PHYS 616 Multifractals and Turbulence

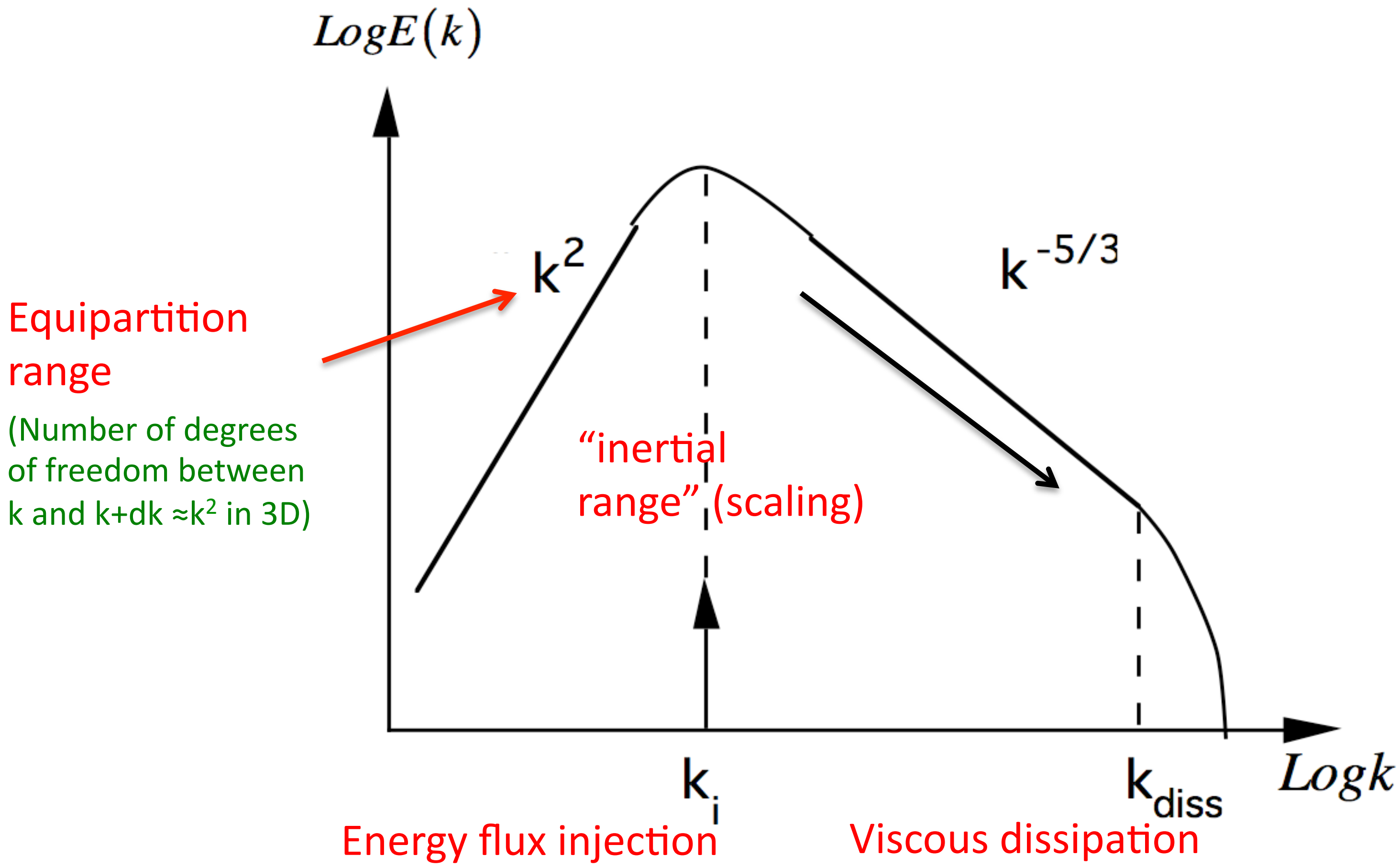
Lecture 4:

Spectra, fractal sets

Feb. 5, 2014

Recap 3D cascades

Schematic diagram of 3-D energy cascade



The Special Case of 2-D Turbulence

The vorticity equation is obtained by taking the curl of the velocity equation:

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\nabla)\underline{v} + \nu\nabla^2\underline{\omega} + \underline{b}; \quad \underline{b} = \frac{1}{\rho^2}\nabla\rho\times\nabla p$$

Vorticity eq.

Vortex stretching term

Baroclinicity vector

Recall:

$$\frac{D\underline{\omega}}{Dt} = \frac{\partial\underline{\omega}}{\partial t} + \underline{v}\cdot\nabla\underline{\omega}$$

In 2-D:

For a two dimension flow the vorticity $\underline{\omega}$ must beis perpendicular to $\underline{v}$ (*i.e.*, $\underline{\omega} = \omega_z\hat{z}$, $\omega_z = \partial v_y/\partial x - \partial v_x/\partial x$, $\omega_x = \omega_y = 0$ since $v_z = 0$, $\partial/\partial z = 0$; consequently:. We therefore can no longer have any vortex stretching since:

$$(\underline{\omega}\cdot\nabla)\underline{v} \equiv 0$$

No Vortex stretching

i.e. there is no longer any vortex stretching and the incompressible vorticity equation reduces to In 2-D we thus obtain an advection-dissipation equation for the vorticity:

$$\frac{D\underline{\omega}}{Dt} = \nu\nabla^2\underline{\omega}$$

No Vortex stretching

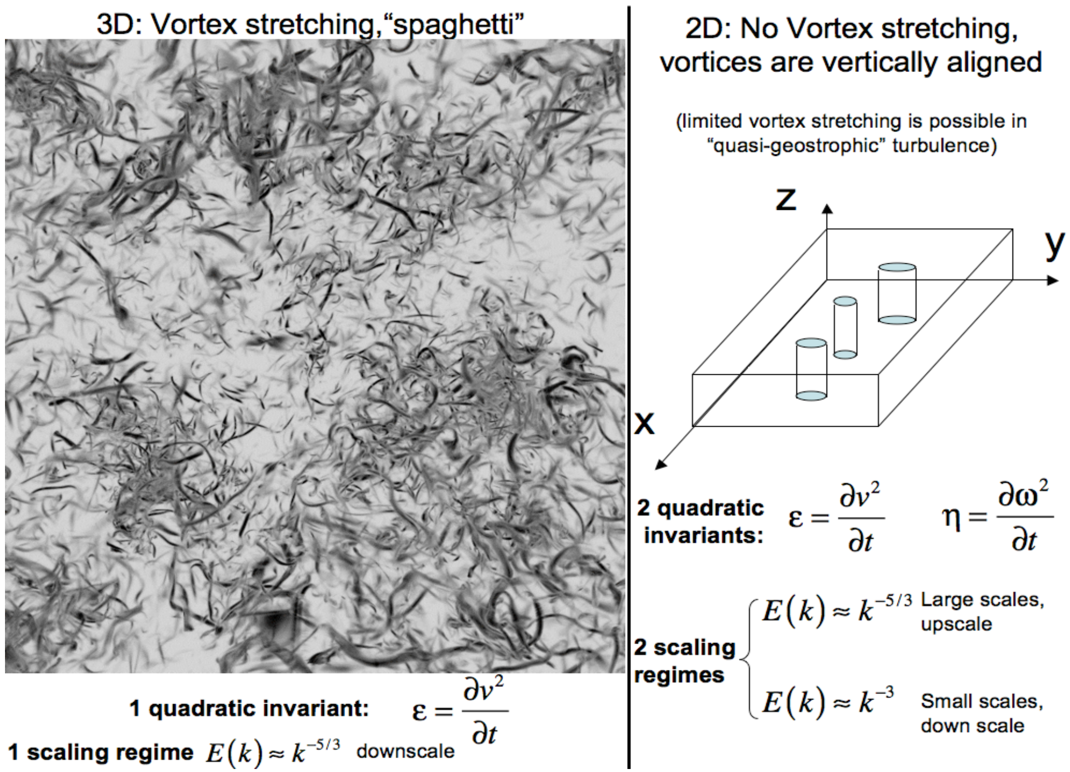
when the dissipation is negligible, any power of the vorticity is conserved, not only the enstrophy which is its square. We can define the enstrophy flux density:

$$\eta = -\frac{1}{2}\frac{\partial\omega^2}{\partial t}$$

Enstrophy flux density

In 3D, Vortex stretching implies the direction of the cascade is from large to small scales

Read section 2.4.3



Vortex tubes (surfaces bounded by lines of vorticity) are material tubes (section 2.4.3) hence if the ends of the tube move further apart as the flow evolves (a kind of "drunkard's walk in a complex flow), then since the tubes are incompressible, the cross-section must tend to get smaller, hence the creation of small structures from large due to vortex stretching.

Two-Dimensional Enstrophy Cascades 1

Returning to the ideal case of two dimensional turbulence we have seen that both energy and enstrophy are conserved by the nonlinear terms, hence both will be cascaded. We have:

$$\Omega = \text{enstrophy} = \langle \omega^2 \rangle = \int_0^\infty dp\ E_\omega(p) = \int_0^\infty dp\ p^2 E(p)$$

enstrophy

Spectrum of vorticity

Spectrum of velocity

where the enstrophy Ω. The enstrophy in the n^{th} octave is therefore:

$$\Omega_n = \int_{k_n/\sqrt{2}}^{\sqrt{2}k_n} dp\ p^2 E(p) \approx \int_{k_n/\sqrt{2}}^{\sqrt{2}k_n} dp\ p^2 p^{-\beta} \approx p^{3-\beta}\Big|_{k_n/\sqrt{2}}^{\sqrt{2}k_n} \sim k_n^3\ E(k_n)$$

From the spectrum we can estimate the lifetime/ "eddy turn over-time" of a structure size $l_n = 1/k_n$ as:

$$\tau_n \sim \left(\int_0^{k_n} dp\ p^2 E(p) \right)^{-1/2} \sim \left(k_n^3 E(k_n) \right)^{-1/2}$$

In analogy with the energy cascade, we can also define:

$$\Pi_n^{(\Omega)} = \frac{\Omega_n}{\tau_n}$$

as the Fourier-space enstrophy flux (which is constant for a quasi-steady process) through the n^{th} octave.

Two-Dimensional Enstrophy Cascades 2

Finally we obtain the enstrophy flux through the n^{th} octave in Fourier space:

$$\Pi_n^{(\Omega)} = \frac{\Omega_n}{\tau_n} \sim \frac{k_n^3 E(k_n)}{\left(k_n^3\, E(k_n)\right)^{-1/2}}$$

Enstrophy flux

If we assume that this is constant in a steady state and independent of n, then $\eta \sim \Pi_n^{(\Omega)}$ and we obtain the spectrum in the constant enstrophy flux regime:

$$\boxed{E(k) \sim \eta^{2/3} k^{-3}}$$

Kraichnan law Fourier space

Using either dimensional analysis or the Tauberian theorems (section 2.4.5), we can obtain the corresponding real space result:

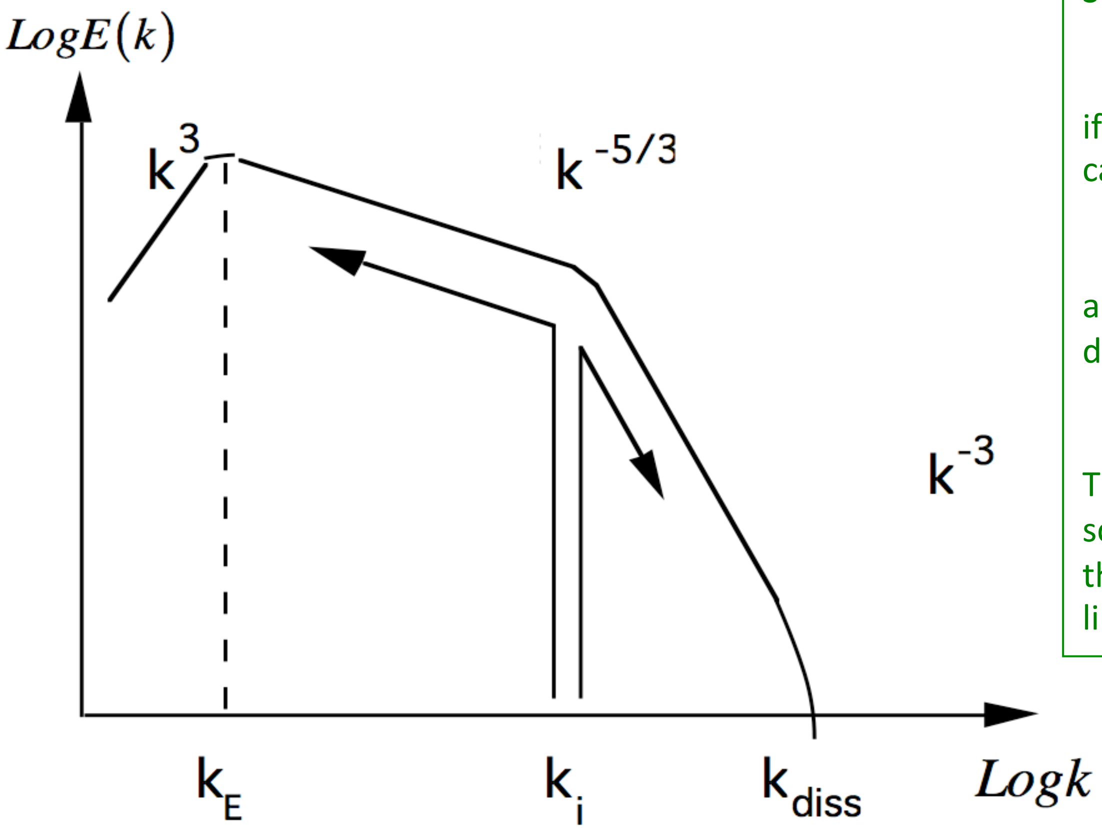
$$\boxed{\Delta v \approx \eta^{1/3} \Delta x}$$

Kraichnan law real space

These formulae (sometimes called the real and in Fourier space "Kraichnan" laws;

Two-Dimensional Enstrophy Cascades

Read: 2.5.2



If both ε and η are cascaded, which direction?

Since

$$E_\omega(k) = k^2 E(k)$$

if the small scale were dominated by an energy cascade, then

$$E_\omega(k) = k^2 k^{-5/3}$$

and this would yield a small scale (large k) divergence of enstrophy since:

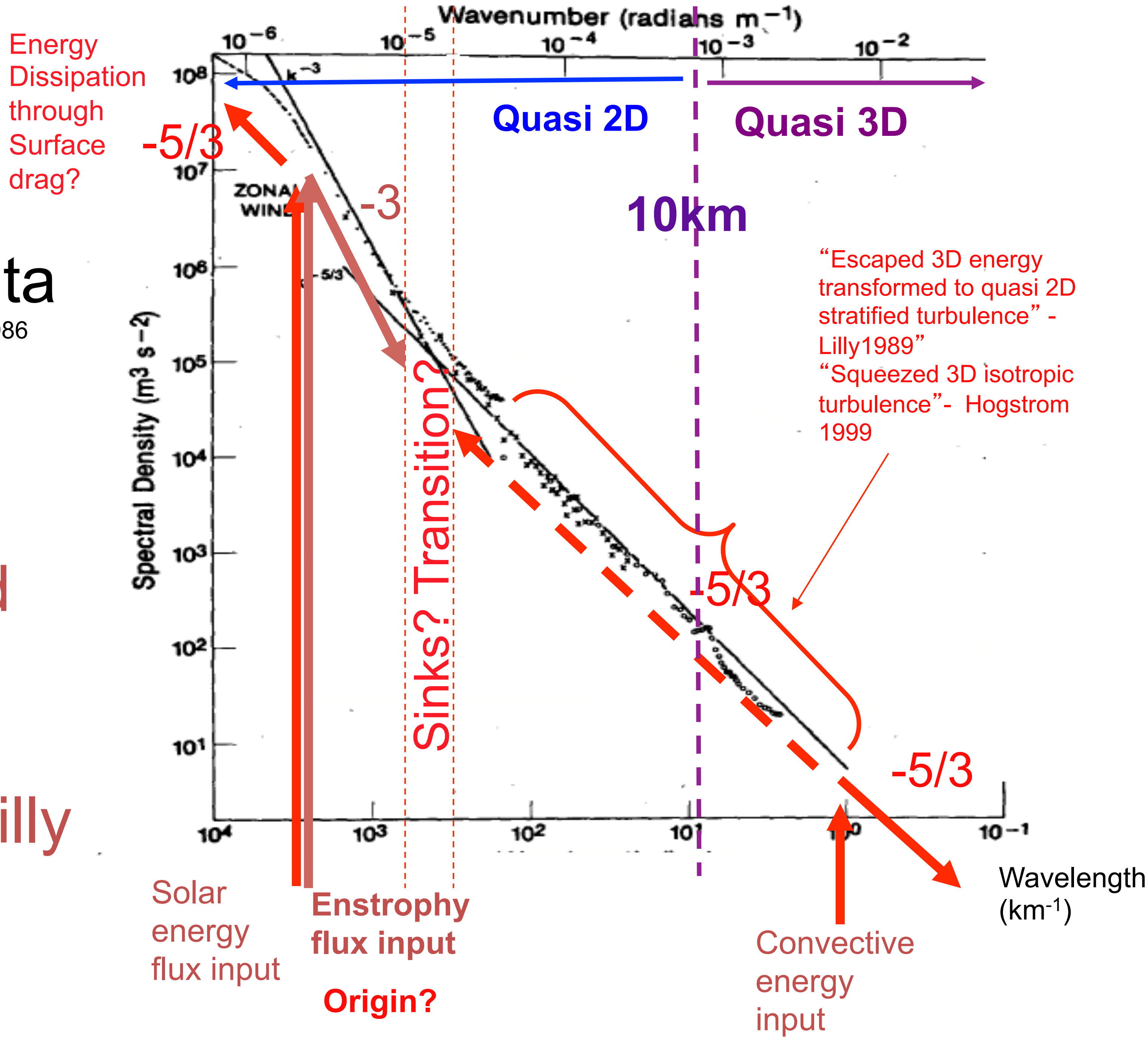
$$\Omega = \langle \omega^2 \rangle = \int_0^\infty dp\ E_\omega(p)$$

This implies that the enstrophy is cascades to small scales and the energy to large scales (otherwise the energy flux would vanish in the small scale limit).

Injection of enstrophy and energy at the same wavenumber k_i

GASP data

Nastrom and Gage 1986

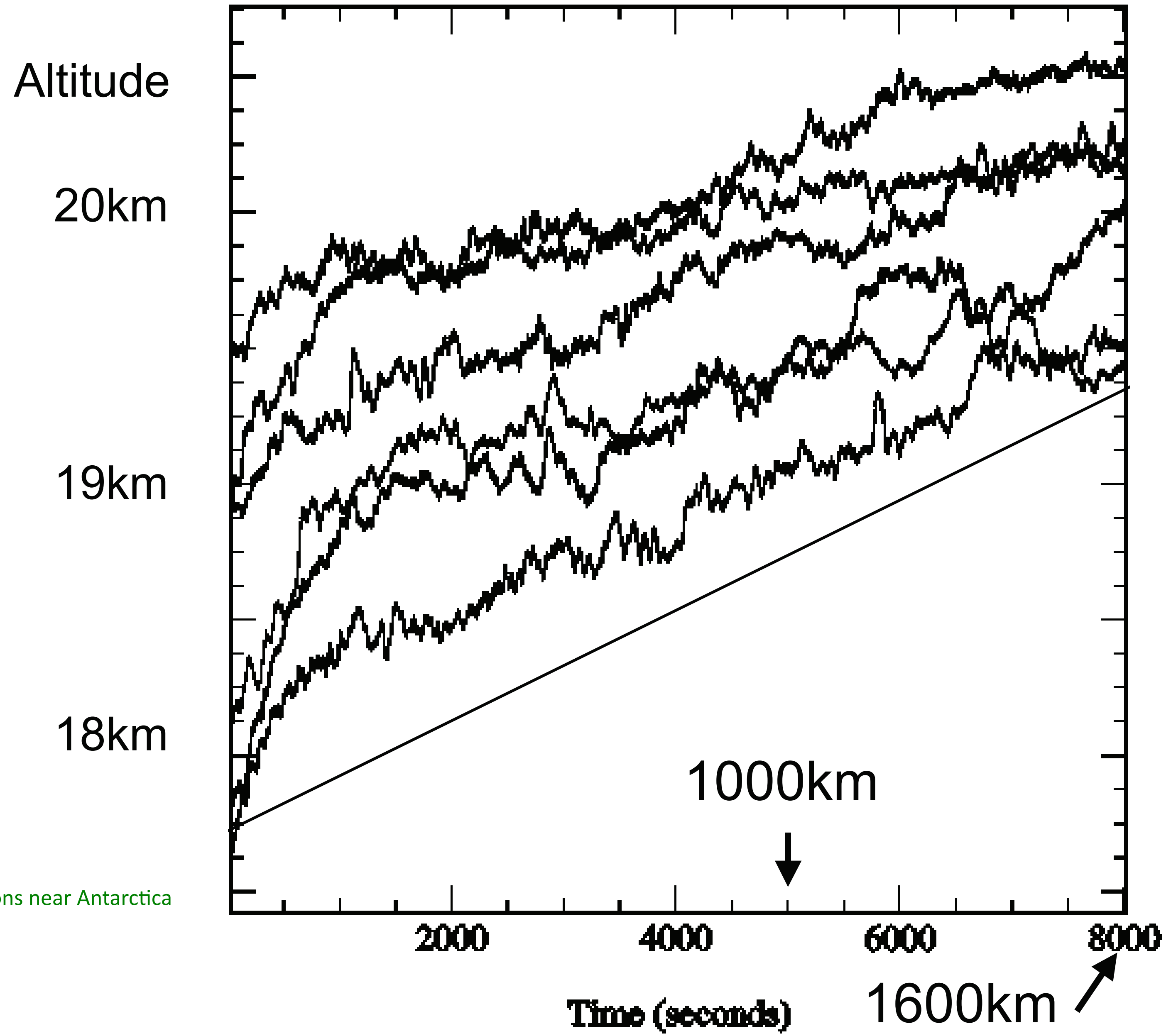


The Standard Model v2.0, "Gage-Lilly model" 1989

Quick explanation for low frequency spectrum

Aircraft do not fly on flat trajectories: they fly on fractals and they are sloping .

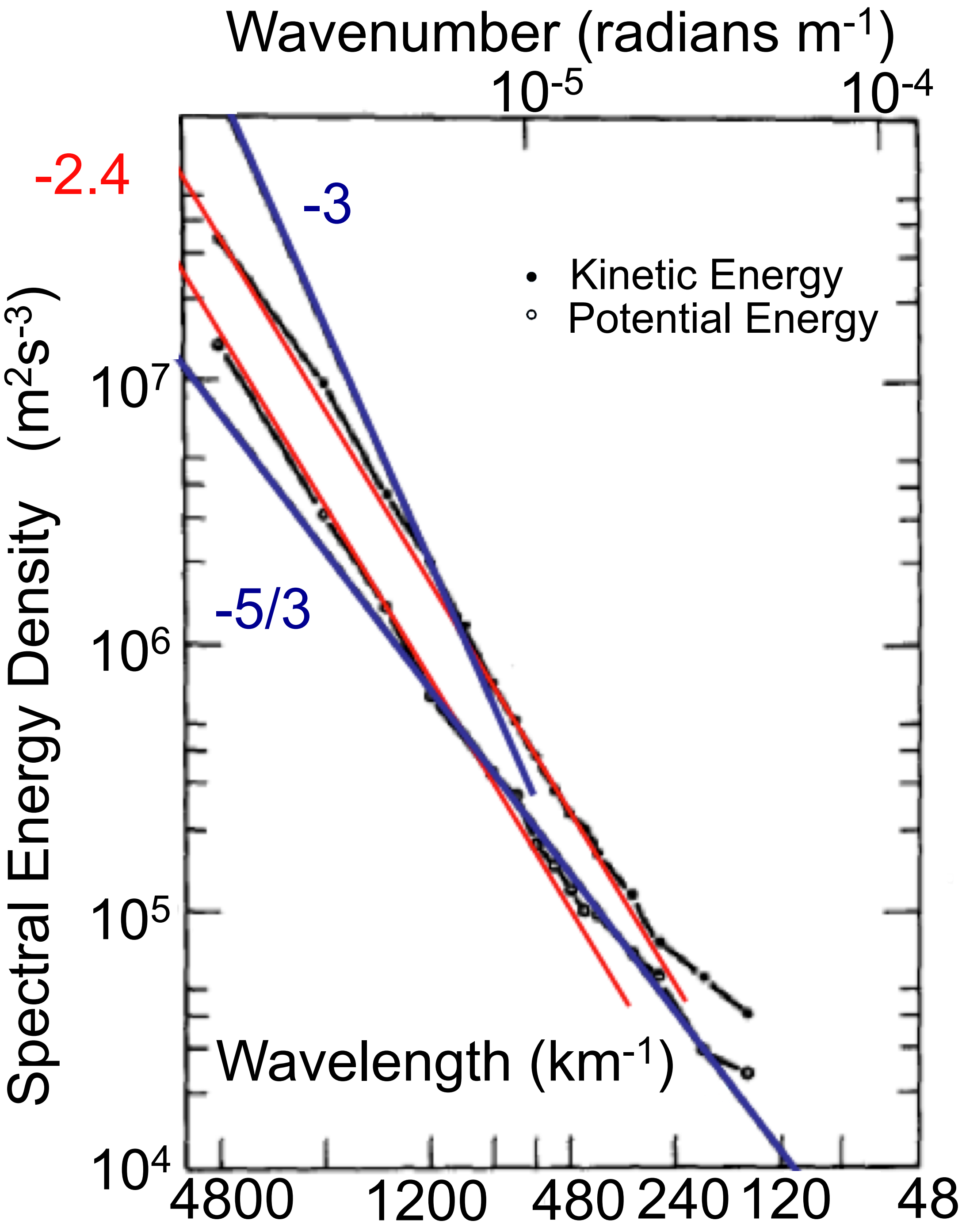
In the vertical the spectrum is $k^{-2.4}$, this becomes dominant at large distances aircraft do not fly on flat trajectories: they fly on fractals and they are sloping



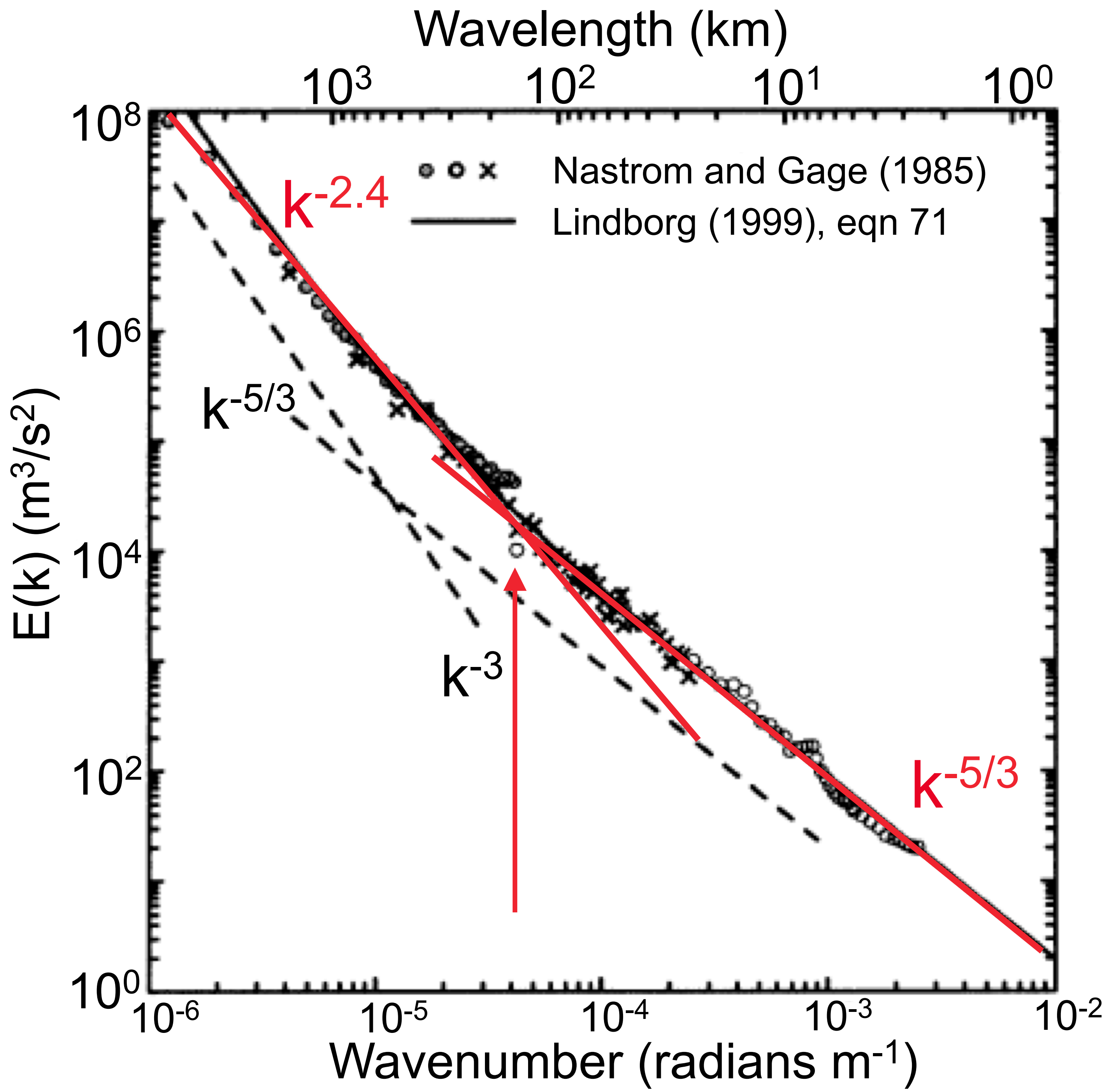
NASA's ER-2 aircraft during missions near Antarctica

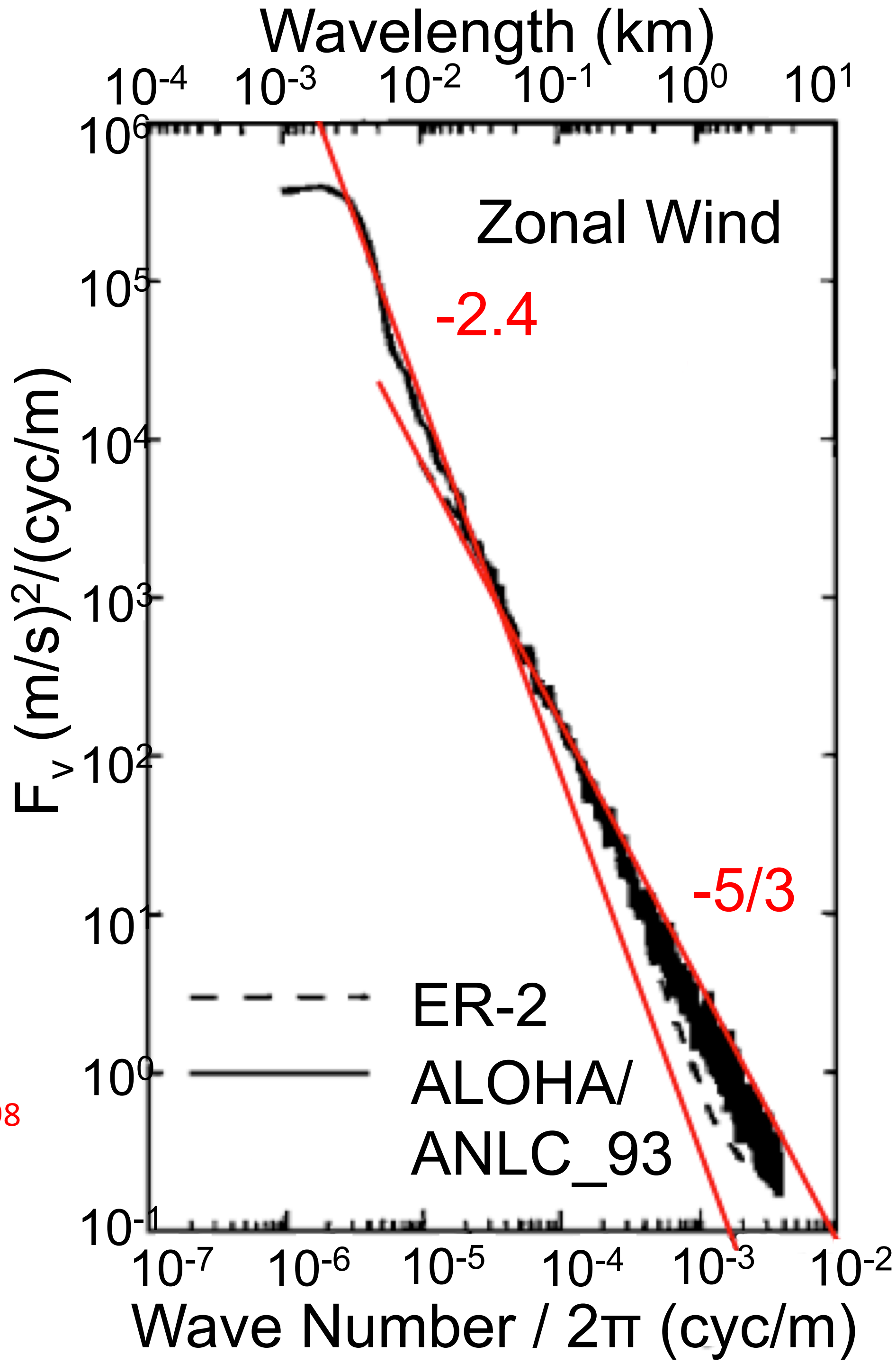
GASP spectrum of long haul flights (> 4800 *km*)

adapted from (Gage and Nastrom, 1986) with the reference lines corresponding to the horizontal and vertical behaviour discussed in the text (exponents 5/3, 2.4, i.e. ignoring intermittency corrections corresponding to H_h = 1/3, H_v = 0.7 as well as to the 2D isotropic turbulence slope -3).

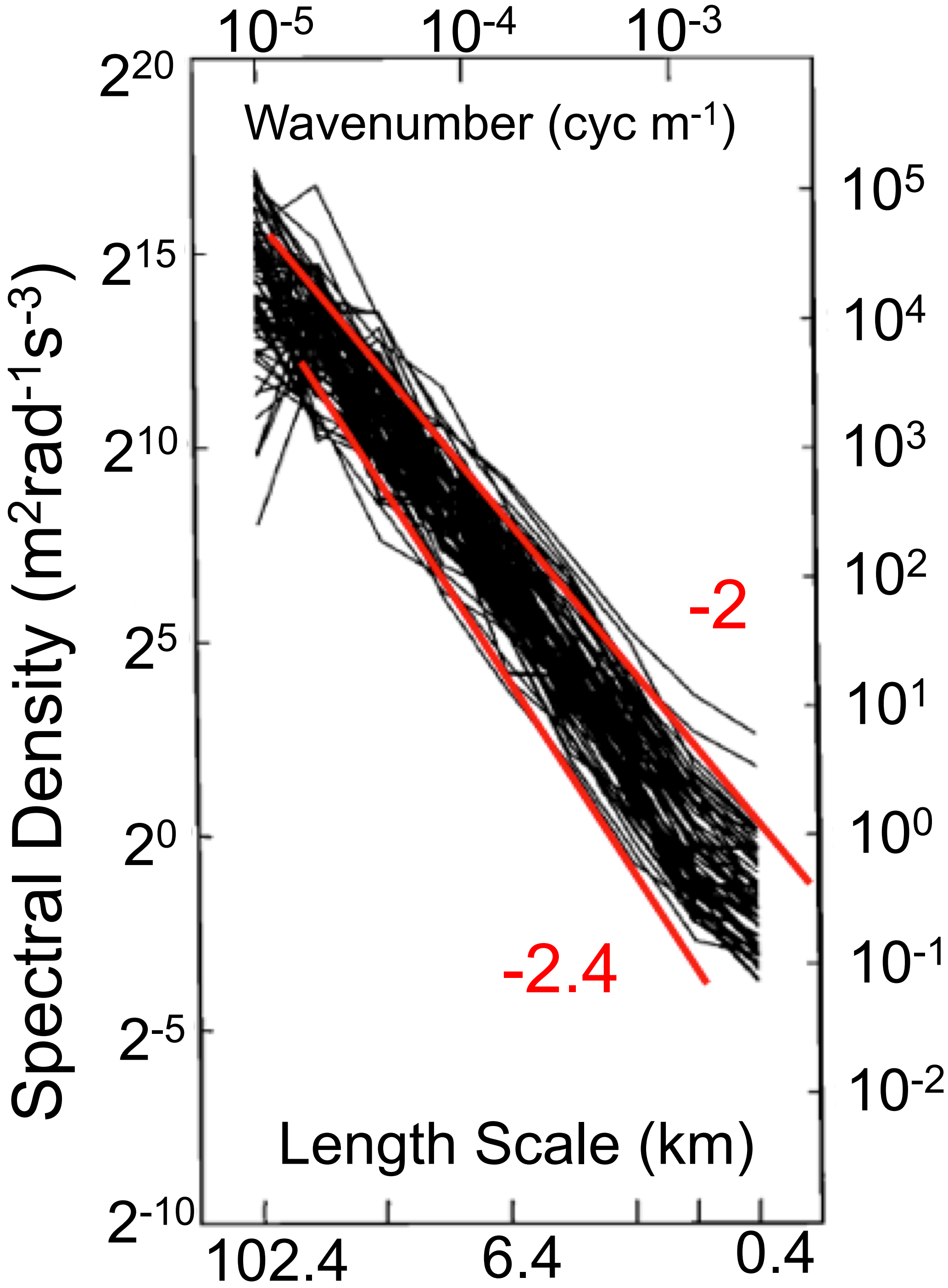


2.10b

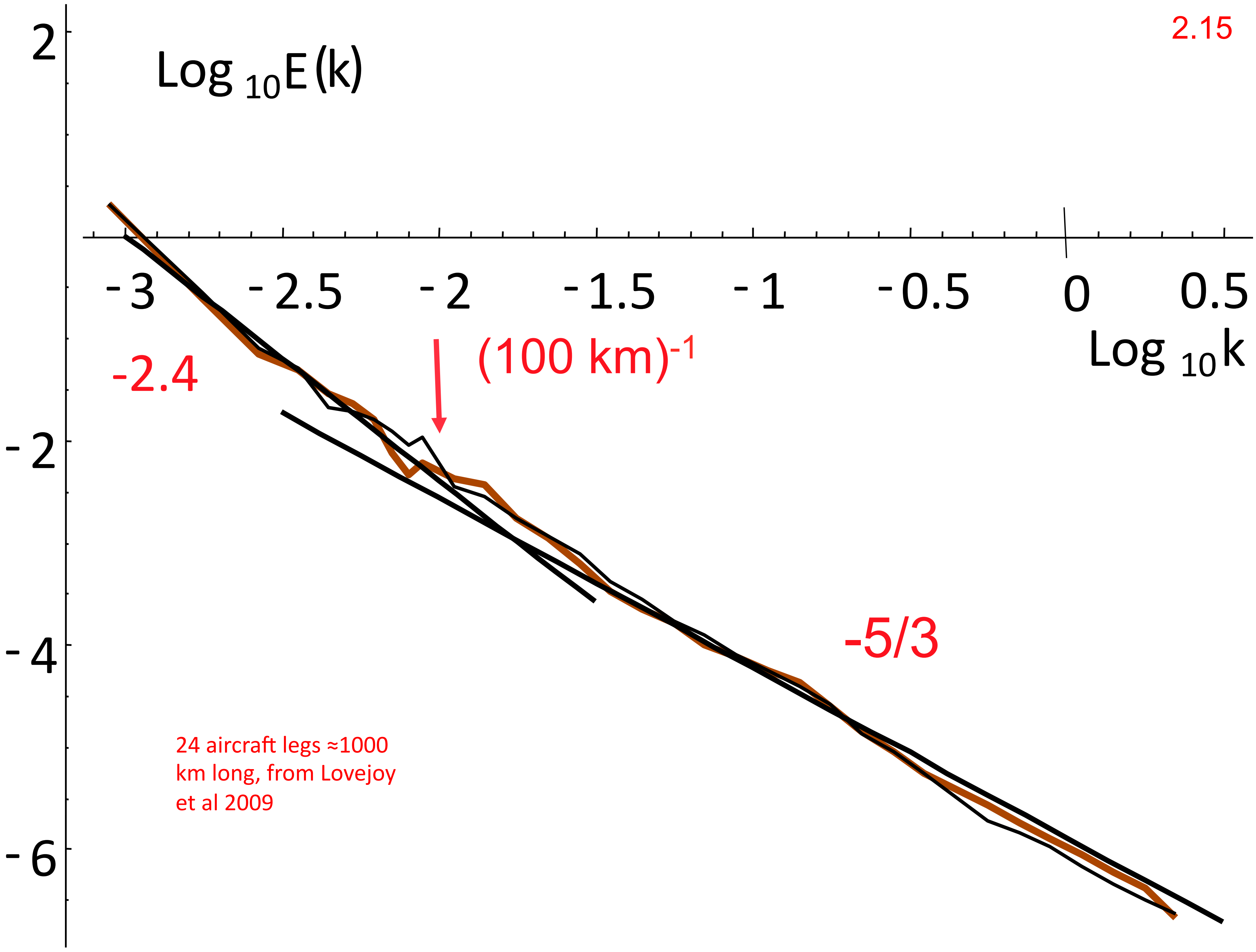

Wavelength (km)
10^3
10^2
10^1
10^0
10^8
10^6
10^4
10^2
10^0
E(k) (m^3/s^2)
Nastrom and Gage (1985)
Lindborg (1999), eqn 71
k^-2.4
k^-5/3
k^-3
k^-5/3
10^-6
10^-5
10^-4
10^-3
10^-2
Wavenumber (radians m^-1)



Gao and Merriweather 1998
at 6km altitude



Stratospheric ER-2 spectra adapted from (Bacmeister et al., 1996)



2.15
2
Log 10E(k)
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
Log 10k
-2.4
(100 km)-1
-2
-4
-5/3
-6
24 aircraft legs ≈1000 km long, from Lovejoy et al 2009

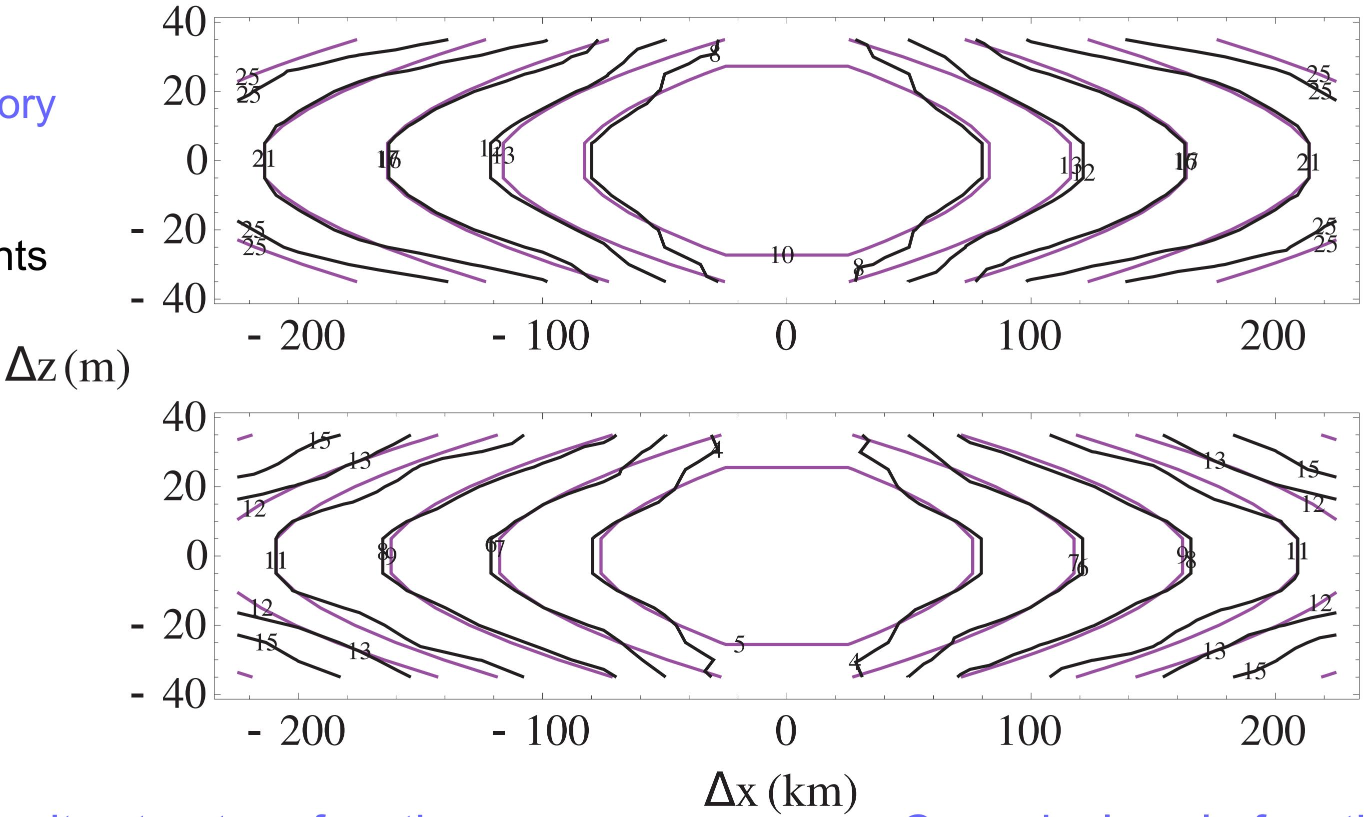
14500 aircraft flights: 5-5.5km altitude, 2009, US (TAMDAR data)

$<|\Delta v(\Delta x, \Delta z)|^2>$ (m/s)^2

Pinel et al 2012

Purple = theory

Black= measurements



Velocity structure function

$$\left\langle \Delta v^2 (\Delta x, \Delta z) \right\rangle = C \left\| (\Delta x, \Delta z) \right\|^{\xi(2)}$$

2-D exponent: =2

$$\xi(2) \approx 0.80$$

Canonical scale function

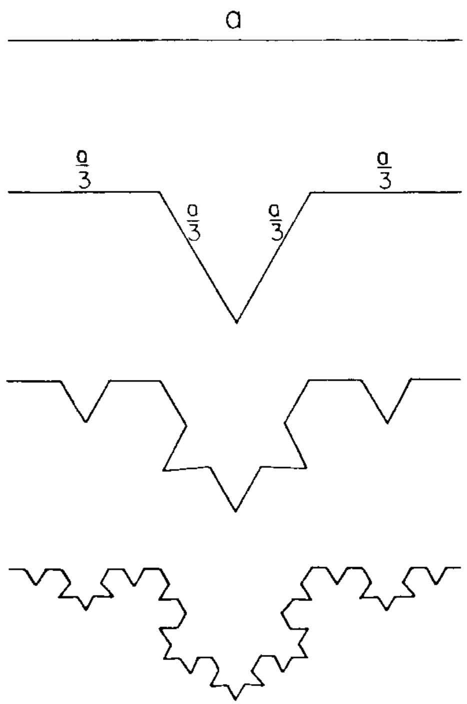
$$\left\| (\Delta x, \Delta z) \right\| = \left(\left(\frac{\Delta x}{l_s} \right)^2 + \left(\frac{\Delta z}{l_s} \right)^{2/H_z} \right)^{1/2}$$

$$H_z \approx 0.57 \pm 0.01$$

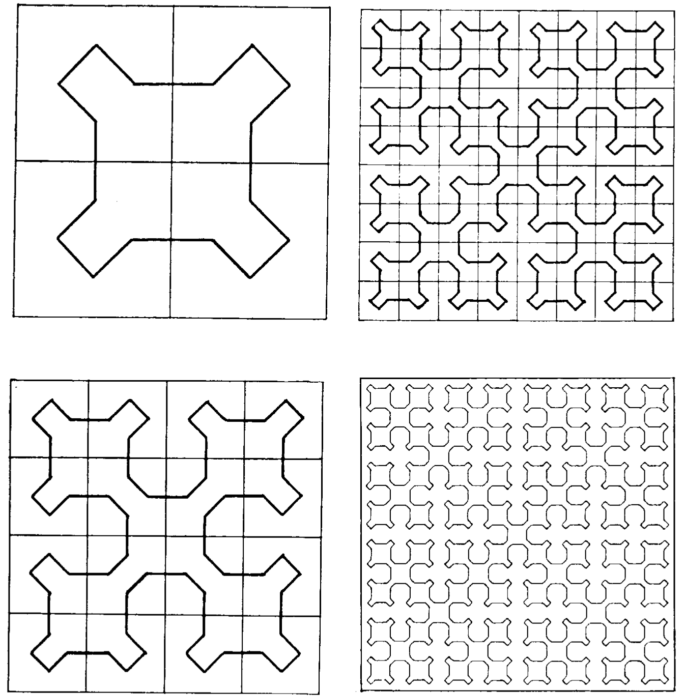
(Theory: 5/9=0.555…)

Fractal sets

Set: Black / white, single fractal dimension



A fractal Koch curve ([*Koch*, 1904]), reproduced from [*Welander*, 1955] to illustrate the mixing of a two dimensional fluid.



A fractal Peano curve, reproduced from [*Steinhaus*, 1960] showing how a line (dimension 1) can literally fill the plane (dimension 2), illustrating how streams can fill a surface.

Early notions of dimension (Greeks)

In the 19th C, It was believed that the dimension was the number of independent coordinates needed to specify the position of a point.

Mapping the unit square onto the unit interval (Cantor, discontinuous)

I.1 Topological Dimensions

I.1.1 Early Ideas of Dimension

Before set theory (≲1870), ideas of dimension were vague. For example it was widely thought that the dimension of the set to which a point belongs is equal to the number of parameters needed to specify its position. For example "a configuration is said to be n-dimensional if the least number of real parameters needed to describe its points in some unspecified way is n". The basic ideas of dimension had hardly evolved since Euclid's definition:

Euclid's definition of Dimension (*circa* 300 B.C.)

1) A point is that which has no part.
2) A line is a breadthless length.
3) The extremities of lines are points.
4) A surface is that which has length and breadth only.
5) The extremities of surfaces are lines.

The idea of dimension as the number of parameters was shown to be inadequate by two developments:

I.1.2 Problems with the Early Definitions

1) In 1872, Cantor found a way to map a unit square $[0,1] \otimes [0,1]$ onto a unit interval using a 1:1 mapping.

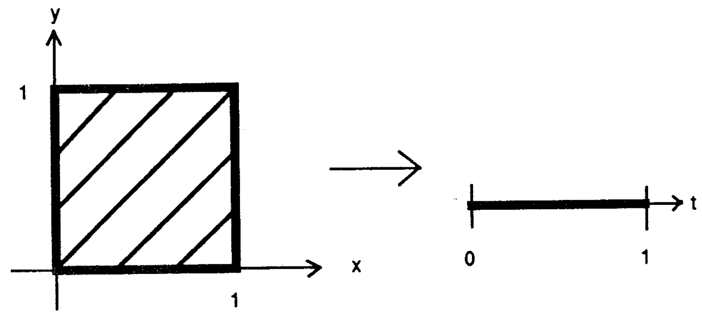
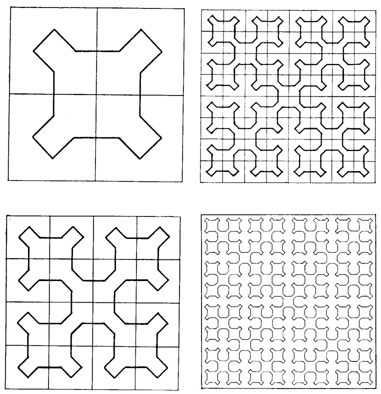


Figure I.1

To define the mapping, write it as $(x,y) \rightarrow t$ and expand x,y coordinates in binary, *i.e.*, $x = 0.n_0n_2n_4n_6\ldots$, and $y = 0.n_1n_3n_5n_7\ldots$, where the n_i are all zeroes and ones. The point (x,y) is mapped to the point in the unit interval with single coordinate t with $t = 0.n_1n_2n_3n_4\ldots$

Continuous mapping a line onto the unit square (Peano, … but not 1:1)

With rounded corners for pedagogy only:



Although this mapping is clearly 1:1, it is far from continuous (neighbouring points on the line are not neighbouring on the square). This mapping clearly showed that a square cannot be considered "two dimensional" simply because two coordinates are usually used—one is sufficient! However it was still hoped that the parameter definition of dimension could be useable if such non-continuous transformations were ruled out.

2) However, even this restriction was not enough as Peano (1890) showed by performing a continuous mapping of a line to a square:

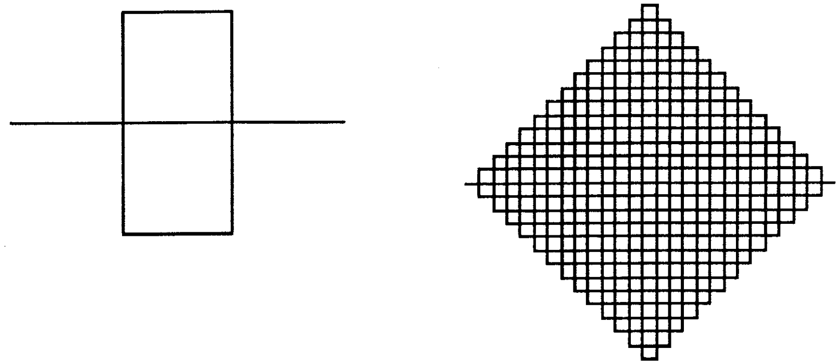


Figure I.2— The Peano curve: shown at left is the generator and its third iteration at right.

By the construction, as the number of steps increases, the line eventually goes through each of the points of the square (this is not hard to show by considering a base 3 expansion of the coordinates of the point). After n iterations the length of the line is 3^n, *i.e.*, it diverges as $n \to \infty$. This mapping is obviously continuous, but it is not a 1:1 mapping (*i.e.*, not invertible—the points of contact shown in figure I.3 are inevitable):

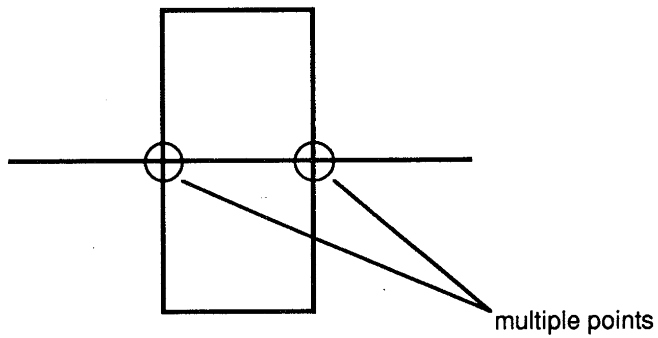


Figure I.3—Multiple contact points on Peano curve.

Modern definition of topological dimension (invariant under Continuous and 1:1 mappings)

The upshot of Cantor's and Peano's mappings was that the dimension of a set is not invariant under either (separately) a 1:1 discontinuous or a non 1:1 continuous transformation.

I.1.3 The Crisis and the Modern Definition of Topological Dimension

These results lead to the following question (Hurwitz): "Is it possible to establish a correspondence between Euclidean n-space (ordinary space of n variables) and Euclidean m-space combining features of both Cantor and Peano constructions, *i.e.*, a correspondence which is both 1:1 *and* continuous?"

"...If the above was possible then Euclidean dimension has no topological sense whatsoever! Hence the class of topological transformations would be too wide to be of any real geometric use..."

The issue was settled by Lebesgue's theorem, and the topological dimension (defined below) replaced Euclidean dimension as a similar but more precise concept.

Lebesgue's Theorem (proved by Brower [1911]) settled the question.

The *topological dimension* (defined below) is invariant under 1:1 *and* continuous transformations.

Modern Definition of Topological Dimension (Menger's definition):

(i) { } or ∅ (*i.e.*, the empty set) has dimension -1.

(ii) the dimension of a space is the least integer, n, for which every point has an arbitrarily small neighbourhood whose boundaries have dimension less than n.

Examples:

1. a point—only ∅ surrounding: maximum boundary dimension $= \tilde{n} = -1$ $\Rightarrow n = 0$.
2. a line—∅ or points surrounding, $\tilde{n} = -1$ or 0 $\Rightarrow n = 1$.
3. a plane—∅, points, line surrounding, $\tilde{n} = -1, 0$ or 1 $\Rightarrow n = 2$.

Summary: The concept of dimension which resulted from the attempt to make earlier definitions precise made no reference to the *size* of the set/system.

Measure based dimensions, intuitive

I.2 Measure Based Fractal Dimensions : The Intuitive "Similarity Dimension"

I.2.1 The Dimension of Cantor's Perfect Set (1883)

An entirely different class of dimension concepts is necessary to deal with the size of an object. These notions of dimension all revolve around the intuitive idea that an object of size L and of dimension D has *content* $n(L)$ (equal to length, area, volume) where

$$n(L) \propto L^D.$$

Mathematically rigourous definitions of the size of a set: coverings

I.3 Coverings: To Measure the Size of a Set

We now seek to put the idea $n(L) \propto L^D$ on a rigorous basis by defining objective methods of determining the "size" or "content" of a set.

I.3.1 Cantor-Minkowski Coverings (1901)

Consider a set, S, embedded in $\mathbb{R}^D$, $D = 1,2,3,\ldots$ (the usual line, plane, volume, *etc.*). Define the usual distance function,

$$d(\mathbf{x},\mathbf{y})^2 = \sum_{i=1}^{D}(x_i - y_i)^2$$

and the δ-sized "ball" $B_\delta(\mathbf{x}_\alpha)$ is a D-dimensional ball—$B_\delta(\mathbf{x}_\alpha) = \{\mathbf{y} \mid d(\mathbf{x}_\alpha,\mathbf{y}) \leq \delta\}$. These balls are now centered at every point of S, defining a smoothed set:

$$S(\delta) = \bigcup_\alpha B_\delta(x_\alpha).$$

Notice that

$$\lim_{\delta \to 0} S(\delta) = S.$$

To see how to measure the dimension of a set, consider sets embedded in three dimensional space ($D = 3$). Then the volume of $B_\delta(x_\alpha) \approx \delta^D$. In this example we will use spheres as balls and cover a serie of sets of varying dimensions.

(i) Consider first $S = $ cube; B_δ is a sphere.

For small enough δ, the volume of $S(\delta) \sim$ volume of S,

(ii) Next, $S = $ square.

For small enough δ, the volume of $S(\delta) \sim 2\delta \cdot$Area of S,

(iii) finally, consider $S = $ line.

For small enough δ, the volume of $S(\delta) \sim \pi \cdot \delta^2 \cdot$Length of S.

Circle centred at each point on the set

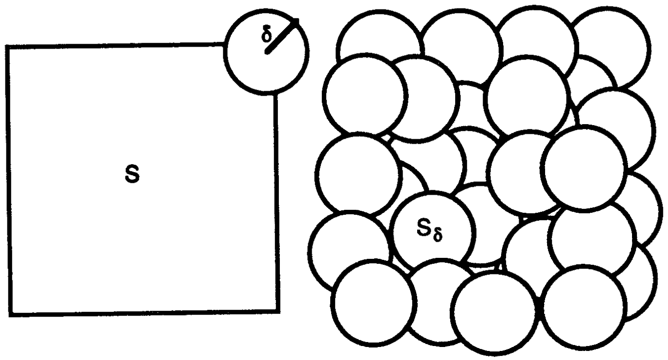


Figure I.9—Illustration of Cantor-Minkowski covering of a square.

Hence, in general, defining the Cantor-Minkowski dimension D(S) as the dimension of S and $C(S) = D - D(S)$ as the "codimension" of S (this will crop up frequently below), we have

Cantor-Minkowski coverings

$$\lim_{\delta \to 0} \left\{ \frac{\text{volume of } S(\delta)}{\delta^C} \right\} \to \begin{cases} 0 & C > C(S) \\ \text{finite} & C = C(S) \\ \infty & C < C(S) \end{cases}$$

So by varying C for a given S and D (dimension of the balls), C(S) can be determined as the value that yields a finite non-zero limit and the $D(S) = D - C(S)$. $S(\delta)$ is the Cantor-Minkowski covering set, D(S) is the Cantor-Minkowski dimension (in general it will be the same as the box-counting dimension defined below).

I.3.2 δ-Coverings

δ coverings

The Cantor-Minkowski covering is not very "efficient" in the sense that for a finite δ, each point on the set is covered by infinitely many balls, so we introduce a different, more general covering called the *δ-covering* with the following definition: a *δ-covering* of a set A is a countable (or finite) collection of sets B_i, of *diameter* at most δ, that cover A, *i.e.*,

$$A \subset \bigcup_{i=0}^{\infty} B_i.$$

Writing $\mathrm{diam}(B_i) = \delta_i$, we require $\delta_i < \delta$. Note the *diameter* need not be defined by a metric, it is some convenient measure of size (such as the square root of the area): this will be useful in generalized scale invariance (section II).

I.4 Hausdorff Measures and Dimensions

I.4.1 Definition

Hausdorff measures, relative to w at resolution δ

The Hausdorff measure (Hausdorff [1919] , also called Hausdorff-Besicovitch measure) of A relative to w at resolution δ is defined as:

$$\mu_{w,\delta}(A) = \inf \left\{ \sum_i w(\delta_i) \right\},$$

where the δ_i are diameters of the δ-covering of A. Note that inf{ } requires that we use the δ-covering which minimizes the sum and provides a unique definition of the measure. The only restriction upon w is that it must be a monotonically increasing positive function of δ. In particular we will be almost exclusively interested in power laws, *e.g.*, $w(t) \sim t^D$; futhermore since we will take $\delta \to 0$, only the behaviour of w near the origin ($t \to 0$) will be important.

Hausdorff measures, relative to w

<u>Definition:</u> **Hausdorff measure of A relative to w:**

$$\mu_w(A) = \lim_{\delta \to 0} \mu_{w,\delta}(A).$$

Hausdorff measures, dimensions D

Definition: **Hausdorff measure of A dimension D**

is obtained by taking $w(t) \sim t^D$ in the above and is denoted $\mu_D(A)$.

I.4.2 Properties of the Hausdorff Measures

Properties of Hausdorff measures

(i) $0 \leq \mu_{w,\delta}(A) \leq \infty$ (since $w(t) > 0$),

(ii) $\mu_{w,\delta}(A)$ increases or stays constant as δ decreases since the restriction $\delta_i < \delta$ becomes more and more stringent as $\delta \to 0$,

(iii) with increasing D, $\mu_D(A)$ has an *infinite jump* from infinity to zero for any set A with at most one finite non-zero intermediate value.

Definition: **Hausdorff dimension D(A) of set A**

is the value of D at which this jump occurs:

$$D(A) = \sup\{D: \mu_D(A) = \infty\} = \inf\{D: \mu_D(A) = 0\}.$$

Why Hausdorff measures have an ∞ jump:

Consider a δ-covering of set A: from the fact that $\delta_i/\delta \leq 1$, $\forall i$, it follows that if $D' > D$:

Infinite jump of Hausdorff measures

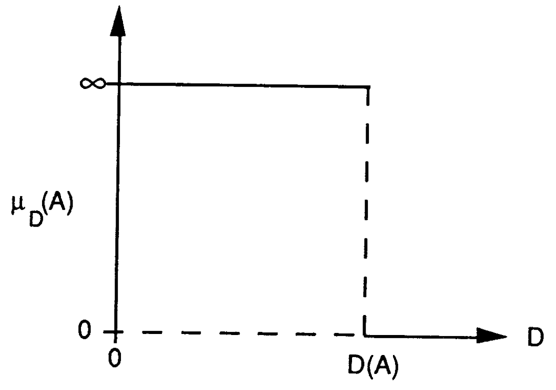


Figure I.10—Illustration of the divergence rule for Hausdorff measures, generalizing the divergence rule "the length of a surface is infinite, its volume is zero...". The transition at $D = D(A)$, from infinity to zero, defines the Hausdorff dimension of the set A.

Demonstration of the Infinite jump

$$\sum_{\substack{i \\ \delta_i<\delta}} \left(\frac{\delta_i}{\delta}\right)^{D'} < \sum_{\substack{i \\ \delta_i<\delta}} \left(\frac{\delta_i}{\delta}\right)^{D} \Rightarrow \sum_{\substack{i \\ \delta_i<\delta}} \delta_i^{D'} < \delta^{D'-D} \cdot \sum_{\substack{i \\ d_i<d}} \delta_i^{D}$$

Next taking infima we obtain:

$$\mu_{D',\delta}(A) \leq \delta^{D'-D} \mu_{D,\delta}(A) \; ; D' > D,$$

hence as $\delta \to 0$ if $\mu_D(A)<\infty$, it follows $\mu_{D'}(A)=0$; *i.e.*, if $\mu_D(A)$ starts off at ∞ for small enough D, then as soon as D is increased to a value where $\mu_D(A)$ is finite or zero, then all further values equal 0. Similarly the argument can be inverted starting with large enough D' such that $\mu_{D'}(A) = 0$ and decreasing until a finite value is obtained.

Remark: Sometimes sets with an infinite number of points have $\mu_D(A)=0$ or ∞ without any non-zero finite value. In these cases, we can introduce *sub-dimensions*, Δ_1, Δ_2, Δ_3,... to measure the size of the set:

$$w(t) = t^D |\log(t)|^{\Delta_1} (\log|\log(t)|)^{\Delta_2} \ldots$$

Example: the trail of a particle undergoing Brownian motion in a space of dimension ≥ 2 will have $D(A)=2$, but requires

$$w(t) = t^2 \log|\log(t)| \, ,$$

for a finite non-zero measure, *i.e.*, $D=2$, $\Delta_1=0$, $\Delta_2=1$ (this is the "law of the iterated logarithm").

Subdimensions, law of iterated logarithm

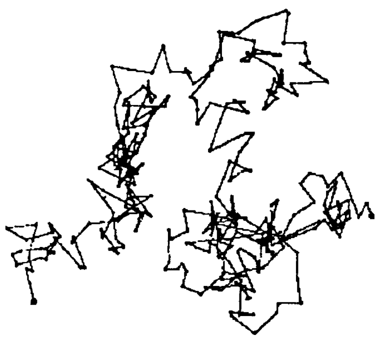


Figure I.11—A 2–D random walk (Brownian motion).

I.5 More Properties of Hausdorff Measures

I.5.1 Relation Between Hausdorff and Lebesgue Measures

For standard Euclidean sets the Hausdorff measure reduces to Lebesgue measure (the usual integral).

Example: a planar set. Take the square norm

$$\|x\| = \sup_j |x_j| \qquad \text{(which is the same as } \max_j |x_j| \text{ or } L_\infty \text{ norm).}$$

the sup is over the coordinates of the vector x.

The "balls" are squares in $\mathbf{R}^2$. Also take $w(t) = t^2$. We will argue that $\mu_D(A)$ = Lebesgue measure of $A = \int_A d^2x$. Now

$$\mu_{D,\delta}(A) = \inf\left\{\sum_{i=1}^{N} \delta_i^2\right\} ,$$

with $D = 2$, *i.e.*, squares. However, the $\inf\{\}$ requires that we use disjoint (non-overlapping) squares,

$$\mu_{2,\delta}(A) = \sum_{i=1}^{N} \delta_i^2 ,$$

with $\delta_i < \delta$ over disjoint squares. In the limit as $\delta \to 0$, the sum approaches the Lebesgue measure and $\mu_{2,\delta}(A) = \mu_2(A)$ which is finite and positive if the set has a finite area, hence $\mu_2(A) = \int_A d^2x$ ~~and $D(A) = 2$.~~

I.5.2 Scale Invariance and Scaling

Scale invariance, scaling

The Hausdorff measure provides a simple example of scale invariance/scaling. By construction it satisfies:

$$\mu_D(\lambda^{-1}A) = \lambda^{-D}\, \mu_D(A) ,$$

where $\lambda^{-1}A$ represents a reduction of set A by a factor of λ (see figure I.12).

Proof: $\mu_{D,\lambda^{-1}\delta}(\lambda^{-1}A) = \inf\left\{\sum_{\delta_i'<\delta'} \delta_i'^D\right\}$.

However, clearly $\delta_i' = \lambda^{-1}\delta_i$ (same as the δ_i's used to cover A), hence:

$$\mu_{D,\lambda^{-1}\delta}(\lambda^{-1}A) = \inf\left\{\sum_{\delta_i'<\delta'} (\lambda^{-1}\delta_i)^D\right\}$$

$$= \lambda^{-D}\cdot\inf\left\{\sum_{\delta_i<\delta} \delta_i^D\right\} = \lambda^{-D}\,\mu_{D,\delta}(A)$$

$$\Longrightarrow \quad \mu_D(\lambda^{-1}A) = \lim_{\delta\to 0} \mu_{D,\lambda^{-1}\delta}(\lambda^{-1}A) = \lambda^{-D}\,\mu_D(A).$$

Therefore the measure:

(i) $\lambda^D\mu_D(\lambda^{-1}A)$ is *scale invariant* (independent of λ).

(ii) $\mu_D(\lambda^{-1}A)$ is *scaling* (power law dependence).

Iote: This is a simple case where the scale changing operator is $T_\lambda = \lambda^{-1}\mathbf{1}$, where $\mathbf{1}$ is the lentity matrix (see section II for generalized scale invariance and more general scale hanges). Hence the Hausdorff measures defined above are scale invariant under isotropic ilatations; they will be generalized to anisotropic "balls", hence to anisotropic Hausdorff neasures below.

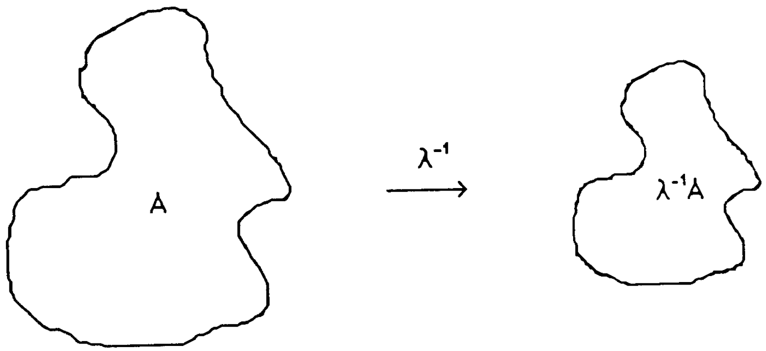


Figure I.12—The set A under the action of the scale changing operator $T_\lambda = \lambda^{-1}\mathbf{1}$.

1.5.3 Further Properties of Hausdorff Dimensions

The Hausdorff dimension (denoted with subscript H only when confusion with other dimensions may result) D_H satisfies the following properties (which might be expected to hold for any reasonable definition of dimension):

(i) Open sets: If $A \subset \mathbf{R}^D$ is open, then $D(A) = D$ since A contains a ball of positive D-dimensional volume.

(ii) Smooth sets: If A is a smooth (*i.e.*, continuously differentiable) m-dimensional submanifold (*i.e.*, an m-dimensional surface) of $\mathbf{R}^m$, then $D(A) = m$. In particular, smooth curves have dimension 1 and smooth surfaces have dimension 2 (this follows from the relationship between Hausdorff and Lebesgue measures).

(iii) Monotonicity: If $A \subset E$ then $D(E) \geq D(A)$. This follows immediately from the fact that μ_D is a measure hence $\mu_D(E) \geq \mu_D(A)$.

(iv) Countable stability: If $A_1, A_2, \ldots$ is a (countable) sequence of sets then

$$D\left(\bigcup_{i=1}^{\infty} A_i\right) = \sup_i \{D(A_i)\}$$

(*i.e.*,the Hausdorff dimension of a set is the maximum of its non-trivial subsets).

(v) Countable sets: If A is countable (*e.g.*, the set of rationals in [0,1]) then $D(A) = 0$ since we take A_i as single points and then use countable stability. See Falconer [1990] Chapter 2 for more details.

(vi) Hausdorff dimensions and fractals: although he disowned it later as being too restrictive, Mandelbrot [1977] originally defined a fractal set as a set for which $D_H > D_{top}$. The basic problem was to give a definition which ruled out "standard sets" such as lines, planes, *etc.*, while including all the complex "nonstandard" fractal sets. The reason $D_H > D_{top}$ is inadequate is that it rules out certain "obvious" fractals such as Brownian motion (which when embedded in a plane has $D_H = D_{top} = 2$, but which is distinguished from the set of points on the plane by the fact that $\Delta_2 = 1$). A more useful way to define a fractal is probably one which relies on some aspect of scale invariance, *i.e.*, invariance under zooms, although no formal definition based on this idea exists. Our point of view is that such a goal is in any case not too important: we will be much less interested in the detailed (and often complicated) properties of fractals sets, than we will be with the more fundamental idea of scale invariance.