PHYS 616 Multifractals and Turbulence

Lecture 10: Multifractal simulations

March 26, 2014

Box 5.1 Flux dynamics and statistical mechanics

Up until now, we have followed a "constructivist" approach. We have constructed specific cascade models and studied their properties. It is also possible to follow a more abstract "flux dynamics" approach (Schertzer and Lovejoy, 1992) because it parallels classical (or quantum) statistical mechanics, but the quantity of interest is the flux of energy whereas in thermodynamics/statistical mechanics the corresponding quantity is just the energy E. It turns out that the analogy is not only formal, since it corresponds to mappings from cascade models to Hamiltonian systems.

Analogy with thermodynamics. The correspondences between flux dynamics and thermodynamics with Boltzmann's constant = 1, $\Sigma(\beta)$ the Massieu potential, $F(\beta)$ the Helmholtz free energy. The implications are that just as one can discuss thermodynamic processes without reference to any specific microscopic model of matter, one can similarly discuss multifractal processes without reference to specific models such as cascades.

Thermo-dynamics analogues

Flux dynamics	Thermodynamics
probability space	phase space
q	$\beta = \frac{1}{T}$
γ	$-E$
$c(\gamma)$	$-S(E)$
$K(q) = \max_{\gamma}\left(q\gamma - c(\gamma)\right)$	$\Sigma(\beta) = \max_{E}\left(S(E) - \beta E\right)$
$C(q) = \frac{K(q)}{q-1}$	$F(\beta) = -\Sigma(\beta)/\beta$

In order to establish the analogy, recall that in thermodynamics (taking Boltzmann's constant =1) we have:

$$\begin{aligned} Z(T) &= \exp\left(\frac{-F(T)}{T}\right) = \left\langle \exp\left(\frac{-E}{T}\right)\right\rangle \\ &= \sum_i \exp\left(\frac{-E_i}{T}\right) \\ &= \int dE \exp\left(S(E) - \frac{E}{T}\right) \end{aligned} \tag{5.42}$$

Partition function

F= free energy

S= entropy

Box 5.1 (*cont.*)

where $Z(T)$ is the partition function, $S(E) = \log(p(E))$ is the entropy and $p(E)$ is the probability density of states with energy E. In the sum over i we sum over all states, whereas in the sum over j it is only over states with different energy (p_j is the degeneracy associated with the state of energy E_j): the integral form is obtained when the density of states goes to a continuous limit.

In flux dynamics the analogous equations are:

$$Z_\lambda(q) = e^{K_\lambda(q)} = \langle \varepsilon_\lambda^q \rangle = \left\langle e^{q\Gamma_\lambda} \right\rangle = \sum_{\{i\}} \lambda^{q\gamma_i} p(\gamma_i) = \int d\gamma \lambda^{q\gamma - c(\gamma)} \tag{5.43}$$

A summary of the analogies is shown in the table. Such analogies have been discussed in the literature on multifractals, with notably different points of view. For example, our treatment differs somewhat from that of Tél (1988); we rather follow Schuster (1988).

Using the trace moments introduced earlier, we can make parallels with the grand canonical ensemble. The grand canonical ensemble $Z_G(T)$ is obtained by summing not only over all energy states with a fixed number of particles, but also over the number of particles, each weighted by e^{-N}:

$$Z_G(T) = \sum_N \sum_{\{j\}} p_j e^{-E_j/(T-N)} = Tr\left\{ e^{-E_j/(T-N)} \right\} \tag{5.44}$$

where the trace indicates the sum over all states with energy E_j and N particles. In flux dynamics, the sum over energy states is replaced by sums over probability spaces (ensemble averaging, therefore "superaveraging") and the sum over various numbers of particles is replaced by integrals over various observing sets:

$$Z_{G,\lambda}(q) = Tr_{A_\lambda} \varepsilon_\lambda^q \tag{5.45}$$

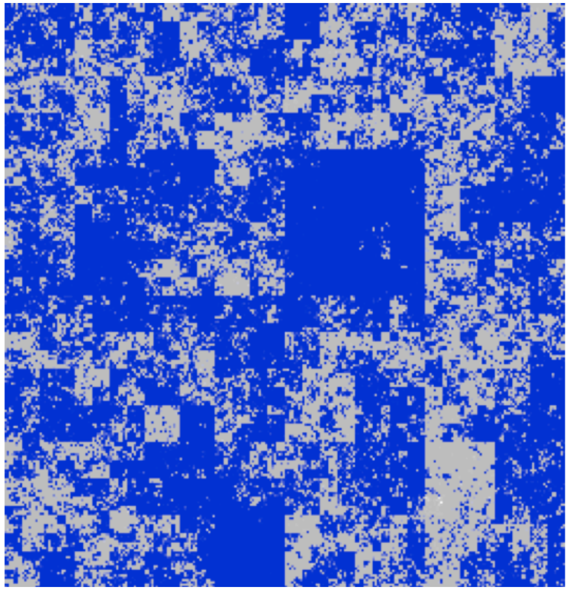
Hence, formally, we have:

$$e^{-N} = 1_{A_\lambda} d^{D(A)} \underline{r} \tag{5.46}$$

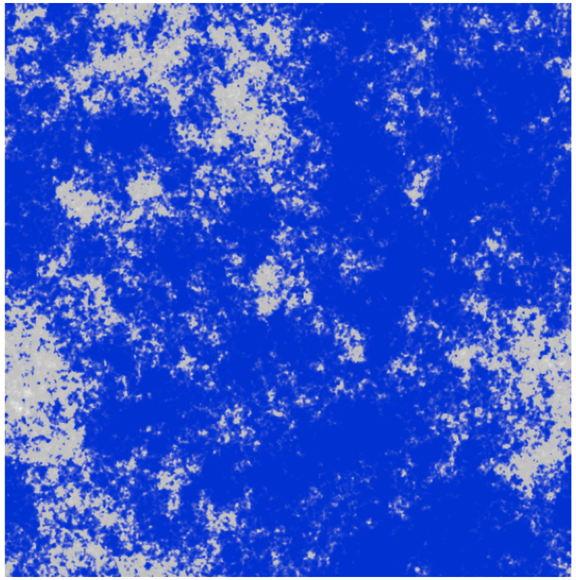
where 1_{A_λ} is the indicator function for the set A_λ.

Simulating isotropic continuous in scale multifractals

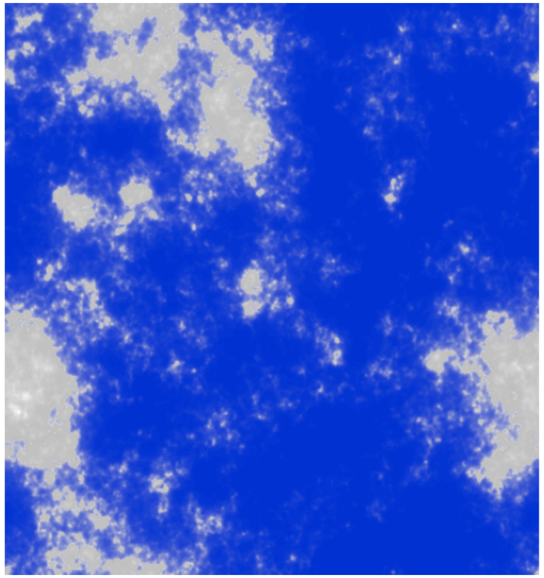
Continuous in scale multifractal modeling



A discrete in scale simulation of a universal multifractal with basic scale ratio $\lambda_0 = 2$, $\lambda = 2^9$, $\alpha = 1.8$, $C_1 = 0.1$.



The corresponding continuous in scale simulation.



Same as at left but with an additional fractional integration of order $H = 1/3$ (a scale invariant smoothing); to simulate a turbulent passive scalar density; notice that the structures are smoothed.

Levy variables

To explain continuous in scale cascade processes in a rather elementary manner, let us first introduce the "unit" (and extremal) Lévy random variable γ_α whose probabilities are implicitly defined by the following characteristic function:

$$\left\langle e^{q\gamma_\alpha}\right\rangle = e^{\frac{q^\alpha}{(\alpha-1)}}; \qquad q \geq 0$$

$$\left\langle e^{q\gamma_\alpha}\right\rangle = \infty; \qquad q < 0; \quad \alpha < 2$$

Unit extremal Levy variables

Note that:

a) for $\alpha = 2$ we have the familiar Gaussian case and the $q \geq 0$ formula is valid for all q:

b) for $\alpha = 1$ we have $\left\langle e^{q\gamma_\alpha}\right\rangle = e^{q\log q}$ ($q > 0$, otherwise $= \infty$).

An extremal Lévy random variable A with amplitude $a>0$ and Lévy index α therefore satisfies:

$$A = a\gamma_\alpha$$

$$\left\langle e^{qA}\right\rangle = e^{\frac{a^\alpha q^\alpha}{\alpha-1}}; \qquad q \geq 0$$

$$\left\langle e^{qA}\right\rangle = \infty; \qquad q < 0; \quad \alpha < 2$$

Extremal Levy variables amplitude A

(with corresponding exception for $\alpha = 1$).

Second characteristic functions

Consider the problem of determining the probability density $p_c(C)$ of the sum C of two independent random variables A,B, with probability densities $p_A(A)$, $p_B(B)$:

C = A+B

Direct method: $p_C(C) = \iint p_A(A)\, p_B(B)\, \delta(C-(A+B))\, dA\, dB$

Or: $p_C(C) = \int p_A(A)\, p_B(C-A)\, dA$ i.e. convolution

Solution with characteristic functions

Multiply both sides by q and the exponentiate the result and average:

$$\left\langle e^{Aq} \right\rangle = \left\langle e^{Bq} e^{Cq} \right\rangle = \left\langle e^{Aq} \right\rangle \left\langle e^{Bq} \right\rangle$$

Now, define the first characteristic function ψ: $\psi_B(q) = \left\langle e^{Bq} \right\rangle \quad \psi_A(q) = \left\langle e^{Aq} \right\rangle \quad \psi_C(q) = \left\langle e^{Cq} \right\rangle$

$$\psi_C(q) = \psi_A(q)\, \psi_B(q)$$

Finally defined the second characteristic function K as the log: $K(q) = \log \psi(q)$

Thus: $K_C(q) = K_A(q) + K_B(q)$

Hence for independent r.v.'s, 2nd characteristic functions add

If needed, $p_C(C)$ can be found by inverse Laplace transform of $\psi_C(q) = e^{K_C(q)}$

Characteristic functions for extremal Levy variables

α=2 is Gaussian case and this is valid for all q

For unit Levy variable

$$\psi_\alpha(q)=\langle e^{q\gamma_\alpha}\rangle = e^{\frac{q^\alpha}{\alpha-1}}; \qquad q\geq 0$$

$$\psi_\alpha(q)=\langle e^{q\gamma_\alpha}\rangle = \infty; \qquad q<0; \quad \alpha<2$$

First characteristic functions

(If α>2, the densities are not positive)

Hence

For unit Levy variable

$$K_\alpha(q)=\log\langle e^{q\gamma_\alpha}\rangle = \frac{q^\alpha}{\alpha-1}; \qquad q\geq 0$$

$$K_\alpha(q)=\log\langle e^{q\gamma_\alpha}\rangle = \infty; \qquad q<0; \quad \alpha<2$$

Second characteristic functions

Due to the additivity of second characteristic functions for any independent, identically distributed random variables, this implies that for the sum of two statistically independent Lévy variables A, B we have:

$$C=A+B$$

with $A=a\gamma_\alpha$ and $B=b\gamma_\alpha$

and

$$K_A(q)=\log\langle e^{qa\gamma_\alpha}\rangle = \frac{a^\alpha q^\alpha}{\alpha-1}$$

$$K_B(q)=\log\langle e^{qb\gamma_\alpha}\rangle = \frac{b^\alpha q^\alpha}{\alpha-1}$$

Hence: $K_C(q)=K_A(q)+K_B(q)$

substituting $K_C(q)=\frac{a^\alpha q^\alpha}{\alpha-1}+\frac{b^\alpha q^\alpha}{\alpha-1}=\frac{(a^\alpha+b^\alpha)q^\alpha}{\alpha-1}$

This shows that $C=c\gamma_\alpha$; $c^\alpha = a^\alpha + b^\alpha$

i.e. the sum of two extremal Levy's is an extremal Levy

"Stability under addition"

(Sum of Levy variables= Levy variable of same type)

Probabilities of Levy variables

General Levy variables

$$\langle |l_\alpha|^q \rangle \to \infty; \quad q \ge \alpha$$

Power law tails for both large positive and negative values

$$p(l_\alpha) \approx A_+ l_\alpha^{-\alpha-1}; \quad l_\alpha >> 1$$

$$p(l_\alpha) \approx A_-(-l_\alpha)^{-\alpha-1}; \quad l_\alpha << -1$$

Generally different weights of the "tails"

$A_+ = 0$: "extremal" or maximally skewed

Extremal Levy variables (with A_+=0)

$$p(\gamma_\alpha) \approx \exp\left[-\left(\frac{\gamma_\alpha}{\alpha'}\right)^{\alpha'}\right]; \quad \left(\frac{\gamma_\alpha}{\alpha'}\right)^{\alpha'} >> 0; \quad \frac{1}{\alpha'} + \frac{1}{\alpha} = 1$$

Power law tails for large negative values only

$$p(\gamma_\alpha) \approx (-\gamma_\alpha)^{-\alpha-1}; \quad \gamma_\alpha << 0; \quad 0 < \alpha < 2$$

Reason for extremals:

$$\langle \varepsilon_\lambda^q \rangle = \langle \lambda^{q\gamma} \rangle = \langle e^{q\gamma_\alpha \log\lambda} \rangle = \int_{-\infty}^{\infty} e^{q\gamma_\alpha \log\lambda} p_\alpha(\gamma_\alpha) d\gamma_\alpha$$

$$\langle \varepsilon_\lambda^q \rangle \approx A_- \int_{-\infty}^{-1} e^{q\gamma_\alpha \log\lambda} (-\gamma_\alpha)^{-\alpha-1} d\gamma_\alpha + A_+ \int_{1}^{\infty} e^{q\gamma_\alpha \log\lambda} (\gamma_\alpha)^{-\alpha-1} d\gamma_\alpha$$

Only if A_+ =0 will the moments of ε_λ converge for q>0

Continuous in scale cascade processes

With the help a suitably normalized convolution kernel $g_\lambda(\underline{r})$, it then possible to colour this white noise to obtain a generator $\Gamma_\lambda = \log \varepsilon_\lambda$ such that $\langle \varepsilon_\lambda^q \rangle = \langle e^{q\Gamma_\lambda} \rangle = e^{K(q)\log\lambda}$.

We now show how to obtain a generator with the appropriate properties (including a mean codimension C_1) by convolving a Levy noise $\gamma_\alpha(\underline{r})$ with a kernel $g_\lambda(\underline{r})$:

$$\Gamma_\lambda = C_1^{1/\alpha} g_\lambda * \gamma_\alpha = C_1^{1/\alpha} \int g_\lambda\left(\underline{r} - \underline{r}'\right) \gamma_\alpha\left(\underline{r}\right) d\underline{r}$$

Put i.i.d. Levy r.v.'s on a grid, then take small scale limit

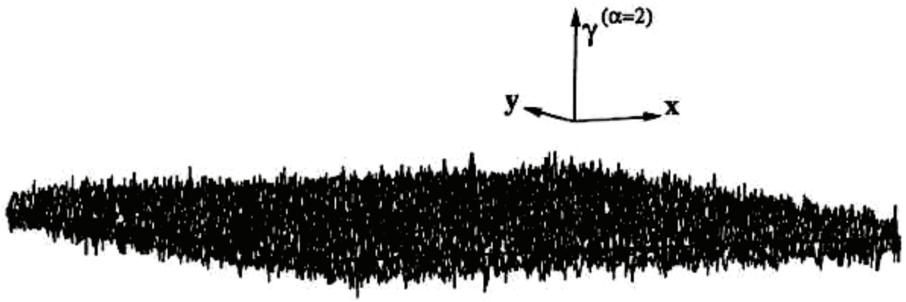
We now review, step by step the different properties that the kernel $g_\lambda(\underline{r})$ and its domain of integration must satisfy in order to obtain the announced result, in particular that the multifractal ε_λ:

$$\varepsilon_\lambda = e^{\Gamma_\lambda}$$

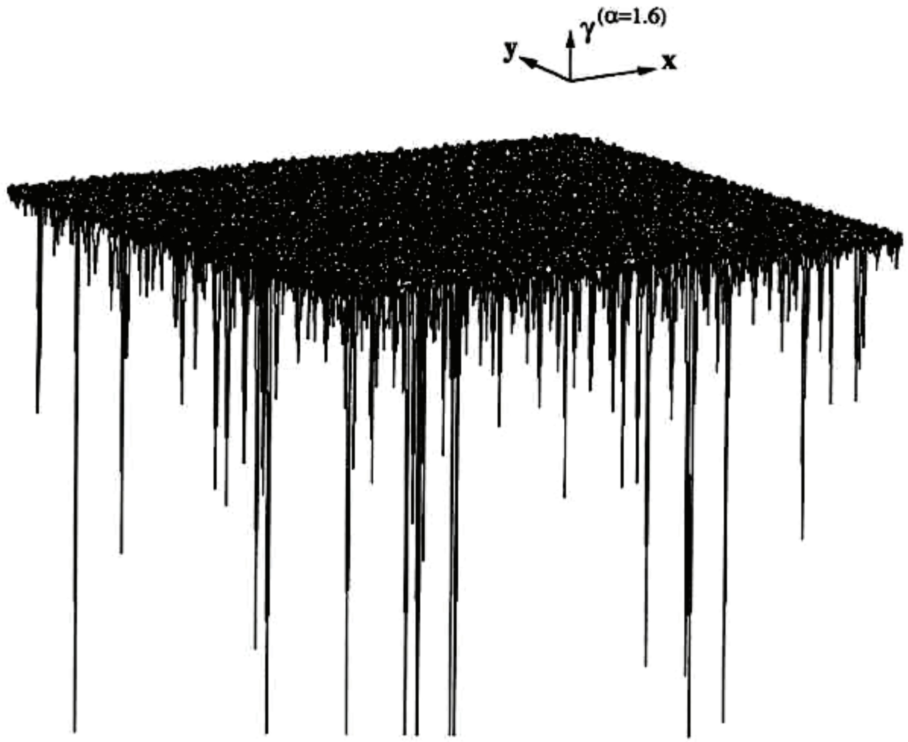
will be indeed be multiscaling: $\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)}$.

Levy sub generators $\gamma_\alpha(\underline{r})$

Gaussian (α =2)



Levy (α =1.6)



A comparison of the Gaussian (α =2, top) and Levy (α =1.6, bottom) subgenerators γ_α showing that whereas the former is both positive – negative symmetric with low amplitude excursions, the latter is asymmetric with huge (algebraic) excursions for negative values.

How to obtain log divergent generator second characteristic function

This is the first property to be respected:

$$K_\Gamma(q,\lambda) = Log(<e^{q\Gamma_\lambda}>) = Log(\lambda)K(q)$$

Criterion for choosing g

i.e. $g_\lambda(\underline{r})$ must be chosen so as to yield a logarithmic divergence of the second characteristic function of the generator Γ_λ.

Since $\gamma_\alpha(\underline{r})$ is statistically homogeneous, the statistics of Γ are independent of $\underline{r}$ so that one can take $\underline{r} = 0$ and apply the additivity of the second characteristic function:

$$K_\Gamma(q) = C_1 \frac{q^\alpha}{\alpha - 1}\left|g_\lambda\right|_\alpha^\alpha; \quad \left|g_\lambda\right|_\alpha^\alpha \equiv \int \left|g_\lambda(\underline{r})\right|^\alpha d^d\underline{r}$$

Recall:

$$\Gamma_\lambda(\underline{r}) = C_1^{1/\alpha} g_\lambda * \gamma_\alpha = C_1^{1/\alpha}\int g_\lambda\left(\underline{r} - \underline{r}'\right)\gamma_\alpha(\underline{r}')d\underline{r}'$$

where the α-1 in the denominator comes from the definition of the unit Levy variables,

In order to obtain the desired Log(λ) divergence for K_Γ it suffices to choose g_λ to be an isotropic power law with the appropriate power law from the larger scale L to the resolution L/λ:

$$g_\lambda(\underline{r}) = N_{d,\alpha}^{-1/\alpha} 1_{L/\lambda \le |\underline{r}| \le L} \left|\underline{r}\right|^{-d/\alpha}$$

Check: $$\int_{L \le |\underline{r}| \le L/\lambda} \left(\left|\underline{r}'\right|^{-d/\alpha}\right)^\alpha d^d\underline{r}' = \int_{L/\lambda}^{L} \left(\left|\underline{r}'\right|^{-d/\alpha}\right)^\alpha \Omega_d \left|\underline{r}'\right|^{d-1} d\left|\underline{r}'\right| = \Omega_d \log\lambda$$

Note: take g(r)=g(-r)

where 1_B is the indicator function of the subset B *i.e.* $1_B(\underline{r}) = 1$ if $\underline{r} \in B$, = 0 otherwise. This implies:

$$\left\langle \varepsilon_\lambda^q \right\rangle = \left\langle e^{q\Gamma_\lambda}\right\rangle = e^{\frac{C_1}{\alpha-1} q^\alpha N_d^{-1}\Omega_d \log\lambda}; \quad \Omega_d = \int_{|\underline{r}'|=1} d^d\underline{r}'$$

where Ω_d is the integral over all the angles in the d dimensional space (e.g. $\Omega_1 = 2$, $\Omega_2 = 2\pi$, $\Omega_3 = 4\pi$ etc.).

Normalization

We see that if we choose:

$$N_d = \Omega_d$$

then we obtain the desired nonlinear part of the multiscaling behaviour:

$$\left\langle \varepsilon_{\lambda,u}^q \right\rangle = \lambda^{K_u(q)}; \quad K_u(q) = \frac{C_1}{\alpha - 1} q^\alpha$$

Recall:

$$\left\langle \varepsilon_\lambda^q \right\rangle = \left\langle e^{q\Gamma_\lambda} \right\rangle = e^{\frac{C_1}{\alpha-1} q^\alpha N_d^{-1} \Omega_d \log \lambda}; \quad \Omega_d = \int_{|\underline{r}'|=1} d^d \underline{r}'$$

where K_u is the unnormalized exponent scaling function corresponding to the fact that ε given by is unnormalized (hence we temporarily add the subscript "*u*"). A normalized $\varepsilon_{\lambda,n}$ can now be easily obtained using:

$$\varepsilon_{\lambda,n} = \frac{\varepsilon_{\lambda,u}}{\left\langle \varepsilon_{\lambda,u} \right\rangle}$$

so that:

$$\left\langle \varepsilon_{\lambda,n}^q \right\rangle = \lambda^{K(q)}; \quad K(q) = K_u(q) - qK_u(1) = \frac{C_1}{\alpha - 1}\left(q^\alpha - q\right)$$

as required (we temporarily add the subscript "*n*" to distinguish it from the unnormalized process).

Fractional Brownian and fractional Lévy noises

We have seen that the typical observables such as the wind have fluctuations (Δv) whose statistics are related to the fluxes by a lag Δx raised to a power, the prototypical example being the Kolmogorov law: $\Delta v = \varphi \Delta x^{H}$ with $\varphi = \varepsilon^{1/3}$, $H = 1/3$. If we take the q^{th} moments of this equation, we obtain:

$$S_q(\Delta x) \propto \Delta x^{\xi(q)}; \qquad S_q(\Delta x) = \left\langle |\Delta v|^q \right\rangle; \quad \xi(q) = qH - K(q)$$

More precisely, these are written as convolutions of noises with power laws which are extensions of integration/differentiation to fractional orders; "fractional integrals":

Power law convolution

$$v(\underline{r}) = \gamma_\alpha * |\underline{r}|^{-(d-H')} = \int \frac{\gamma_\alpha(\underline{r}')}{|\underline{r}-\underline{r}'|^{d-H'}} d^d \underline{r}'; \quad H' = H + d/\alpha$$

fBm = Fractional Brownian Motion (α=2)

fLm=Fractional Levy Motion ($0 \le \alpha < 2$)

where γ_α is again a Lévy noise made of uncorrelated Lévy random variables (here they need not be extremal) and H' is the order of fractional integration (as usual, the Gaussian case is recovered with $\alpha = 2$).

"Fractionally Integrated Flux", FIF model:

$$v_\lambda(\underline{r}) = \varepsilon_\lambda * |\underline{r}|^{d-H} = \int \frac{\varepsilon_\lambda(\underline{r}') d^d \underline{r}'}{|\underline{r}-\underline{r}'|^{d-H}}$$

FIF= Fractionally Integrated Flux model

we already saw that the flux itself can be modelled in the same (power convolution/fractional integration) framework

Fractional derivatives, integrals

The power law convolution is easier to understand if we consider it in Fourier space. Since the Fourier transform ("F.T.") of a singularity is another singularity (the Tauberian theorem):

$$|\underline{r}|^{-(D-H)} \overset{F.T.}{\leftrightarrow} |\underline{k}|^{-H}$$

We can use the basic property that a convolution is Fourier transformed into a multiplication, to obtain simply:

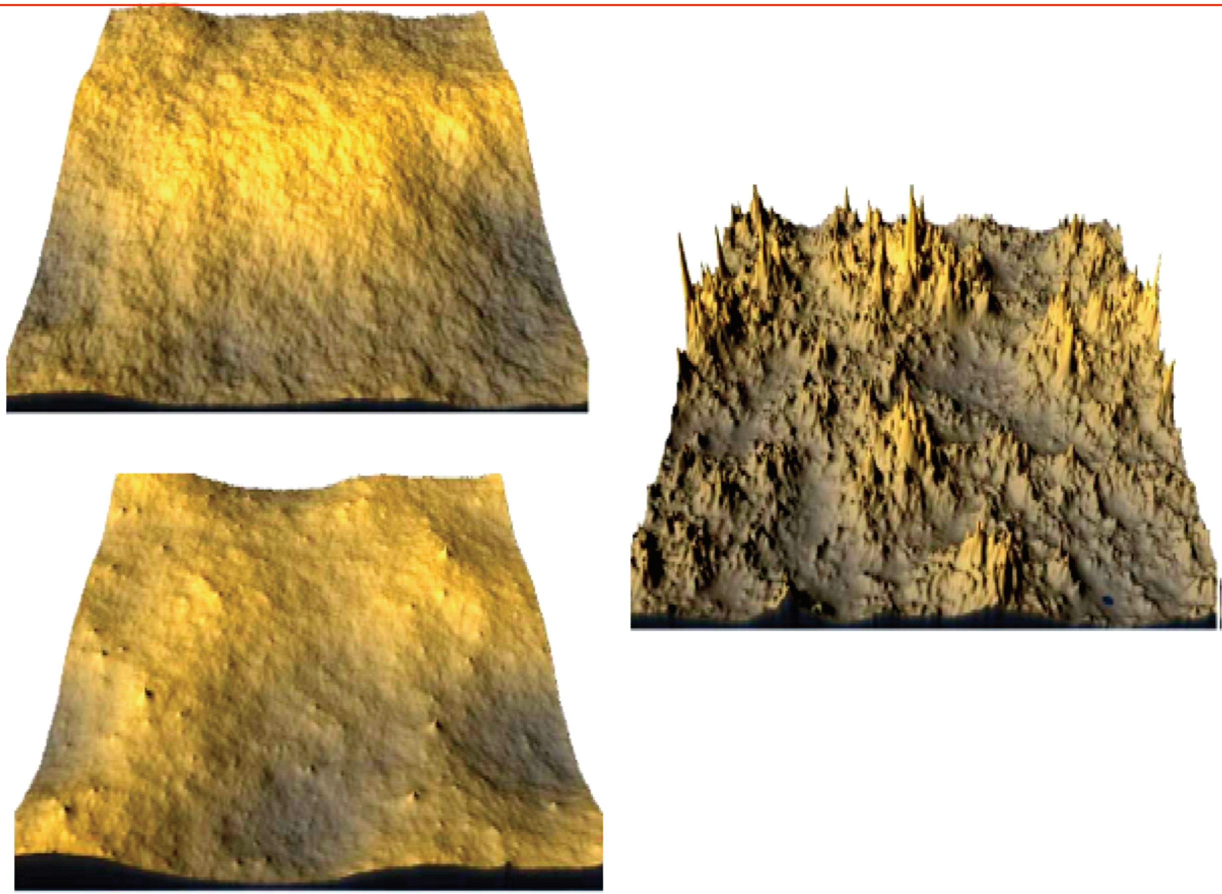
$$\tilde{v}(\underline{k}) \propto \widetilde{\gamma_\alpha}(\underline{k})|\underline{k}|^{-H}$$

where:

$$v(\underline{r}) \overset{F.T.}{\leftrightarrow} \tilde{v}(\underline{k}); \quad \gamma_\alpha(\underline{r}) \overset{F.T.}{\leftrightarrow} \widetilde{\gamma_\alpha}(\underline{k})$$

We thus see that the convolution with power law $|\underline{r}|^{-(D-H)}$ is the equivalent to a power law filter $|\underline{k}|^{-H}$. However, such filters are themselves generalizations of differentiation (H<0) or integration (H>0). To see this, recall the Fourier transform of the Laplacian: $(\nabla^2\gamma_\alpha) \overset{F.T.}{\leftrightarrow} -|\underline{k}|^2\widetilde{\gamma_\alpha}$ so (ignoring constant factors) that $|\underline{k}|^{-H}$ corresponds to real space $(\nabla^2)^{-H/2}$, i.e. for H>0 it corresponds to a negative order differentiation (i.e. integration) of order H.

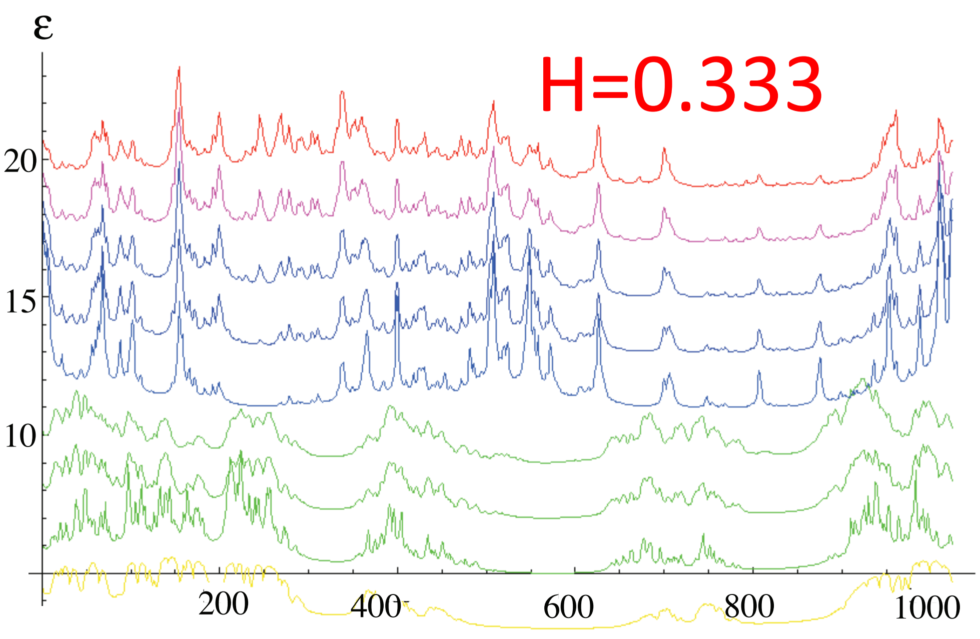
fBm, fLm, FIF examples

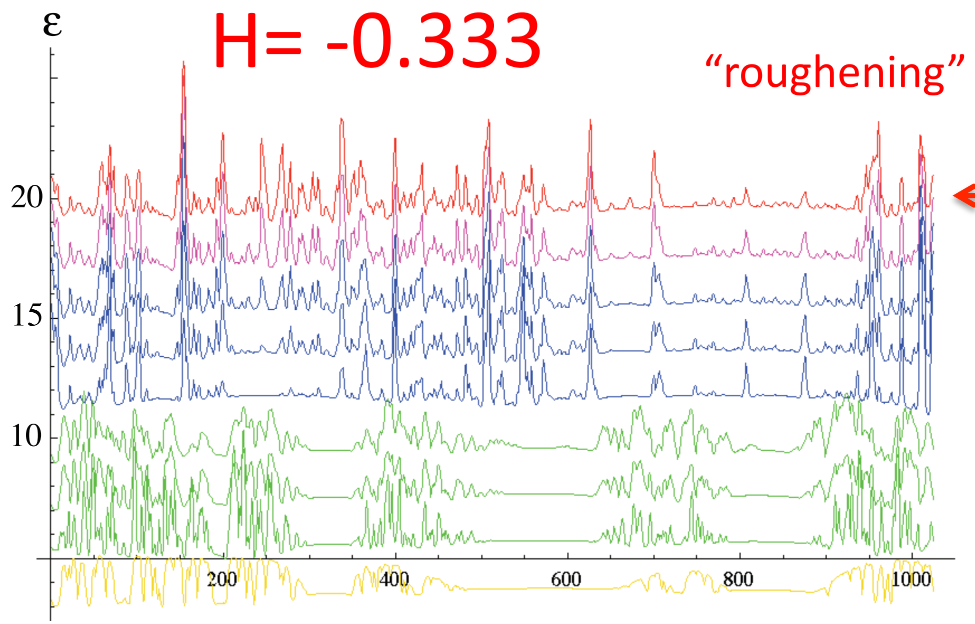


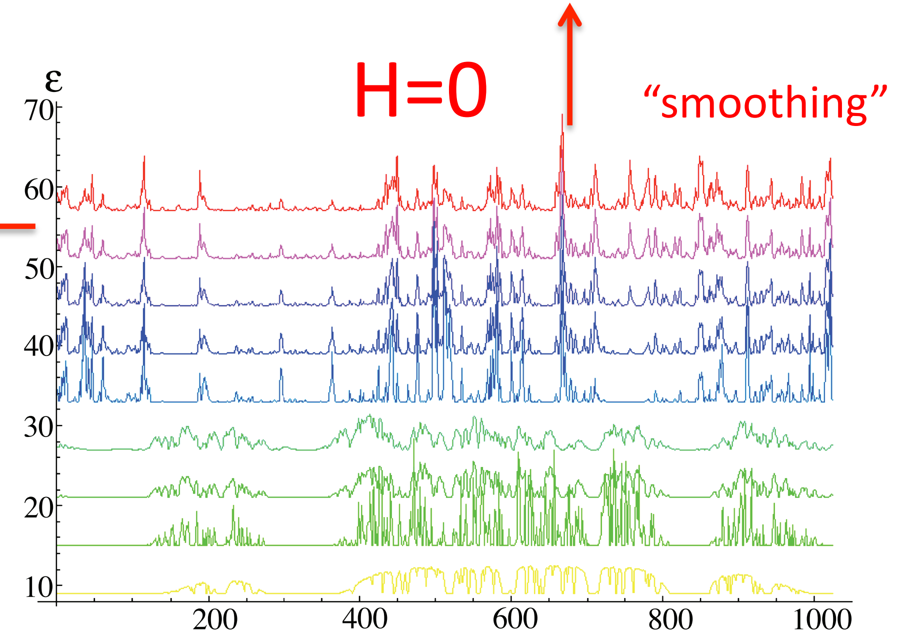
The upper left simulation shows fBm , with H = 0.7, lower left fLm with H = 0.7, α =1.8, and the right the Multifractal FIF with H = 0.7, α =1.8, C_1= 0.12 (close to observations for topography). Note the occasional "spikes" in the FLm which are absent in the fBm; these are due to the extreme power law tails (In this fLm positive extremal Levy variables were used, hence there are no corresponding "holes") .

FIF simulations 1-D

FIF simulations with α = 0.3, 0.5, …1.9, (bottom to top), C_1 = 0.1, each offset for clarity, each with the same random seed.

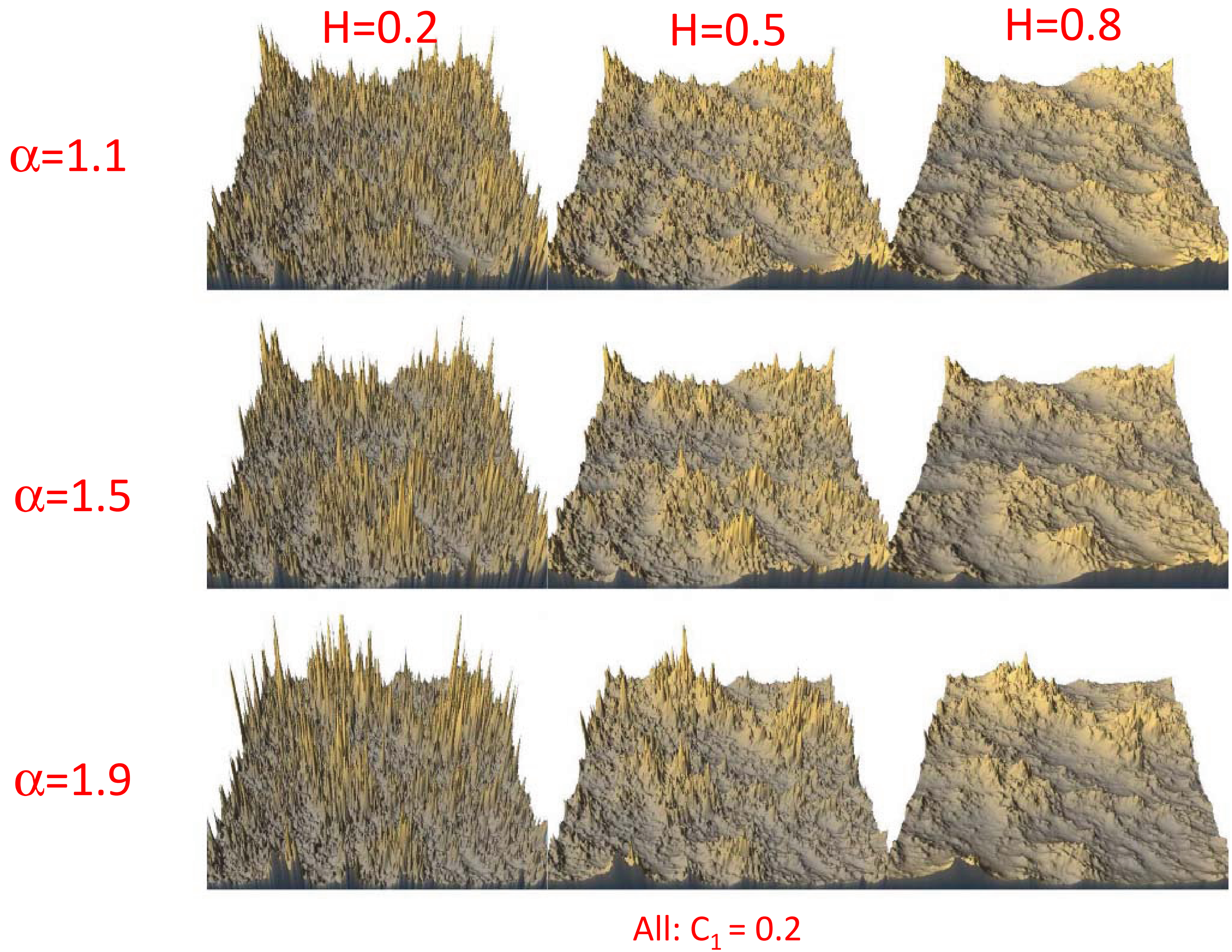






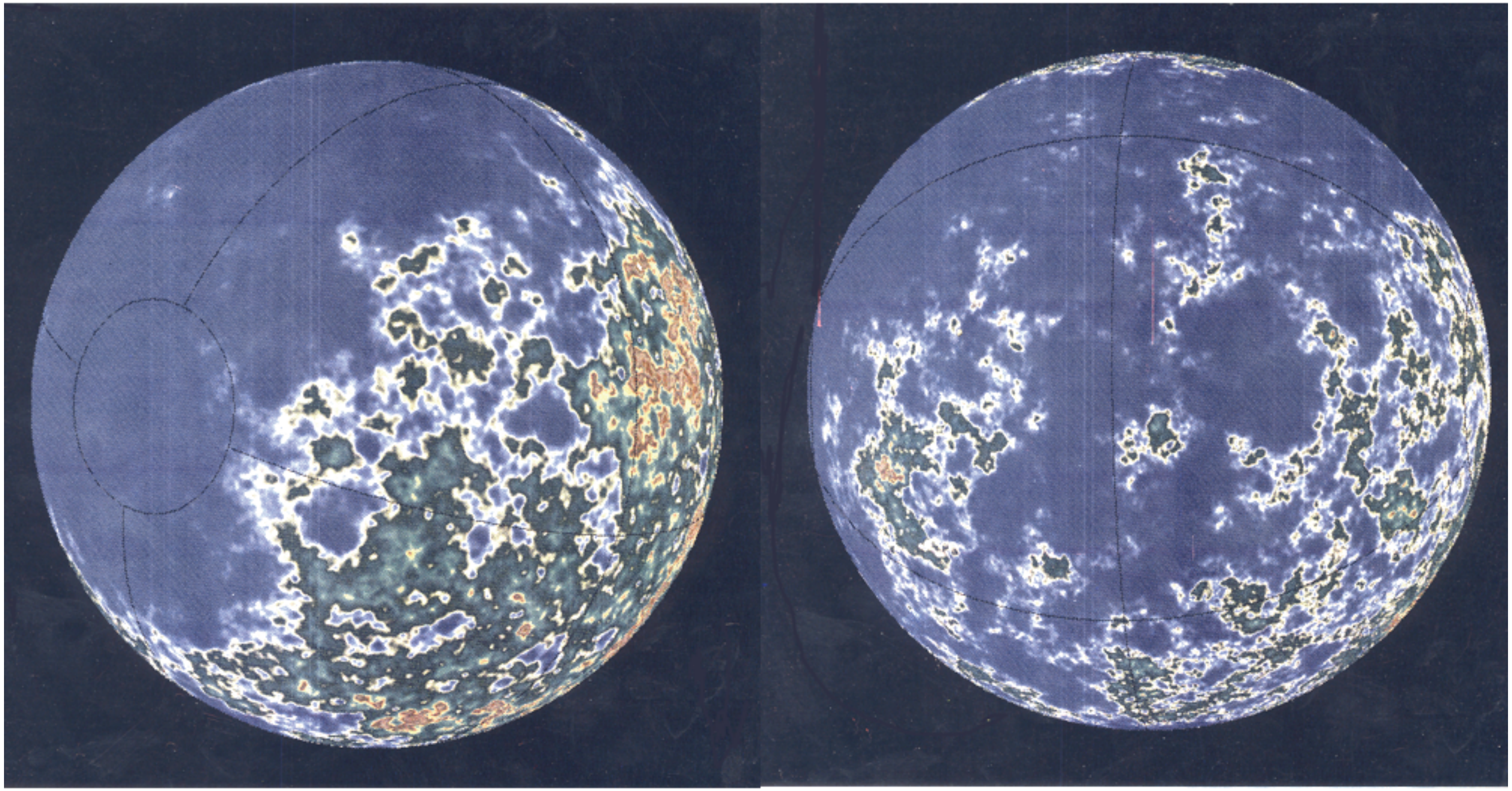
Theoretical Comparison of fBM, fLm, FIF

Model	Field	Increments	Codimensions of level sets
Monofractal fBm	$v(\underline{r}) = \gamma_2 * \lvert\underline{r}\rvert^{-(d-H')}$ $H' = H + H_2;$ $H_2 = d/2$	$\Delta v \overset{d}{=} \gamma_2 \lvert\underline{\Delta r}\rvert^{H}$ $\langle \lvert\Delta v\rvert^{q} \rangle \propto \lvert\underline{\Delta r}\rvert^{\xi(q)}$ $\xi(q) = qH$	$C = H$ $D = d - C$
Monofractal fLm	$v(\underline{r}) = \gamma_\alpha * \lvert\underline{r}\rvert^{-(d-H')}$ $H' = H + H_\alpha;$ $H_\alpha = d/\alpha$	$\Delta v \overset{d}{=} \gamma_\alpha \lvert\underline{\Delta r}\rvert^{H}$ $\langle \lvert\Delta v\rvert^{q} \rangle \propto \lvert\underline{\Delta r}\rvert^{\xi(q)}$ $\xi(q) = \begin{matrix} qH; & q < \alpha \\ \infty; & q > \alpha \end{matrix}$	$C = H$ $D = d - C$
Multifractal FIF	$\Gamma_\lambda \propto {C_1}^{1/\alpha} \gamma_\alpha * \lvert\underline{r}\rvert^{-(d-H_{\alpha'})};$ $\varphi_\lambda = e^{\Gamma_\lambda}$ $v_\lambda(\underline{r}) = \varphi_\lambda * \lvert\underline{r}\rvert^{-(d-H)}$ $H_{\alpha'} = d/\alpha'; \quad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1$	$\Delta v = \varphi_\lambda \lvert\underline{\Delta r}\rvert^{H}$ $\langle \lvert\Delta v\rvert^{q} \rangle \propto \lvert\underline{\Delta r}\rvert^{\xi(q)}$ $\xi(q) = qH - K(q)$	$c(\gamma) = \max_q \left(q\gamma - K(q)\right)$ $D(\gamma) = d - c(\gamma)$



Isotropic (i.e. self-similar) multifractal simulations showing the effect of varying the parameters α and H (C_1=0.1 in all cases). From left to right, H = 0.2, 0.5 and 0.8. From top to bottom, α =1.1, 1.5 and 1.8. As H increases, the fields become smoother and as α decreases, one notices more and more prominent "holes" (i.e. low smooth regions). The realistic values for topography (α=1.79, C_1=0.12, H=0.7) correspond to the two lower right hand simulations. All the simulations have the same random seed.

FIF on a sphere

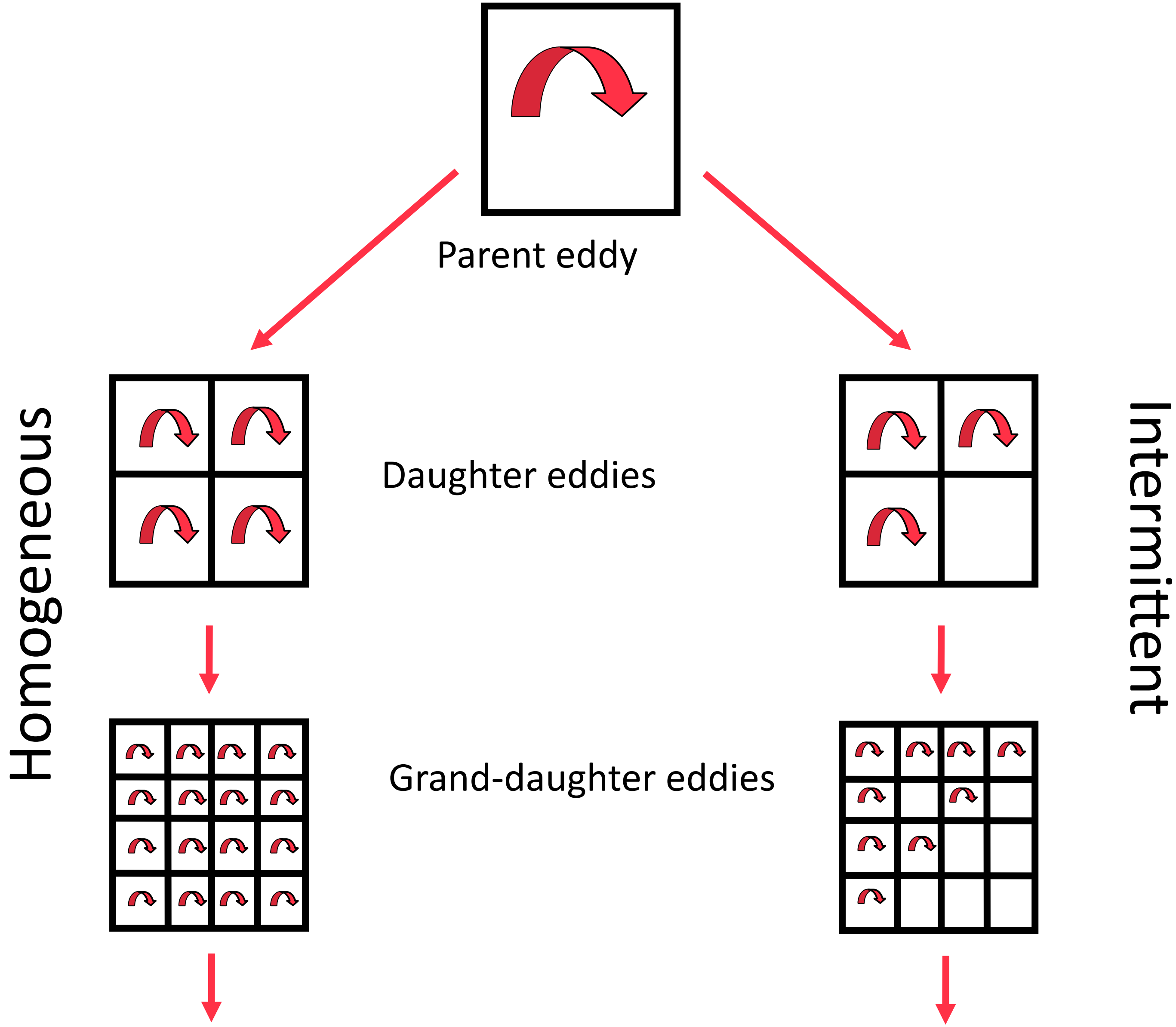


A simulation of an (isotropic) multifractal topography on a sphere using the spherical harmonic method discussed in the appendix (both sides of a single simulation are shown, using false colours). The simulation parameters are close to the measured values: α =1.8, C_1= 0.1, H = 0.7. The absence of mountain "chains" and other typical geomorphological features are presumably due to the absence of anisotropy.

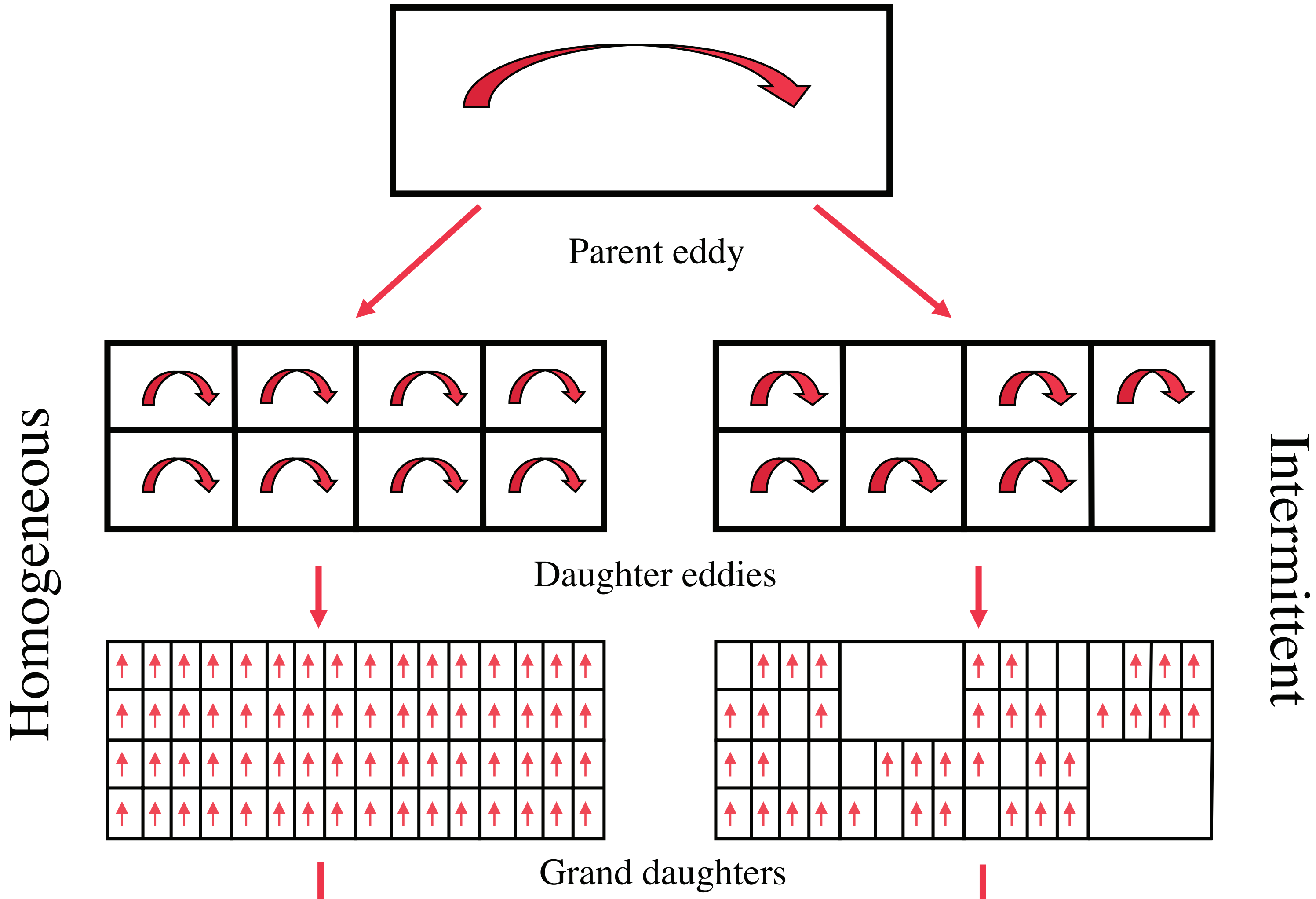
Generalized Scale Invariance

CASCADES

(isotropic)



Stratified CASCADES



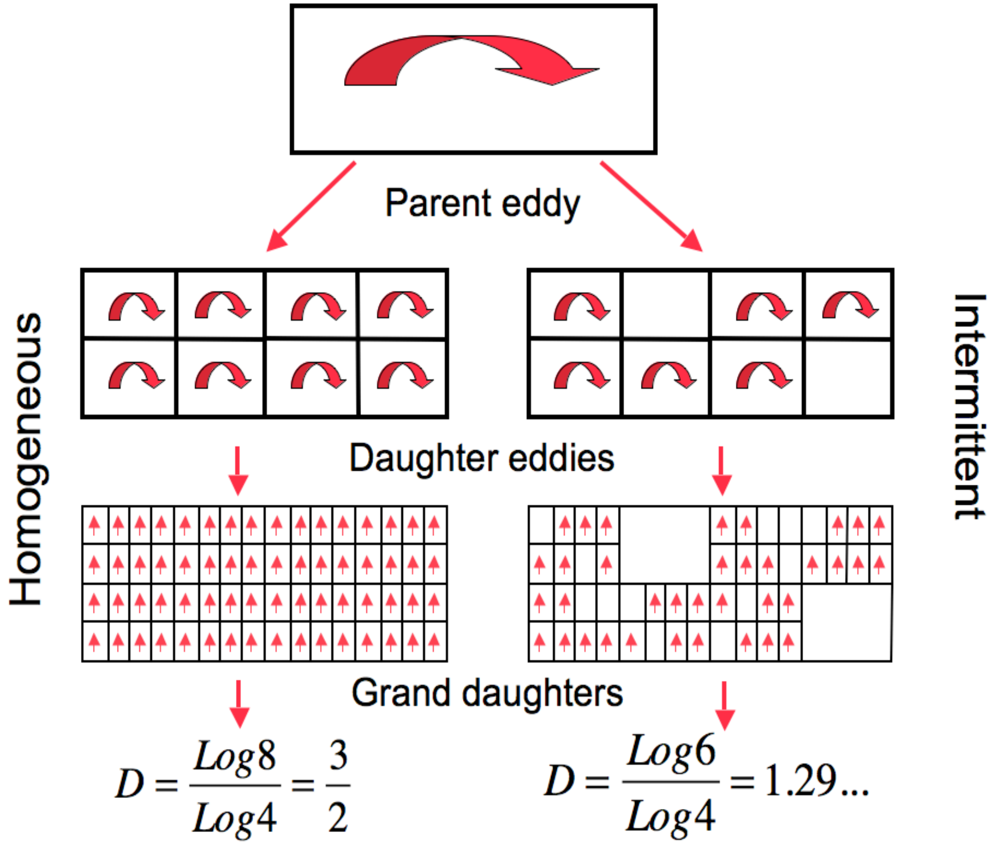
$$D = \frac{Log8}{Log4} = \frac{3}{2}$$

$$D = \frac{Log6}{Log4} = 1.29...$$

Anisotropic cascades, elliptical dimensions

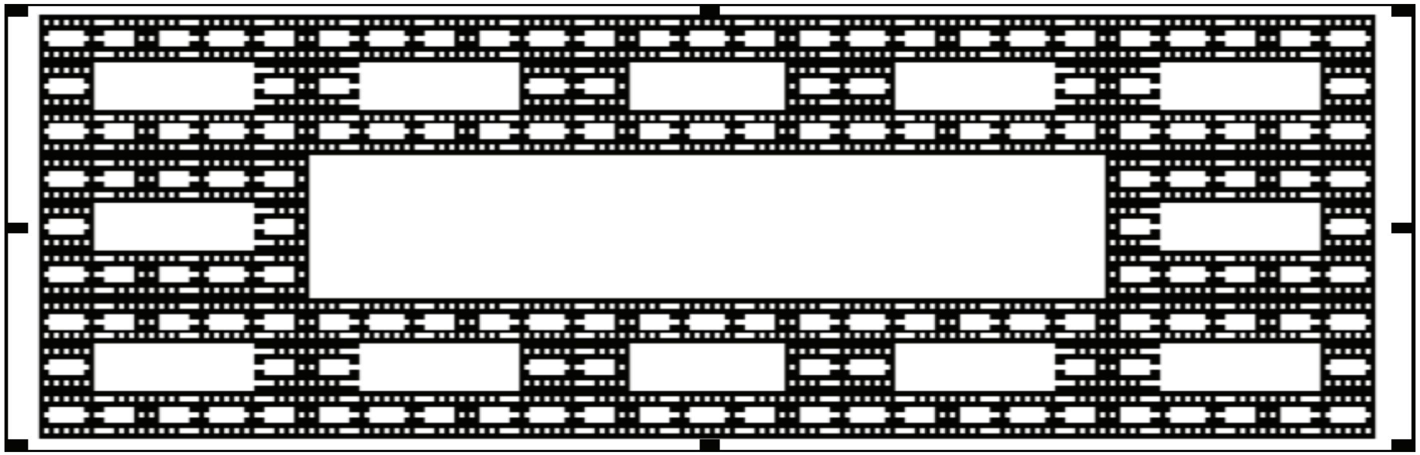
In the figure, $\lambda_0 = 4$ and $\lambda_0^{Hz} = 2$ respectively so that $H_z = \log2/\log4 = 1/2$). Indeed, the reduction of the areas at each iteration is by the factor:

$$\frac{A_{n-1}}{A_n} = \lambda_0 \lambda_0^{H_z} = \lambda_0^{D_{el}}; \quad D_{el} = 1 + H_z$$



A schematic of an anisotropic cascade; compare with its isotropic counterpart. The exponent governing the decrease in area (equivalently the increase in number) of the subeddies with each iteration is D_{el} = log8/log4=3/2. On the right hand side we illustrate the inhomogeneous (intermittent) anisotropic cascade in which $\lambda_0^C = 2$ of the 8 subeddies on average are killed off so the corresponding elliptical (anisotropic) dimension of the active regions of $D = D_{el} - C$ = log6/log2 = 1.29...

Anisotropic "Sierpinski carpet"



An example of a deterministic β model an anisotropic "Sierpinski carpet" obtained by dividing the horizontal by factors of 5 and the vertical by factors of 3 at each iteration and removing the 3 middle rectangles (keeping the 12 outer ones);

Turbulence laws

(for horizontal velocity fluctuations, Δv)

Kolmogorov: dynamically driven Isotropic 3D energy cascades

$$\Delta v(\underline{\Delta r}) \approx \varepsilon^{1/3} |\underline{\Delta r}|^{1/3}$$

Energy flux

Bolgiano-Obukhov: buoyancy driven isotropic 3D Buoyancy variance flux cascades

$$\Delta v(\underline{\Delta r}) \approx \phi^{1/5} |\underline{\Delta r}|^{3/5}$$

Buoyancy variance flux

Isotropic 2D enstrophy cascades

$$\Delta v(\Delta x) \approx \eta^{1/3} \Delta x$$

Enstrophy flux

Isotropic 3D pseudo-potential enstrophy cascades (Charney 1971)

$$\Delta v(\underline{\Delta r}) \approx \eta_p^{1/3} \Theta(\widehat{\underline{\Delta r}}) |\underline{\Delta r}|^{1}$$

Pseudo-potential Enstrophy flux

Trivial anisotropy

Isotropic exponents

$D_{el} = 3$

Anisotropic Quasi-linear Gravity waves

$$\Delta v(\Delta x) \approx \varepsilon^{1/3} \Delta x^{1/3}; \quad \Delta v(\Delta z) \approx N \Delta z$$

Brunt-Vaisaila frequency

Anisotropic BO/K: Unified scaling model

$$\Delta v(\Delta x) \approx \varepsilon^{1/3} \Delta x^{1/3}; \quad \Delta v(\Delta z) \approx \phi^{1/5} \Delta z^{3/5}$$

anisotropic

$D_{el} = 7/3$

$D_{el} = 23/9$

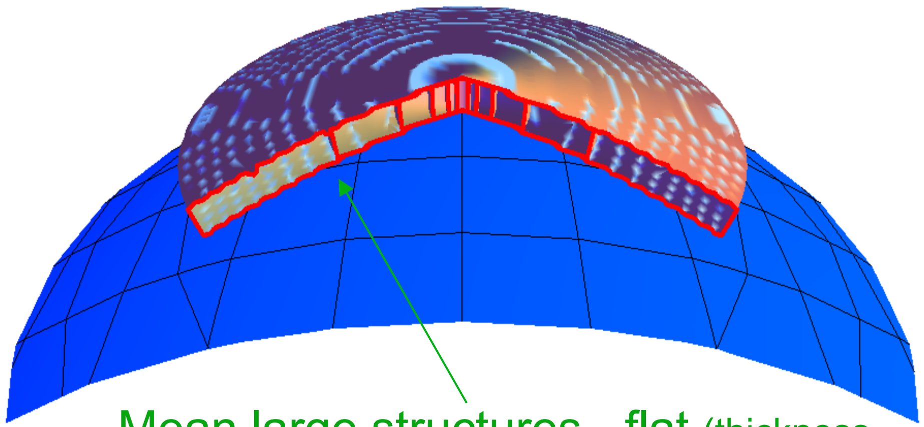
The Standard (2D/3D) Model

"Weather"

Size notion:

$$\left|(\Delta x, \Delta y)\right| = \left(\Delta x^2 + \Delta y^2\right)^{1/2}$$

Large scale 2D



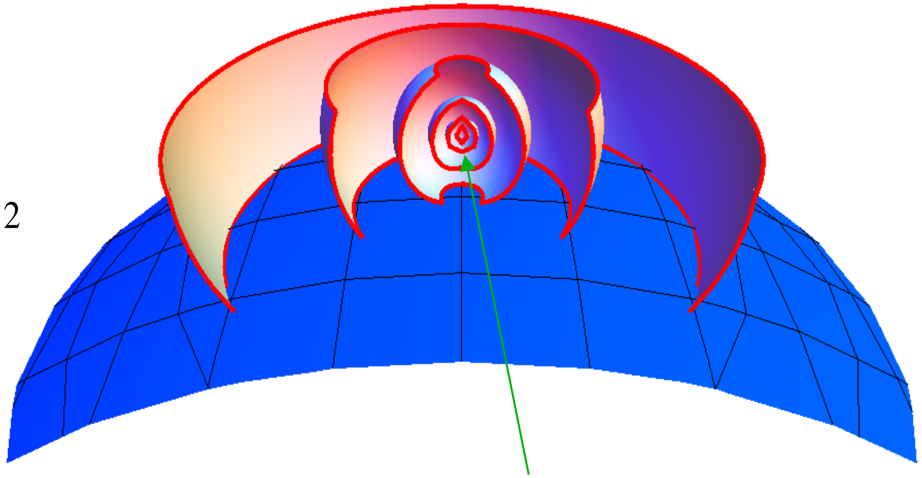
Mean large structures - flat (thickness independent of scale)

"Turbulence"

Size notion:

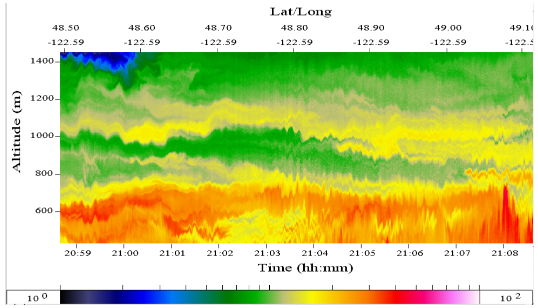
$$\left|(\Delta x, \Delta y, \Delta z)\right| = \left(\Delta x^2 + \Delta y^2 + \Delta z^2\right)^{1/2}$$

Small scale 3D



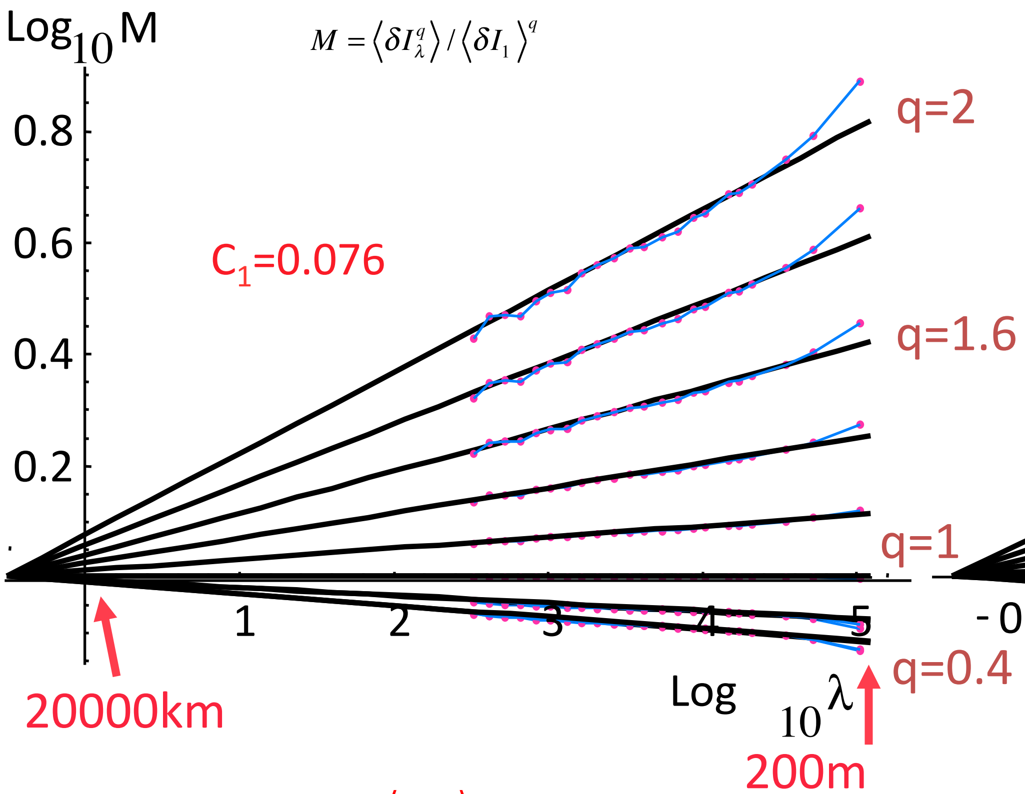
Mean structures - spherical (only smalll ones are physically possible due to finite thickness)

Vertical cascades:
lidar backscatter

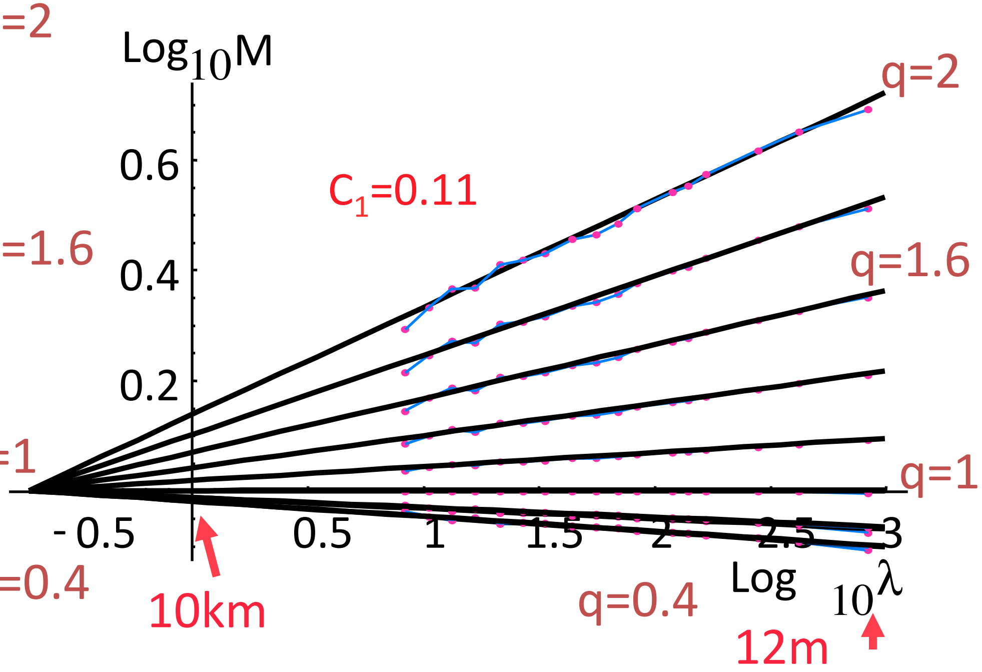


From 10 airborne lidar cross-sections near Vancouver B.C.

Horizontal cascade



Vertical cascade



$$M = \langle \varphi_\lambda^q \rangle / \langle \varphi \rangle^q$$

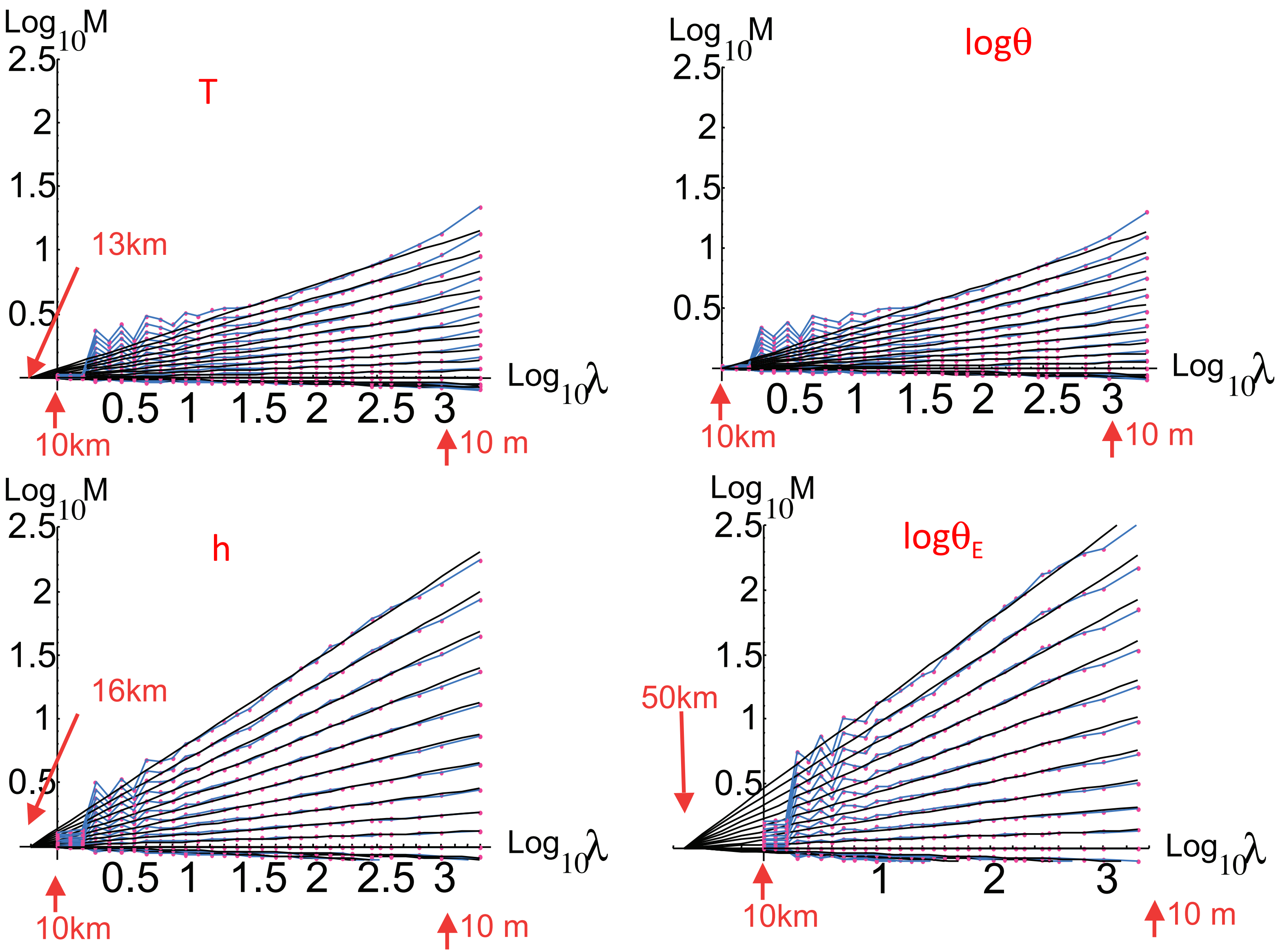
$$M_q \approx \lambda^{K(q)}$$

q=0, 0.2, 0.4..., 2

Vertical cascades:

Thermodynamic fields

$$M = \langle \varphi_\lambda^q \rangle / \langle \varphi \rangle^q$$
$$M_q \approx \lambda^{K(q)}$$



The physical scale function and differential scaling

$$|\underline{\Delta r}| \rightarrow \|\underline{\Delta r}\|$$

Usual distance (=vector norm)

Scale function (scale notion)

Scale symmetry $\quad \|\lambda^{-G}\underline{r}\| = \lambda^{-1}\|\underline{r}\|$

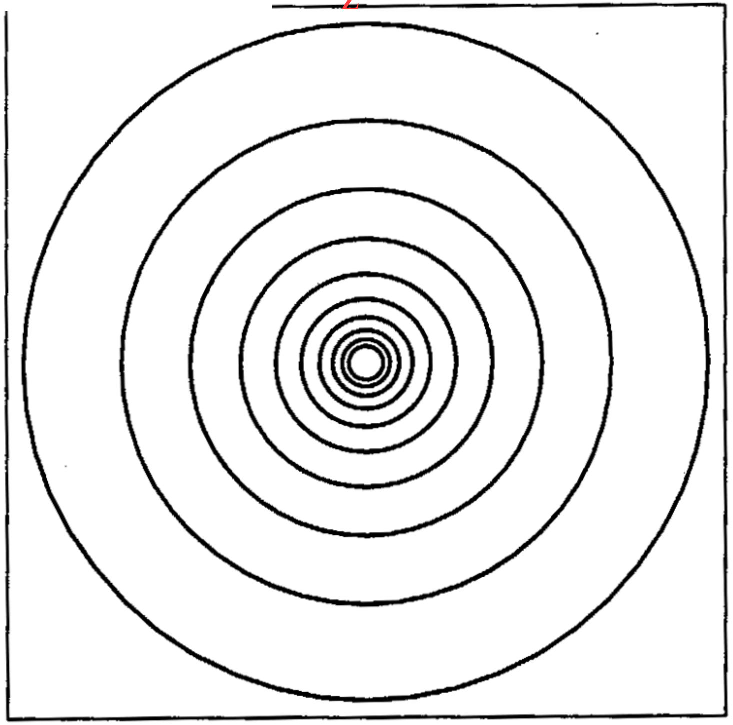
"canonical" scale function:

$$\|(\Delta x,\Delta z)\| = l_s\left(\left(\frac{\Delta x}{l_s}\right)^2+\left(\frac{\Delta z}{l_s}\right)^{2/H_z}\right)^{1/2}$$

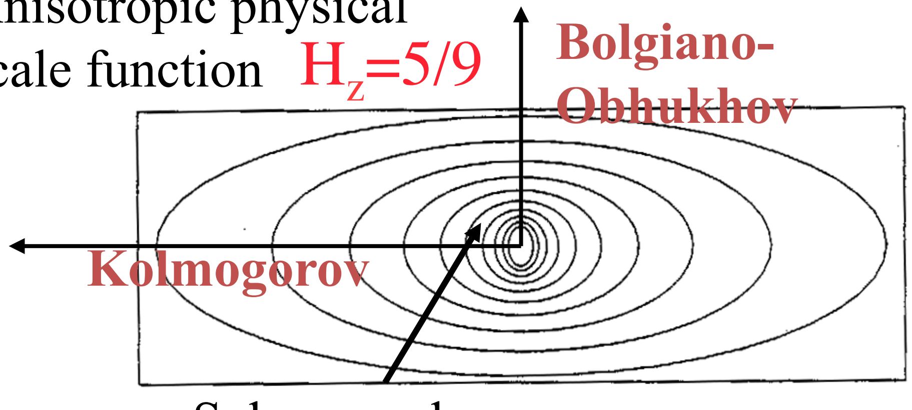
$$G=\begin{pmatrix}1 & 0\\ 0 & H_z\end{pmatrix}$$

Vertical sections

Isotropic function $H_z=1$

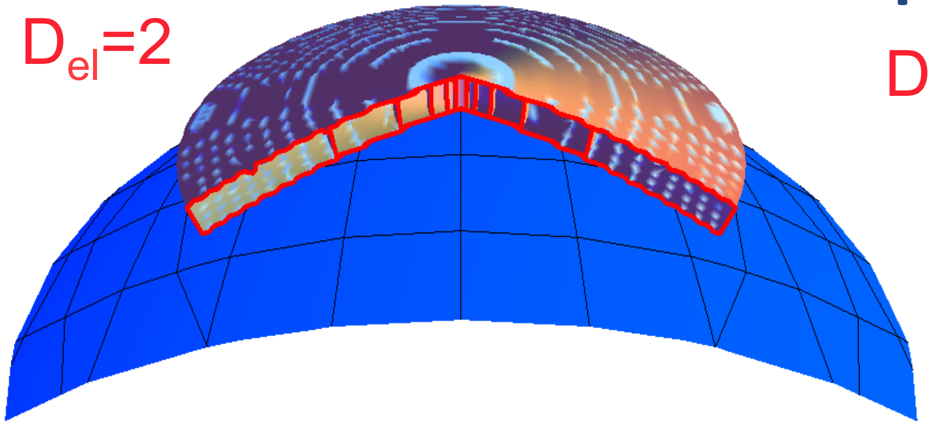


Anisotropic physical scale function $H_z=5/9$

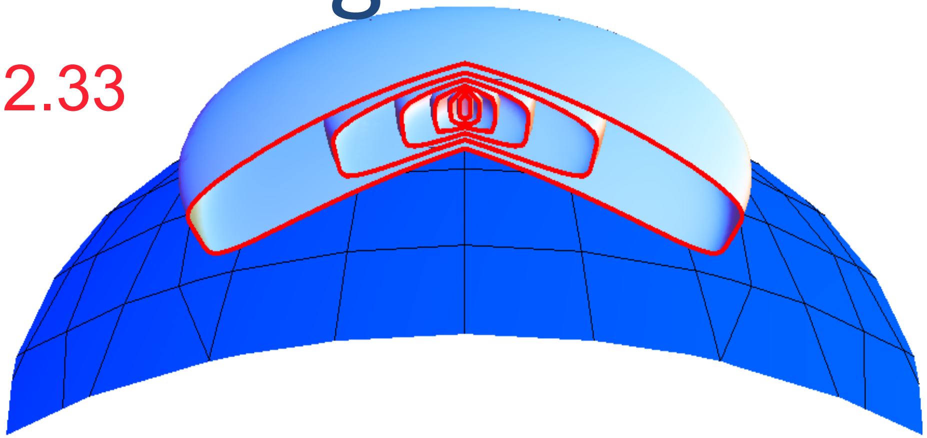


Anisotropic Scaling

D_{el}=2

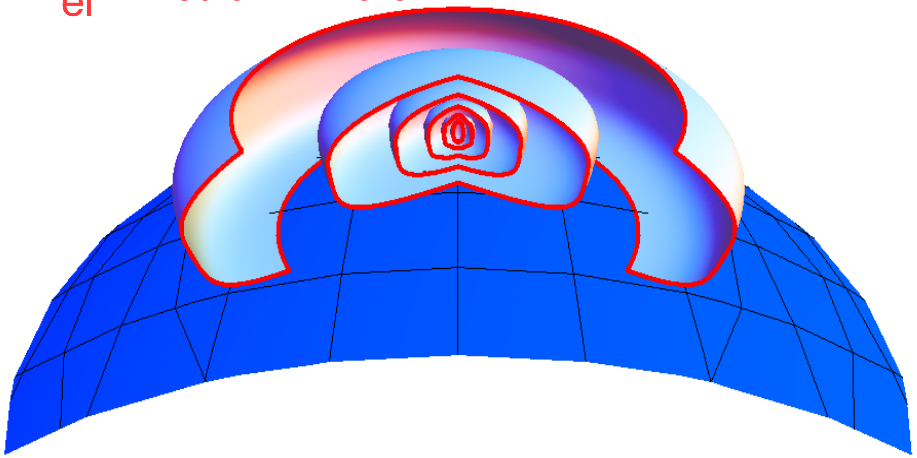


D_{el}=2.33

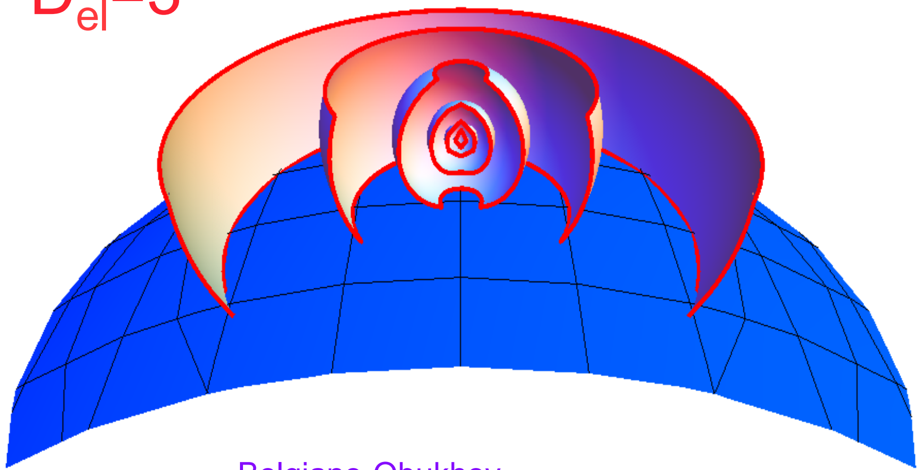


D_{el}=23/9=2.55

c.f. empirical: 2.57



D_{el}=3



The 23/9D model:

Bolgiano-Obukhov

$$\Delta v(\Delta x) = \varepsilon^{1/3}\Delta x^{1/3}; \quad \Delta v(\Delta z) = \phi^{1/5}\Delta z^{3/5}$$

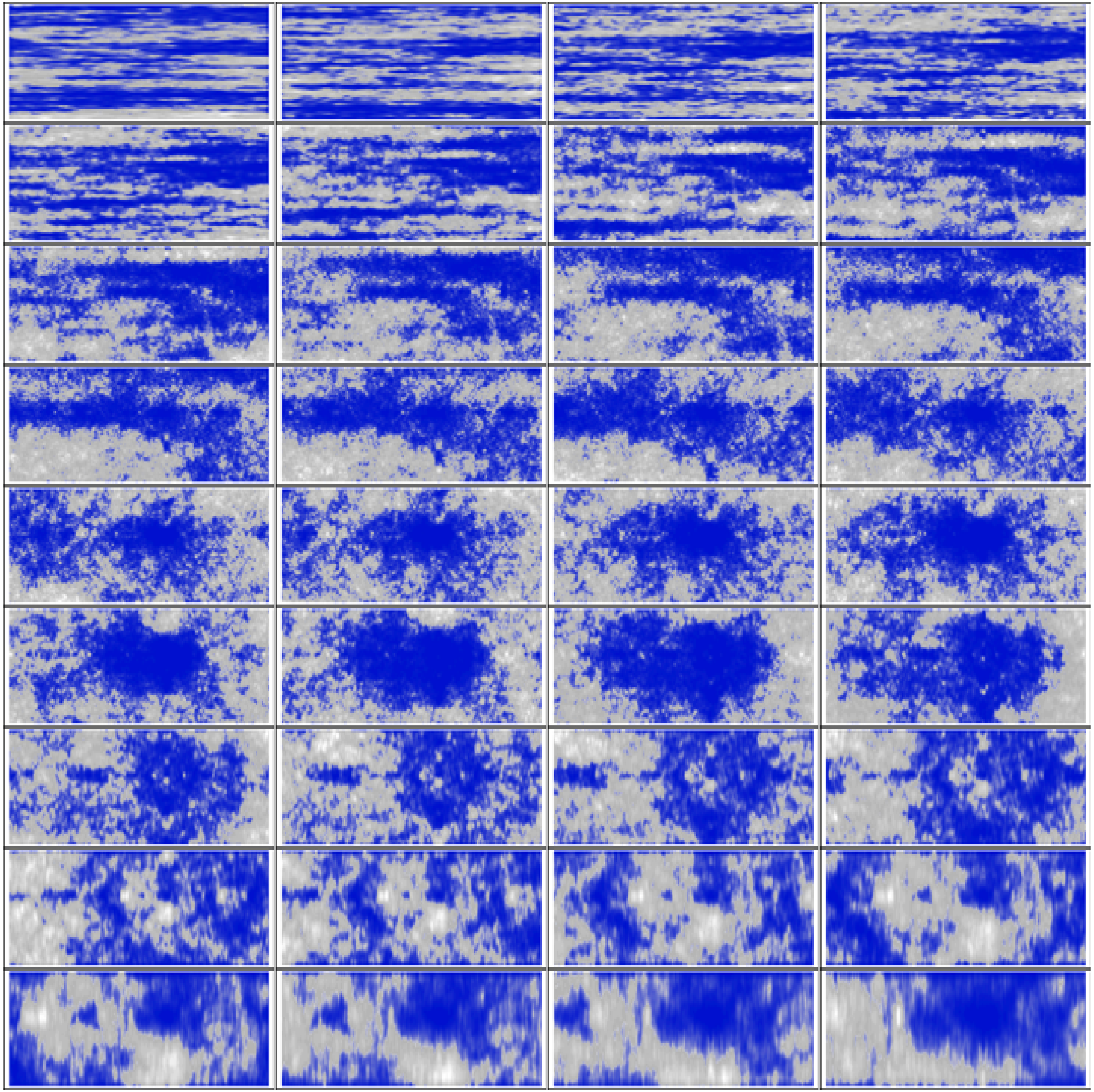
Kolmogorov

H_z=(1/3)/(3/5)=5/9

Volume≈LxLxL^{Hz}≈L^{Del}

D_{el}=2+H_z=23/9

Zoom factor 1000



Vertical cross-section