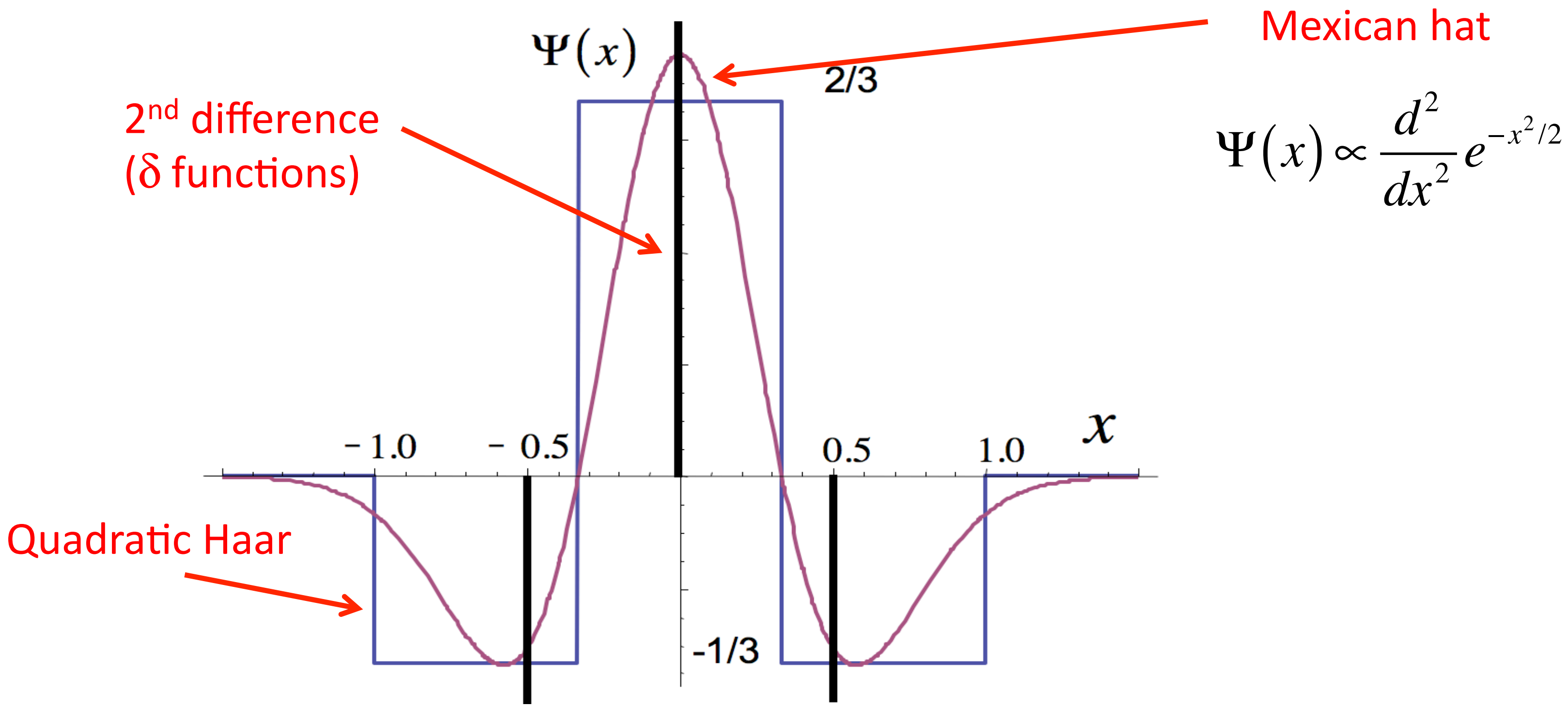
Fourier: $\frac{2}{3k}\left(\sin(k/3)-\sin k\right)$

Haar and poor man’s wavelets



This shows the “poor man’s wavelet black bars representing the amplitudes of Dirac δ functions, (the basis of the usual difference structure function, valid for $0<H<1$), the Haar wavelet (the basis of the Haar structure function, uniform blue shading, the second difference of the running sum, valid for $-1<H<1$), and the wavelet used for the “tendency” structure function valid for $-1<H<0$), stippled shading.

Comparison with Mexican Hat



The popular Mexican hat wavelet (the second derivative of the Gaussian red, valid for $-1<H<2$) compared with the (negative) second finite difference wavelet (black bars representing the relative weights of δ functions, valid for $0<H<2$)), and the second order "quadratic" Haar wavelet (blue) obtained from the third difference of the running sum (i.e. $\Delta v(\Delta x) = \left[\left(s(x+\Delta x) - s(x-\Delta x)\right)/3 - \left(s(x+\Delta x/3) - s(x-\Delta x/3)\right)\right]/\Delta x$ where $s(x)$ is the running sum valid for $-1<H<2$).

Various wavelets

Name	Real space	Fourier	Small k	Large k
Poor man's (first difference)	$\delta(x-1/2)-\delta(x+1/2)$	$2\sin(k/2)$	≈0	≈0
2nd difference	$\frac{1}{2}\left(\delta\left(x+\frac{1}{2}\right)+\delta\left(x-\frac{1}{2}\right)\right)-\delta(x)$	$\sin^2(k/4)$	≈0	≈0
Tendency	$I_{[-1/2,1/2]}(x)-\frac{I_{[-L/2,L/2]}(x)}{L}; \quad L>>1$	$\frac{2}{k}\left(\sin\left(\frac{k}{2}\right)-L^{-1}\sin\left(\frac{kL}{2}\right)\right)$	$\frac{2\sin\left(\frac{kL}{2}\right)}{kL}\approx 0; \quad kL>>1$	$\approx k^{-1}$
Haar	$\Psi(x)=\begin{cases}1/2; & 0\le x<1/2\\ -1/2; & -1/2\le x<0\\ 0; & otherwise\end{cases}$	$2ik^{-1}\sin^2(k/4)$	≈k	$\approx k^{-1}$
Quadratic Haar	$\Psi(x)=\begin{cases}-1/3; & 1/3<x\le 1\\ 2/3; & -1/3\le x\le 1/3\\ -1/3; & -1\le x<-1/3\\ 0; & otherwise\end{cases}$	$\frac{2}{3k}(\sin(k/3)-\sin k)$	$\approx k^2$	$\approx k^{-1}$
Mexican Hat	$\Psi(x)\propto\frac{d^2}{dx^2}e^{-x^2/2}$	$k^2e^{-k^2/2}$	$\approx k^2$	$e^{-k^2/2}$

Range of exponents over which average fluctuations at scale Δt corresponds to frequency $1/\Delta t$

$$\langle \Delta I \rangle = \langle \varphi \rangle \Delta t^{H}$$

Fluctuation → $\langle \Delta I \rangle$; $\langle \varphi \rangle$ = constant

$$E(\omega) = \left\langle \left| \tilde{I}(\omega) \right|^2 \right\rangle = \omega^{-\beta}$$

$$\beta = 1 + 2H - K(2)$$

$K(2)$: Multifractal "correction"

Statistic	Range of H	Range of β	Comment
Spectrum	$-\infty < H < \infty$	$-\infty < \beta < \infty$	$E(\omega) \approx \omega^{-\beta}$
Difference	0<H<1	1<β+K(2)<3	"Poor man's wavelet"
Tendency Fluctuation	-1<H<0	-1<β+K(2)<1	Average with overall mean removed (standard deviation= "Climactogram", also called the "Aggregated Standard Deviation")
Haar	-1<H<1	-1<β+K(2)<3	Difference of means of first and second halves of interval
Detrended Fluctuation Analysis (DFA, polynomial order n	-1<H<(n+1)	-1<β+K(2)<3+2n	Also multifractal extension (MFDFA), usually linear: n=1, Not a wavelet
Mexican Hat Wavelet	$-\infty < H < 2$	$-\infty < \beta + K(2) < 5$	2nd Derivative of a Gaussian
Generalized Haar	-m<H<n	1-2m<β+K(2)<3+2n	Interpretation not simple

Simple interpretation (Difference, Tendency Fluctuation, Haar)

Difference, Tendency, Haar fluctuations

Differences:

$$\left(\Delta v(\Delta x)\right)_{diff} \equiv \left|\delta_{\Delta x} v\right|; \quad \delta_{\Delta x} v = v(x+\Delta x) - v(x)$$

Difference operator

Tendency:

$$\left(\Delta v(\Delta x)\right)_{tend} = \mathcal{T}_{\Delta x} v = \left|\frac{1}{\Delta x}\sum_{x \le x' \le x+\Delta x} v'(x')\right|$$

Tendency operator

Series with mean removed

Or equivalently:

$$\left(\Delta v(\Delta x)\right)_{tend} = \left|\frac{1}{\Delta x}\delta_{\Delta x} \mathcal{S} v'\right|; \quad \mathcal{S} v' = \sum_{x' \le x} v'(x')$$

Summation operator

Haar:

Haar operator

$$\left(\Delta v(\Delta x)\right)_{Haar} = \mathcal{H}_{\Delta x} v = \frac{2}{\Delta x}\delta^2_{\Delta x/2}\mathcal{S} v = \frac{2}{\Delta x}\left(\left(s(x)+s(x+\Delta x)\right) - 2s(x+\Delta x/2)\right)$$

$$= \frac{2}{\Delta x}\left[\sum_{x+\Delta x/2 < x' < x+\Delta x} v(x') - \sum_{x < x' < x+\Delta x/2} v(x')\right]; \quad s(x) = \mathcal{S} v$$

Higher order Haar

$$\mathcal{H}^{(n)}_{\Delta x} v = \frac{(n+1)}{\Delta x}\delta^{n+1}_{\Delta x/(n+1)} s$$

Relation between tendencies, differences and Haar fluctuations

$$\mathcal{H}_{\Delta x} = 2\mathcal{T}_{\Delta x/2}\delta_{\Delta x/2} = 2\delta_{\Delta x/2}\mathcal{T}_{\Delta x/2}$$

The "saturation" relations:

$$\delta_{\Delta x} v \overset{d}{=} C_{tend} v; \quad H < 0$$

$$\mathcal{T}_{\Delta x} v \overset{d}{=} C_{diff} v; \quad H > 0$$

where C_{tend}, C_{diff} are proportionality constants and $\overset{d}{=}$ indicates equality in the random variables in the sense of probability distributions ($a \overset{d}{=} b$ if $Pr(a>\zeta) = Pr(b>\zeta)$ where "Pr" means "probability" and ζ is an arbitrary threshold).

$$\mathcal{H}_{\Delta x} v = 2\mathcal{T}_{\Delta x/2}\delta_{\Delta x/2} v \overset{d}{=} 2\mathcal{T}_{\Delta x/2} v \overset{d}{=} C'_{tend}\mathcal{T}_{\Delta x} v; \quad H < 0$$

$$\mathcal{H}_{\Delta x} v = 2\delta_{\Delta x/2}\mathcal{T}_{\Delta x/2} v \overset{d}{=} 2\delta_{\Delta x/2} v \overset{d}{=} C'_{diff}\delta_{\Delta x} v; \quad H > 0$$

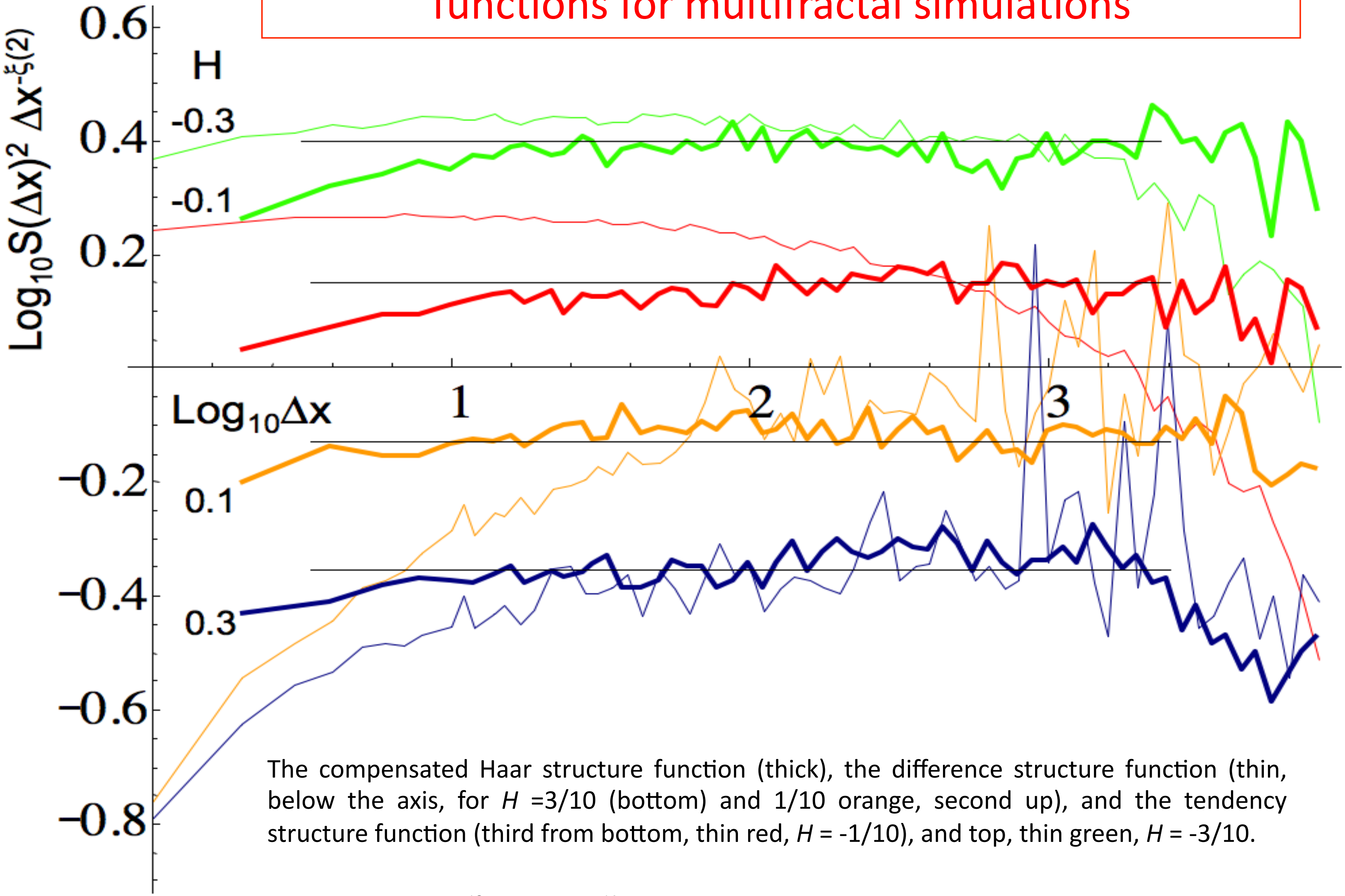
where C'_{tend}, C'_{diff} are "calibration" constants (only a little different from the unprimed quantities – they take into account the factor of two and the change from $\Delta x/2$ to Δx).

$$\frac{\left\langle (\Delta v)^q_{Haar} \right\rangle}{\left\langle (\Delta v)^q_{diff} \right\rangle} = C'^q_{diff}; \quad H > 0$$

$$\frac{\left\langle (\Delta v)^q_{Haar} \right\rangle}{\left\langle (\Delta v)^q_{tend} \right\rangle} = C'^q_{tend}; \quad H < 0$$

This shows that at least for scaling processes that the Haar structure functions will be the same as the difference (H>0) and tendency structure functions (H<0), as long as these are "calibrated" by determining C'_{diff}, and C'_{tend}.

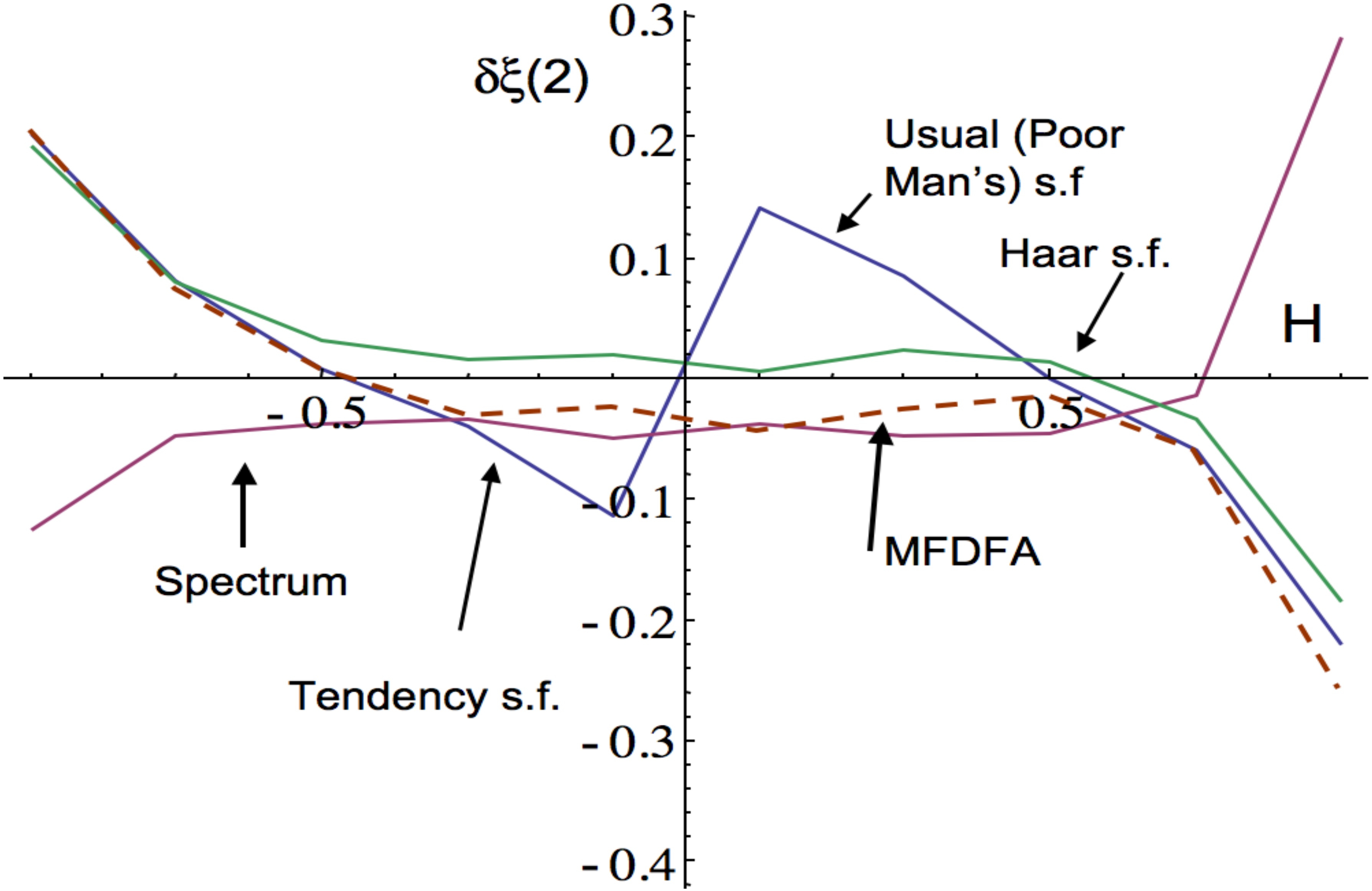
Comparing Haar, difference and tendency structure functions for multifractal simulations



The compensated Haar structure function (thick), the difference structure function (thin, below the axis, for *H* =3/10 (bottom) and 1/10 orange, second up), and the tendency structure function (third from bottom, thin red, *H* = -1/10), and top, thin green, *H* = -3/10.

50 simulations of 2^{16}, reduced to 2^{14} by averaging (improve the high frequencies) , divided in half due to periodicity (improve the low frequencies)

Comparison of different methods for estimating scaling exponents



This shows the regression estimates of the compensated exponents for the spectra $\delta\xi(2) = \xi(2)_{numerics} - \xi(2)_{theory}$ (red, perfect methods give $\delta\xi(2) = 0$), Haar structure function (q =2, green), quadratic, q =2 MFDFA (dashed), The usual difference (poor man's) structure function (q =2, blue, for H>0), and the tendency structure function (q =2, same line, blue for H<0).