PHYS 616 Multifractals and Turbulence

Lecture 3:

Turbulence and spectra

. 29, 2014

Scaling

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\underline{v} = -\frac{\nabla p}{\rho_a} + \nu \nabla^2 \underline{v} + \underline{f}$$

$$\nabla \cdot \underline{v} = 0$$

Incompressible Navier-Stokes equations

where $\underline{v}$ is the velocity, t is the time, p is the pressure, ρ_a is the (fluid) air density, ν is the kinematic viscosity, and $\underline{f}$ represents the body forces (per unit volume) due to stirring, gravity. Eq. 3 expresses conservation of momentum, whereas eq. 4 expresses, conservation of mass in an incompressible fluid: mathematically it can be considered simply as a constraint used to eliminate p.

These equations are known to be formally invariant under isotropic "zooms" $\underline{x} = \lambda^1 \underline{x}'$, as long as one rescales the other variables as:

$$\underline{v} = \lambda^{\gamma_v} \underline{v}'$$

$$t = \lambda^{-\gamma_v + 1} t'$$

$$\nu = \lambda^{\gamma_v + 1} \nu'$$

$$\underline{f} = \lambda^{2\gamma_v - 1} \underline{f}'$$

scaling

γ_v is an arbitrary scaling exponent (singularity; hence the possibility of "multiple scaling" discussed below; we do not consider the pressure since as noted, it is easy to eliminate it with the incompressibility condition.

Indeed, consider the energy flux $\varepsilon = -\partial v^2 / \partial t$ we find:

$$x = \lambda^1 x'$$
$$\varepsilon = \lambda^{-1+3\gamma_v} \varepsilon'$$

The energy dissipation/energy flux ε

If it is scale invariant, we obtain $\gamma_v = 1/3$, hence: for fluctuations in the velocity Δv over distances (lags) Δx, we obtain for the mean shear:

$$\Delta x = \lambda^1 \Delta x'$$
$$\Delta v = \lambda^{1/3} \Delta v'$$

If we eliminate λ this is perhaps more familiar:

$$\Delta v = \left(\frac{\Delta x}{\Delta x'} \right)^{1/3} \Delta v'$$

or in dimensional form:

$$\boxed{\Delta v \approx \varepsilon^{1/3} \Delta x^{H_v}; \quad H_v = \gamma_v = 1/3}$$

The Kolmogorov law real space form

which was first derived by (Kolmogorov, 1941). A similar scaling argument in Fourier space yields the famous $k^{-5/3}$ energy spectrum:

$$\boxed{E(k) = \varepsilon^{2/3} k^{-5/3}}$$

The Kolmogorov law spectral form

This already implies nondifferentiability:

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \approx \Delta x^{-2/3} \to \infty$$

Conservation of Turbulent Fluxes 1: Reynold's number

The ratio of the nonlinear term to the dissipative (viscous) term in the Navier-Stokes equation can be estimated using the "Reynolds" number:

$$\mathrm{Re} \sim \frac{\text{Nonlinear terms}}{\text{Linear damping}} = \frac{|\underline{v} \cdot \nabla \underline{v}|}{\nu |\nabla^2 \underline{v}|} \sim \frac{V \cdot L}{\nu}$$

Reynold's number

where V is a "typical" velocity of the largest scale motions L (the "outer" scale). In the atmosphere, Re is usually estimated by taking $V \approx 10\ m/s$ or structures size $10^4\ km$ (see ch. 8 for more precision, discussion). At standard temperature and pressure the viscosity of air is $\nu = 10^{-5}\ m^2/s$ hence $Re \approx 10^{12}$.

Conservation of Turbulent Fluxes 2: nonlinear terms

We now show that under certain conditions of mathematical regularity, the integral of the energy rate density ε of a fluid parcel is conserved by the non-linear terms of the Navier-Stokes equation:

$$\frac{\partial \underline{v}}{\partial t} = -(\underline{v}\cdot\nabla)\underline{v} - \nabla\left(\frac{p}{\rho_f}\right) + \nu\nabla^2\underline{v}$$

We shall see that this is a flux in Fourier space

Multiplying both sides by $\underline{v}$:

$$\varepsilon = -\frac{1}{2}\frac{\partial v^2}{\partial t} = -\underline{v}\cdot(\underline{v}\cdot\nabla)\underline{v} - (\underline{v}\cdot\nabla)\left(\frac{p}{\rho_f}\right) + \nu\underline{v}\cdot\nabla^2\underline{v}$$

Because of incompressibility (*i.e.,* $\nabla\cdot\underline{v}=0$), this can be written:

$$\varepsilon = -\nabla\cdot\left[\left(\frac{1}{2}v^2 + \frac{p}{\rho_f}\right)\underline{v}\right] + \nu\underline{v}\cdot\nabla^2\underline{v}$$

Conservation of Turbulent Fluxes 3: dissipation

Integrating over a volume of space V, it yields (due to Gauss' divergence theorem that transforms volume integrals of divergences to surface integrals):

$$\int_V \varepsilon \, dV = -\int_V \nabla \cdot \left[\left(\frac{1}{2} v^2 + \frac{p}{\rho_f} \right) \underline{v} \right] dV = -\oint_S \left(\frac{1}{2} v^2 + \frac{p}{\rho_f} \right) \underline{v} \cdot d\underline{S}$$

Nonlinear term only

where the right hand integral is over the surface only. The first term in the surface integral represents the transfer of kinetic energy across the surface, the second is the work done by pressure forces; there is no net source or sink of ε inside the volume.

We now consider the dissipation term $\nu \nabla^2 \underline{v}$. Multiplying by $\underline{v}$, ignoring the surface term we obtain:

$$\int_V \varepsilon \, dV = \nu \underline{v} \cdot \int_V \nabla^2 \underline{v} \, dV$$

viscous term only

Now, using vector identities, we have:

$$\underline{v} \cdot \nabla^2 \underline{v} = -|\nabla \times \underline{v}|^2 - \nabla \cdot [(\nabla \times \underline{v}) \times \underline{v}]$$

The second term an the right hand side is a divergence, when integrated over a volume it can be rewritten as a surface integral (Gauss' theorem):

$$\int_V \varepsilon \, dV = -\nu \int_V |\nabla \times \underline{v}|^2 \, dV - \nu \oint_S [(\nabla \times \underline{v}) \times \underline{v}] \cdot d\underline{S}$$

The change of energy/mass/time in a volume depends only on the viscosity

Since the surface integral vanishes if S is a current surface $(d\underline{S} \perp \underline{v})$ or a rigid boundary ($\underline{v} = 0$); it can be ignored if we take it to infinity. In these cases, the right hand side integrand is a positive definite quantity, $\nu > 0$, and hence the viscosity is always dissipative (decreases the total energy). Conversely, if $\nu = 0$, then ε is "conserved" by the nonlinear terms, and even when ν . 0, the dissipation will only be important at small scales where the derivatives $\nabla \times \underline{v}$ (i.e. the vorticity) are important.

Extensions to passive scalars

If we include the concentration ρ of a passive scalar quantity (*i.e.,* a quantity such as an inert dye or in atmospheric experiments chaff which is advected, transported by the wind without influencing the wind), we obtain the additional equation:

$$\frac{\partial \rho}{\partial t} = -\underline{v} \cdot \nabla \rho + \kappa \nabla^2 \rho + f_\rho$$

← Equation of passive scalar advection (+ Navier-Stokes for v)

where κ is the molecular diffusivity of the fluid.

The passive scalar equations are also formally invariant under the following scale changing operations:

$$\underline{x} = \lambda^1 \underline{x}'; \quad \underline{v} = \lambda^{\gamma_v} \underline{v}'; \quad t = \lambda^{1-\gamma_v} t';$$
$$\rho = \lambda^{\gamma_\rho} \rho'; \quad f = \lambda^{1+2\gamma_v} f'; \quad f_\rho = \lambda^{1+2\gamma_v} f_\rho';$$
$$\nu = \lambda^{1+\gamma_v} \nu'; \quad \kappa = \lambda^{1+\gamma_\rho} \kappa'$$

Scale invariance of passive scalar advection equations

where γ_v, γ_ρ are arbitrary. This arbitrariness allows the possibility of multiple scaling (*i.e.,* weak and intense turbulent regions which scale differently, and have different fractal dimensions), hence the solutions can in principle be multifractals.

By repeating arguments similar to the above for ρ^2 rather than v^2, one can check that the scalar variance flux:

$$\boxed{\chi = -\frac{1}{2}\frac{\partial \rho^2}{\partial t}}$$

New quadratic invariant (conserved by nonlinear terms)

Passive scalars:

convervation of variance rate, variance flux

χ is analogous to ε which will beis conserved by the non-linear terms $\underline{v}\cdot\nabla\rho$. Putting $\kappa = 0$ and recalling $\nabla\cdot\underline{v} = 0$:

$$\chi = -\frac{1}{2}\frac{\partial\rho^2}{\partial t} = -\rho\underline{v}\cdot\nabla\rho = -\frac{1}{2}\nabla\cdot\left(\underline{v}\rho^2\right)$$

Note: the dissipation term $\kappa\rho\nabla^2\rho$ has been set to zero

hence:

$$\int_V \chi\, dV = -\frac{1}{2}\oint_S \rho^2 \underline{v}\cdot d\underline{S}$$

this shows that there is no volume contribution to the passive scalar variance, it will be conserved by the nonlinear $\underline{v}\cdot\nabla\rho$ term.

Using the conservation of χ one obtains $\gamma_\rho = (1-\gamma_v)/2$ and since from eq. 9, (from the conservation of ε), $\gamma_v = 1/3$ so that we find $\gamma_\rho = 1/3$. This yields the result analogous to the Kolmogorov law; the "Corrsin- Obukhov law of passive scalar advection" (Corrsin, 1951), (Obukhov, 1949) which in dimensional form is:

$$\boxed{\Delta\rho = \chi^{1/2}\varepsilon^{-1/6}\Delta x^{H_\rho}; \quad H_\rho = \gamma_\rho = 1/3}$$

Corrsin-Obhukhov law (real space)

i.e. with the same exponent *H* as the Kolmogorov law. Similarly, the Fourier space version is:

$$\boxed{E_\rho(k) = \chi\varepsilon^{-1/3}k^{-5/3}}$$

Corrsin-Obhukhov law (Fourier space)

Classical isotropic 3D turbulence phenomenology: Kolmogorov turbulence and energy cascades

Cascade locality: The last ingredient needed to justify cascade models is to show that the energy transfer is most efficient between neighbouring scales: that it is "local" in Fourier space.

Locality: read appendix 2B

Consider a discrete hierarchy of eddies, broadly defined a fluid "coherent" structure.

v_n = an appropriate characteristic velocity difference
τ_n = the time scale called the "eddy turnover time" which is the typical time necessary for the dynamics to pass energy fluxes from one scale to another
l_n, = the length scale (size of the eddy).

The Navier Stokes equations are Galilean invariant, hence It is the *shear* that is important because it is the difference of velocity across an eddy which intervenes, not the "mean" velocity of an eddy.

The subscript n refers to the number of octaves from the largest "outer scale", thus ℓ_n refers to all values of ℓ in the interval $\left[\frac{\ell_n}{\sqrt{2}}, \sqrt{2}\,\ell_n\right]$.

Typical velocity difference at the nth octave

$$v_n = \sqrt{\left\langle \left|\underline{v}(\underline{r}) - \underline{v}(\underline{r} + \underline{l}_n)\right|^2 \right\rangle}$$

The $\langle\cdots\rangle$ is the ensemble statistical average. Likewise

$$\tau_n \sim \ell_n / v_n,$$

"Eddy turn over" time (lifetime)

is the "eddy turnover time", is the typical time scale of the transfer process. Finally (again using dimensional analysis) the viscous time scale corresponding to the n^{th} octave is:

$$\tau_{n,dis} = \frac{\ell_n^2}{\nu}$$

Viscous time scale

Viscosity can be ignored if $\tau_{n,dis} >> \tau_n$ (*i.e.,* the viscosity is too slow to affect the dynamics).

Quasi-steady energy flux

Denote by Π_n the rate at which energy is transferred out of a low wavenumber octave $k_n/\sqrt{2} \le k \le \sqrt{2}k_n$) into a higher octave $\sqrt{2}k_n \le k \le 2\sqrt{2}k_n$). This is a Fourier-space energy flux. It is given by the energy per unit mass in the octave (E_n) divided by the typical time scale of the transfer, the eddy turn-over time:

$$\Pi_n \approx \frac{E_n}{\tau_n}$$

Flux of energy through the n[th] octave

Now assume that the cascade is local so that the dominant contribution to E_n comes from the velocity gradient at the same scale, i.e. v_n. This implies $E_n \sim v_n^2$ (recall that due to incompressibility all energies are taken per unit mass) and that $\tau_{vis,n} >> \tau_n$ so that there is no energy dissipation in this wave number band.

Assume that the energy injection rate ε (*e.g.,* by stirring) at large scale is balanced by viscous dissipation at small scale then it is possible that the system is stationary (statistically invariant under translations in time) then $\Pi_n \sim \text{constant}$, *i.e.,* there are no viscous losses and no sinks nor sources. This is assumed to be a quasi-steady state: energy flows through the n^{th} octave at a rate ε which is on average equal to the large scale injection rate and to the small scale dissipation (as we will see, such statistical stationarity is quite compatible with violent fluctuations):

$$\varepsilon = \Pi_n \sim \frac{E_n}{\tau_n} \sim \frac{v_n^2}{\left(\dfrac{\ell_n}{v_n}\right)} \sim \frac{v_n^3}{\ell_n} \sim \text{constant}$$

Quasi steady flux of energy from large structures to small

(assuming that the injection rate is constant). Π_n is therefore a scale invariant quantity (it is independent of n). This yields Kolmogorov's law (1941):

$$v_n \sim \varepsilon^{1/3} \ell_n^{1/3}$$

The Kolmogorov-Obukhov Spectrum

Since the fluctuation v_n is a scaling power law function of size l_n, we expect that the spectrum will also be power law (see section 2.4.5 for more details on Tauberian theorems that relate real space and Fourier space scaling). For wavenumber p, we therefore seek the spectral exponent β:

$$E(p) \sim p^{-\beta}$$

corresponding to the real space exponent 1/3 in eq. 2.50. Assuming $\beta > 1$ we get the following expression for the total variance due to all the low wavenumbers in the band:

$$v_n^2 \approx \ell_n^2 \int_{k_n/\sqrt{2}}^{\sqrt{2}k_n} dp\, p^2\, E(p)$$

(since the variance in a spherical shell between p and p+dp is $4\pi\, p^2 \mathrm{d}p$, and we ignore the constant factor). We thus obtain:

$$v_n^2 \approx \ell_n^2 k_n^{3-\beta} \sim \ell_n^{2-3+\beta}$$

(since $\ell_n \sim k_n^{-1}$). This implies $2 - 3 + \beta = \frac{2}{3}$ or:

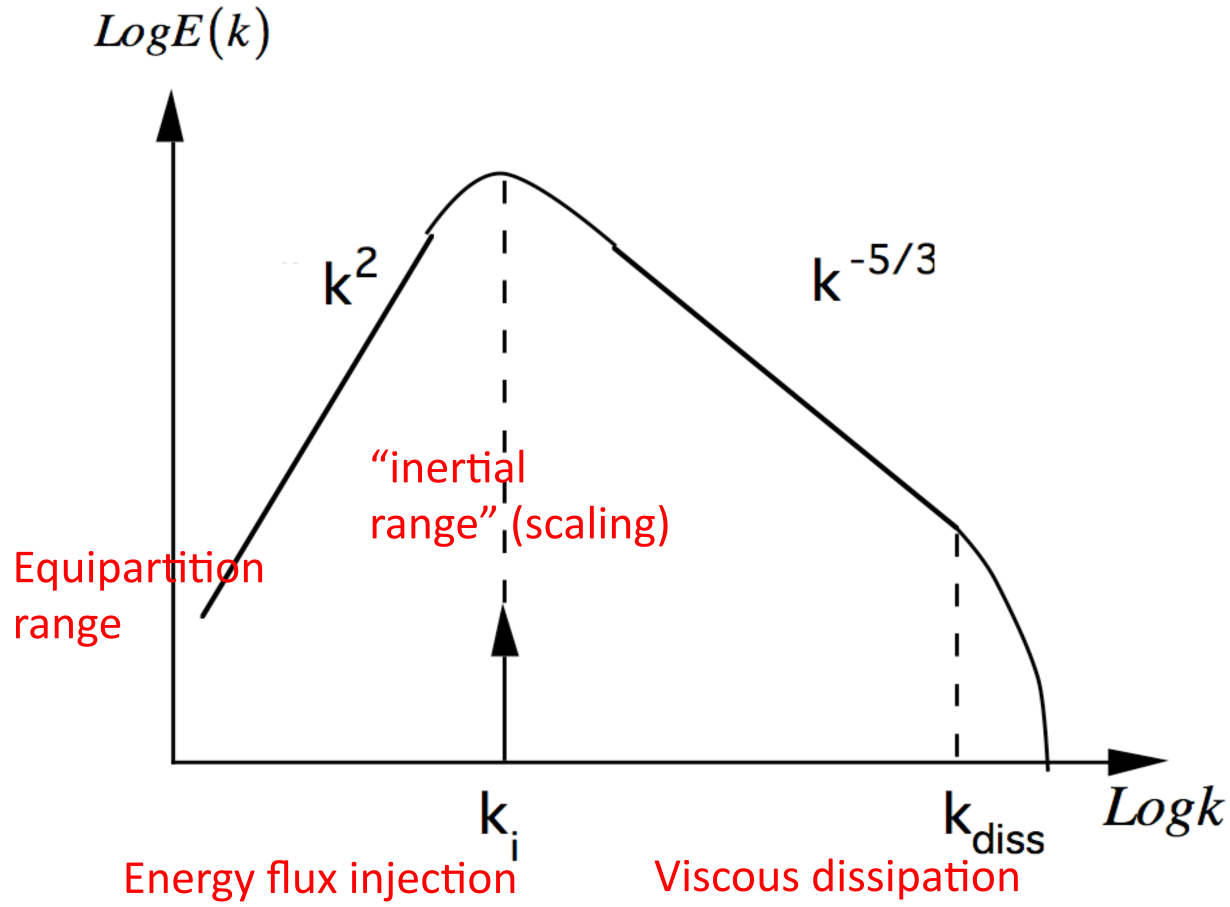
$$\beta = \frac{5}{3}$$

The Kolmogorov-Obukhov spectrum is thus derived:

$$E(k) \sim \varepsilon^{2/3} k_n^{-5/3}$$

The Kolmogorov-Obukhov Spectrum

Schematic diagram of 3-D energy cascade



The Special Case of 2-D Turbulence

The vorticity equation is obtained by taking the curl of the velocity equation:

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\nabla)\underline{v} + \nu\nabla^2\underline{\omega} + \underline{b};\quad \underline{b} = \frac{1}{\rho^2}\nabla\rho\times\nabla p$$

Vorticity eq.

Vortex stretching term

Baroclinicity vector

Recall:

$$\frac{D\underline{\omega}}{Dt} = \frac{\partial\underline{\omega}}{\partial t} + \underline{v}\cdot\nabla\underline{\omega}$$

In 2-D:

For a two dimension flow the vorticity $\underline{\omega}$ must beis perpendicular to $\underline{v}$ (*i.e.*, $\underline{\omega} = \omega_z\hat{z}$, $\omega_z = \partial v_y/\partial x - \partial v_x/\partial x$, $\omega_x = \omega_y = 0$ since $v_z = 0$, $\partial/\partial z = 0$; consequently:. We therefore can no longer have any vortex stretching since:

$$(\underline{\omega}\cdot\nabla)\underline{v} \equiv 0$$

No Vortex stretching

i.e. there is no longer any vortex stretching and the incompressible vorticity equation reduces to In 2-D we thus obtain an advection-dissipation equation for the vorticity:

$$\frac{D\underline{\omega}}{Dt} = \nu\nabla^2\underline{\omega}$$

No Vortex stretching

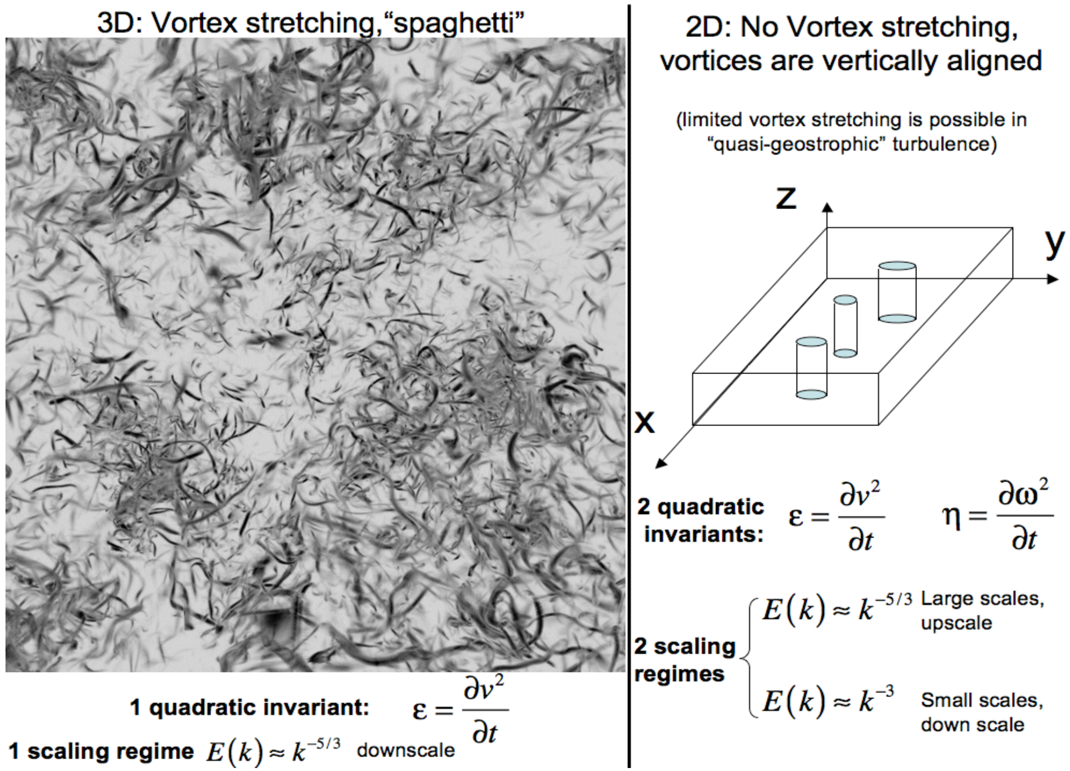
when the dissipation is negligible, any power of the vorticity is conserved, not only the enstrophy which is its square. We can define the enstrophy flux density:

$$\eta = -\frac{1}{2}\frac{\partial\omega^2}{\partial t}$$

Enstrophy flux density

In 3D, Vortex stretching implies the direction of the cascade is from large to small scales

Read section 2.4.3



Vortex tubes (surfaces bounded by lines of vorticity) are material tubes (section 2.4.3) hence if the ends of the tube move further apart as the flow evolves (a kind of "drunkard's walk in a complex flow), then since the tubes are incompressible, the cross-section must tend to get smaller, hence the creation of small structures from large due to vortex stretching.

Spectral Analysis in arbitrary dimensions 1

We generalize the 1-D results to higher dimensions, here we consider space rather than time. Consider the second order velocity correlation tensor:

$$u_{ij}(\underline{r}) = \langle v_i(\underline{r}') v_j(\underline{r}' + \underline{r}) \rangle = \langle v_i(\underline{r}') v_j(\underline{r}' - \underline{r}) \rangle$$

Statistical homogeneity: translational invarance

the symmetry under inversion follows from translational invariance (this can be a function of time but we will not denote this explicitly). We will go on assuming statistical homogeneity, *i.e.,* independence of translation when it applies to translation in time and space.

Furthermore, we will also assume that the turbulence is statistically isotropic (independent of direction). Then we have $u_{ij}(\underline{r}) = u_{ji}(\underline{r})$ and $u_{ij}(\underline{r}) = u_{ij}(r)$ where $r = |\underline{r}|$. We can define $u(r) \equiv u_{ii}(r)$ the trace of the velocity correlation tensor (using Einstein's notation convention for summing over a repeated index) and the average energy per unit mass is thus:

$$e = \frac{1}{2}\left\langle |\underline{v}(0)|^2 \right\rangle = \frac{1}{2} u(0)$$

(by spatial homogeneity, there is no $\underline{r}$ dependence).

Spectral Analysis in arbitrary dimensions 2

Introducing the *d*-dimensional Fourier transform and its inverse:

$$\tilde{u}(\underline{k}) = \int d^d \underline{k}\; e^{-i\underline{k}\cdot\underline{r}} u(\underline{r}); \quad u(\underline{r}) = \int d^d \underline{k}\; e^{i\underline{k}\cdot\underline{r}} \tilde{u}(\underline{k})$$

we obtain:

$$\tilde{u}(0) = \int d^d \underline{r}\; u(\underline{r})$$

(setting $\underline{k}$ = 0). We now wish to exploit the isotropy by performing the *d*-dimensional Fourier space integral above over (*d* −1)-dimensional "annuli" or "shells". We obtain

$$e = \int_0^\infty dk\; E(k)$$

where *e* is the total energy per unit mass and

$$E(k) \sim k^{d-1}\tilde{u}(k)$$

is the (isotropic) "energy spectrum" and where $k = |\underline{k}|$ (in one dimension the integral is $\int u dk$, in two dimensions $\int u 2\pi k \mathrm{d}k$, in three dimensions $\int u 4\pi k^2 dk$).

Spectral Analysis in arbitrary dimensions 3

Consider $\underline{\tilde{v}}(\underline{k})$, the Fourier transform of $\underline{v}(\underline{r})$ then the inverse transform gives:

$$\underline{\tilde{v}}(\underline{k}) = \int d\underline{r}\ e^{-i\underline{k}\cdot\underline{r}}\underline{v}(\underline{r}); \quad \underline{v}(\underline{r}) = \int d\underline{k}\ e^{i\underline{k}\cdot\underline{r}}\underline{\tilde{v}}(\underline{k})$$

Taking complex conjugate of the right hand equation and assuming $\underline{v}(\underline{r})$ is real, where we obtain: $\underline{\tilde{v}}(\underline{k}) = \underline{\tilde{v}}^*(-\underline{k})$.

For the energy tensor we obtain:

$$u(\underline{r}) = \langle \underline{v}(\underline{r}') \cdot \underline{v}(\underline{r}'+\underline{r}) \rangle = \int d^d\underline{k} d^d\underline{k}'\ e^{i\underline{k}\cdot\underline{r}} e^{i(\underline{k}+\underline{k}')\cdot\underline{r}'} \langle \underline{\tilde{v}}(\underline{k}) \cdot \underline{\tilde{v}}(\underline{k}') \rangle$$

Now, statistical homogeneity means that the right hand side is independent of $\underline{r}$. This implies that the only contribution to the double integral is from $\underline{k} = -\underline{k}'$, hence:

$$\langle \underline{\tilde{v}}(\underline{k}) \cdot \underline{\tilde{v}}(\underline{k}') \rangle = P(\underline{k})\delta(\underline{k}+\underline{k}')$$

This defines the spectral density $P(k)$ and shows that a statistically homogeneous field can be represented as the integral over statistically independent pairs of waves with wavevectors $\underline{k}$ and $-\underline{k}$,and with random amplitudes P.

Using this result we obtain a *d*-dimensional Wiener-Khintchine theorem:

$$u(\underline{r}) = \int d^d\underline{k}\ e^{i\underline{k}\cdot\underline{r}} \left\langle \left| \underline{\tilde{v}}(\underline{k}) \right|^2 \right\rangle$$

which relates the autocorrelation function of a stationary process to its harmonic representation via a Fourier transform.

Spectral Analysis in arbitrary dimensions 4

Putting $\underline{r}$ = 0 shows:

$$u(0) = \int d^d \underline{k} \left\langle \left| \tilde{\underline{v}}(\underline{k}) \right|^2 \right\rangle$$

and using isotropy (and ignoring constant factors such as 4π):

$$u(0) = \int_0^\infty dk \, \mathrm{E}(k) = \int_0^\infty dk \, k^{d-1} \mathrm{P}(k)$$

Hence if we attribute an energy $\frac{1}{2} P(k)$ to each wavenumber $\underline{k}$ then the total energy in Fourier space equals $\frac{1}{2} u(0)$ which is the energy per unit mass. We also see immediately that:

$$E(k) = k^{d-1} \frac{1}{2} \left\langle \left| \tilde{\underline{v}}(k) \right|^2 \right\rangle$$

Hence if the d-dimensional spectral density $P(\underline{k})$ is:

$$P(\underline{k}) = \left\langle \left| \tilde{\underline{v}}(\underline{k}) \right|^2 \right\rangle \approx \left| \underline{k} \right|^{-s}$$

then:

$$E(k) = k^{-\beta}; \quad k = \left| \underline{k} \right|; \quad \beta = s + 1 - d$$

Spectral Analysis in arbitrary dimensions 5

Concerning the enstrophy spectrum, we now repeat the above arguments, but for $\left\langle \nabla^2 \underline{v} \right\rangle$ (recalling that in Fourier space $\nabla^2 \rightarrow -k^2$ and using: the identity (assuming statistical translational invariance) $\left\langle \omega^2 \right\rangle = -\left\langle \underline{v} \cdot \nabla^2 \underline{v} \right\rangle$ we obtain:

$$\left\langle \left| \underline{\omega} \right|^2 \right\rangle = \left\langle \underline{v} \cdot \nabla^2 \underline{v} \right\rangle = \int d^d \underline{k} \; k^2 \left\langle \left| \tilde{\underline{v}}\left(\underline{k}\right) \right|^2 \right\rangle$$

and integrating as usual over angles in Fourier space:

$$\left\langle \left| \underline{\omega} \right|^2 \right\rangle = \int_0^\infty dk \; k^2 E(k)$$

hence:

$$E_\omega(k) = k^2 E(k)$$