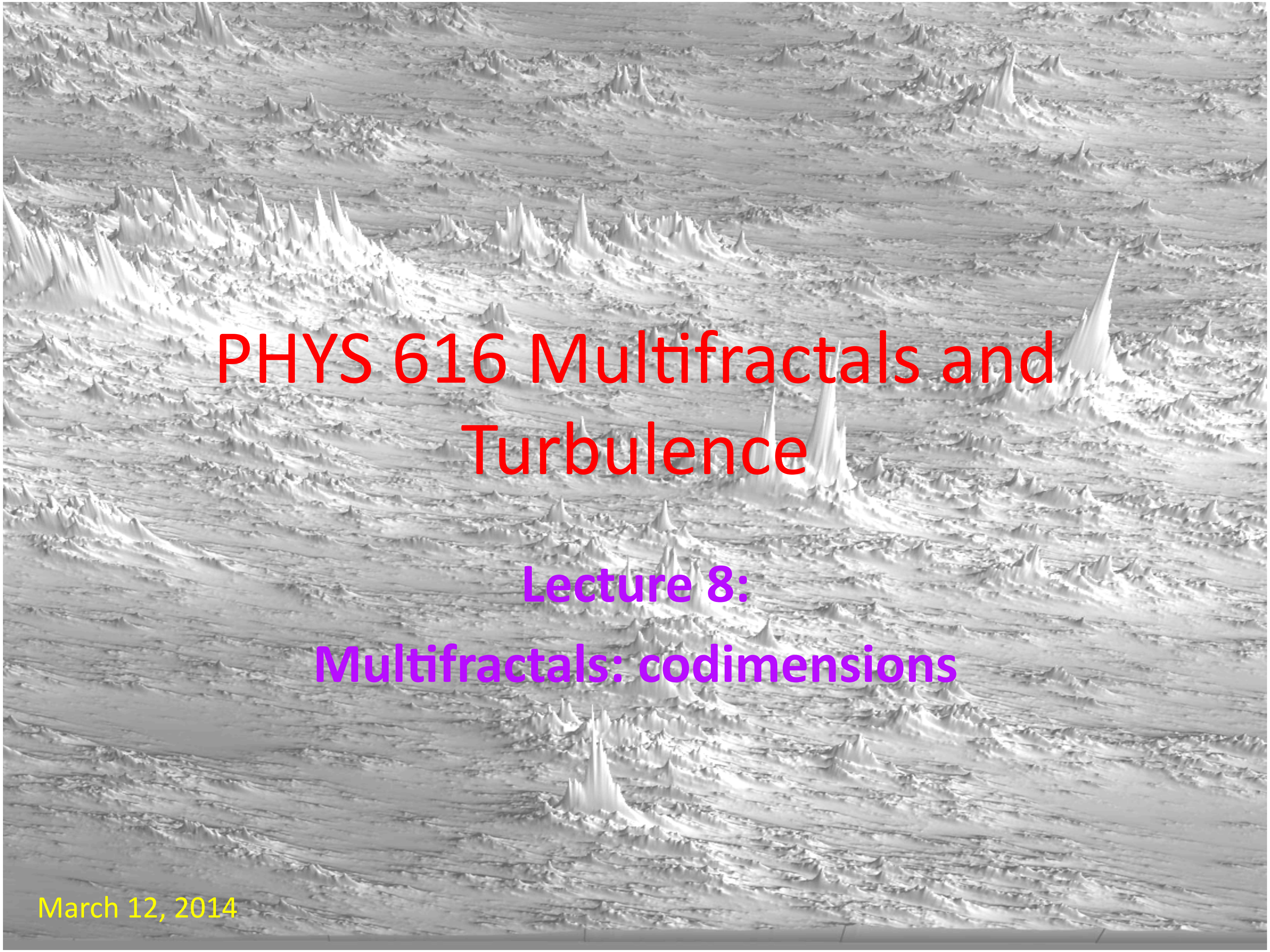


PHYS 616 Multifractals and Turbulence

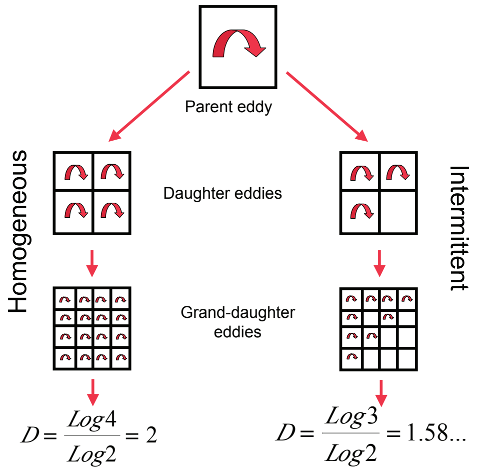
Lecture 8:

Multifractals: codimensions

March 12, 2014

Probabilities and codimensions

Revisiting the β Model



Recall the β model has one parameter $c>0$ and that two states specifally the statistics of the multipliers $\mu\varepsilon$:

$$\Pr(\mu\varepsilon = \lambda_0^c) = \lambda_0^{-c} \qquad \text{(alive)}$$

$$\Pr(\mu\varepsilon = 0) = 1 - \lambda_0^{-c} \qquad \text{(dead)}$$

where λ_0 is the single step (integer) scale ratio. Recall that the magnitude of the boost $\mu\varepsilon = \lambda_0^c$ >1 is chosen so that at each cascade step the ensemble averaged ε is conserved:

$$\langle \mu\varepsilon \rangle = 1 \;\Leftrightarrow\; \langle \varepsilon_n \rangle = \langle \varepsilon_0 \rangle$$

Indeed at each step in the cascade the fraction of the alive eddies decreases by the factor $\beta = \lambda_0^{-c}$ (hence the name "β model") and conversely their energy flux density is increased by the factor $1/\beta$ to assure (average) conservation.

n steps β model

After n steps, the effect of the single-step dichotomy of "dead" or "alive" is amplified by the total (n step) scale ratio $\lambda = \lambda_0^n$:

$$\Pr\left(\varepsilon_n = \left(\lambda_0^n\right)^c = \lambda^c\right) = \left(\lambda_0^n\right)^{-c} = \lambda^{-c} \qquad \text{(alive)}$$

$$\Pr(\varepsilon_n = 0) = 1 - \left(\lambda_0^n\right)^c = 1 - \lambda^{-c} \qquad \text{(dead)}$$

hence either the density diverges ε_n with an (algebraic) order of singularity c, but with an (algebraically) decreasing probability, or is "calmed" down to zero.
After n steps the average number of alive eddies in the β- model is:

$$\langle N_n \rangle = \lambda^d \Pr\left(\varepsilon_\lambda = \lambda^c\right) = \lambda^{d-c}$$

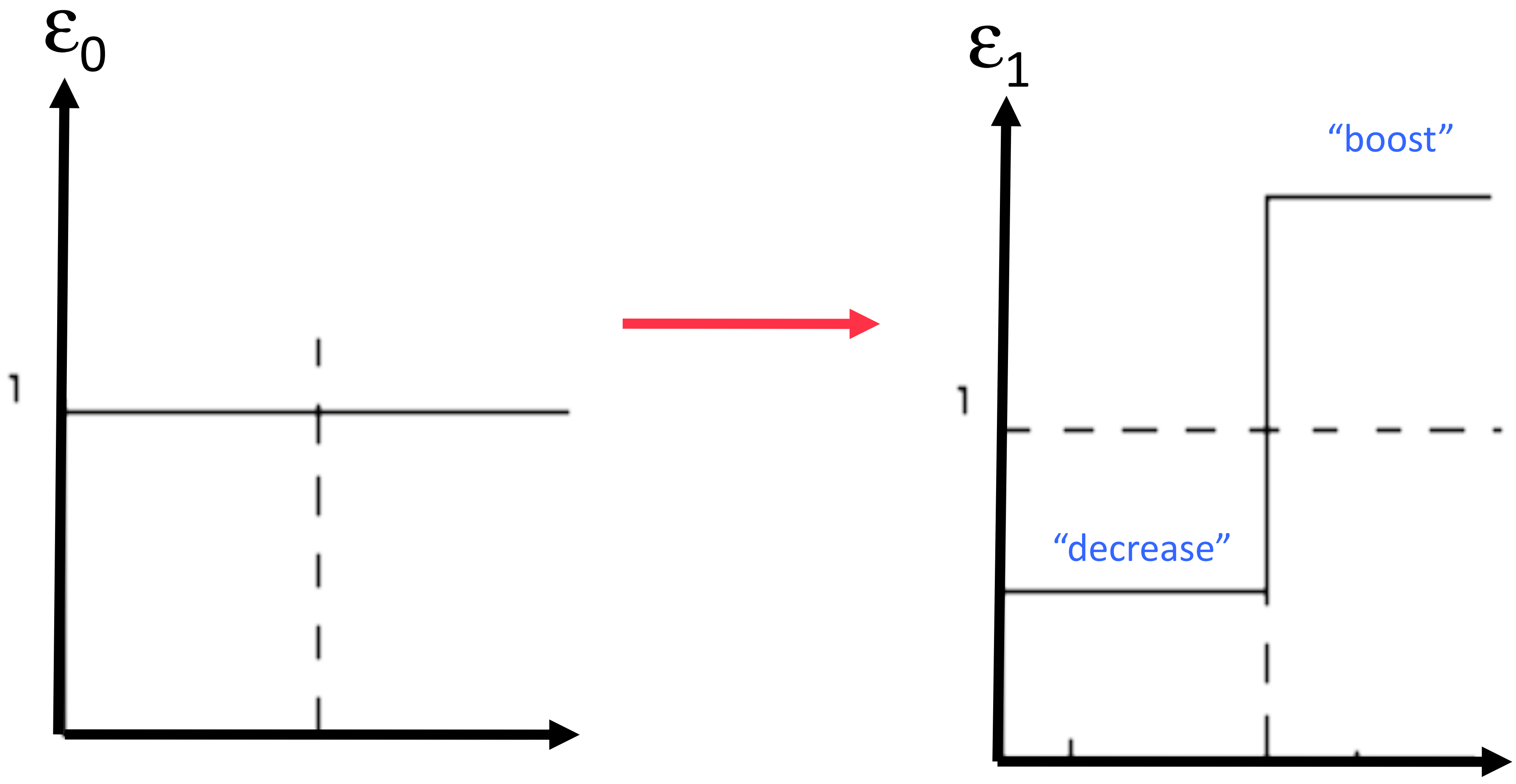
This is the number of λ^{-1} sized boxes" needed to cover the alive regions, hence their corresponding (box counting, fractal) dimension D is:

$$D = d - c$$

Number as function of scale

(d is the dimension of the embedding space) so that the statiical codimension c is indeed equal to the geometric codimension (as long as $D \geq 0$). This is the dimension of the "support" of turbulence, corresponding to the fact that

Revisiting the α Model



Revisiting the α Model

The α model which more realistically allows eddies to be either "more active" or "less active" according to the following binomial process:

$$\Pr(\mu\varepsilon = \lambda_0^{\gamma_+}) = \lambda_0^{-c} \qquad (>1 \;\Rightarrow\; \text{INCREASE})$$

$$\Pr(\mu\varepsilon = \lambda_0^{\gamma_-}) = 1 - \lambda_0^{-c} \qquad (<1 \;\Rightarrow\; \text{DECREASE})$$

Conservation constraint

where γ_+, γ_- correspond to boosts and decreases respectively, the β model being the special case where $\gamma_- = -\infty$ and $\gamma_+ = c$ (due to conservation $<\mu\varepsilon> = 1$, there are only two free parameters):

$$\lambda_0^{\gamma_+ - c} + \lambda_0^{\gamma_-}\left(1 - \lambda_0^{-c}\right) = 1$$

Taking $\gamma_- > -\infty$, the pure orders of singularity $\gamma_-$and γ_+ lead to the appearance of mixed orders of singularity, of different orders $\gamma\ (\gamma_- \le \gamma \le \gamma_+)$. These are built up step by step through a complex succession of γ_- and γ_+, values.

α Model after 2 steps

What is the behaviour as the number of cascade steps, $n \to \infty$? Consider two steps of the process, the various probabilities and random factors are:

$$\Pr(\mu\varepsilon = \lambda_0^{2\gamma_+}) = \lambda_0^{-2c} \qquad \text{(two boosts)}$$

Two steps: an equivalent 3 state model with $\lambda = \lambda_0^2$

$$\Pr(\mu\varepsilon = \lambda_0^{\gamma_+ + \gamma_-}) = 2\lambda_0^{-c}(1-\lambda_0^{-c}) \qquad \text{(one boost and one decrease)}$$

$$\Pr(\mu\varepsilon = \lambda_0^{2\gamma_-}) = (1-\lambda_0^{-c})^2 \qquad \text{(two decreases)}$$

Rewriting:

$$\Pr\left(\mu\varepsilon = \left(\lambda_0^2\right)^{\gamma_+}\right) = \left(\lambda_0^2\right)^{-c} \qquad \text{(one large)}$$

$$\Pr\left(\mu\varepsilon = \left(\lambda_0^2\right)^{(\gamma_+ + \gamma_-)/2}\right) = 2\left(\lambda_0^2\right)^{-c/2} - 2\left(\lambda_0^2\right)^{-c} \qquad \text{(intermediate)}$$

$$\Pr\left(\mu\varepsilon = \left(\lambda_0^2\right)^{\gamma_-}\right) = 1 - 2\left(\lambda_0^2\right)^{-c/2} + \left(\lambda_0^2\right)^{-c} \qquad \text{(large decrease)}$$

α Model after n steps

Iterating this procedure, after $n = n_+ + n_-$ steps we find:

$$\gamma_{n_+,n_-} = \frac{n_+\gamma_+ + n_-\gamma_-}{n_+ + n_-} \quad , \qquad n_+ = 1,\ldots,n$$

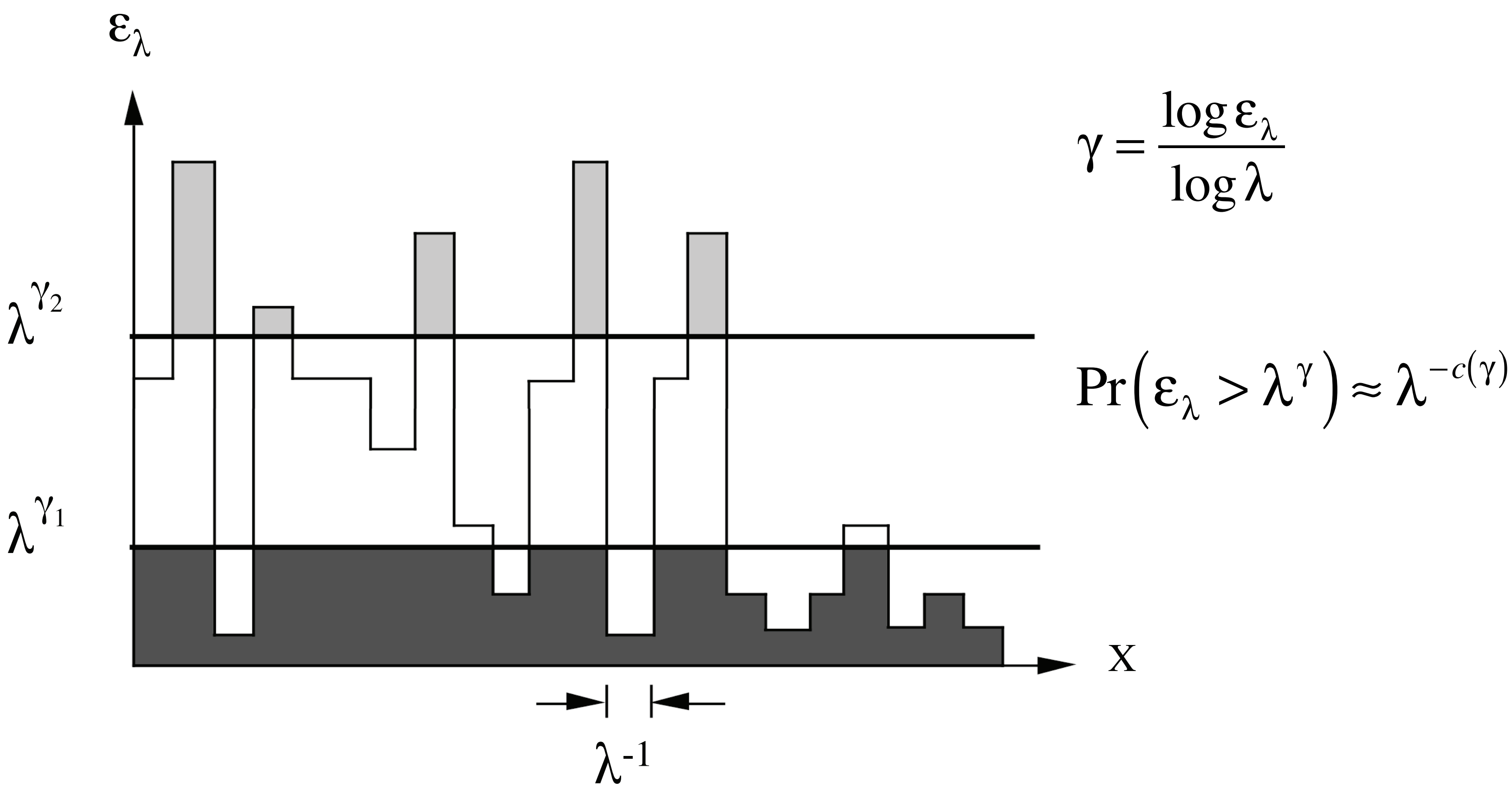
$$\Pr\left(\mu\varepsilon = \left(\lambda_0^n\right)^{\gamma_{n_+,n_-}}\right) = \binom{n}{n_+}\left(\lambda_0^n\right)^{-cn_+/n}\left(1-\left(\lambda_0^n\right)^{-c/n}\right)^{n_-}$$

where $\binom{n}{n_+}$ is the number of combinations of n objects taken n^+ at a time. This implies that we may write:

$$\Pr\left(\varepsilon_{\lambda_0^n} \ge \left(\lambda_0^n\right)^{\gamma_i}\right) = \sum_j p_{i,j}\left(\lambda_0^n\right)^{-c_{i,j}}$$

The p_{ij}'s are the "submultiplicities" (the prefactors in the above), c_{ij} are the corresponding exponents ("subcodimensions") and λ_0^n is the total ratio of scales from the outer scale to the smallest scale. Notice that the requirement that $\langle\mu\varepsilon\rangle = 1$ implies that some of the λ^{γ_i} are >1.

Values and singularities

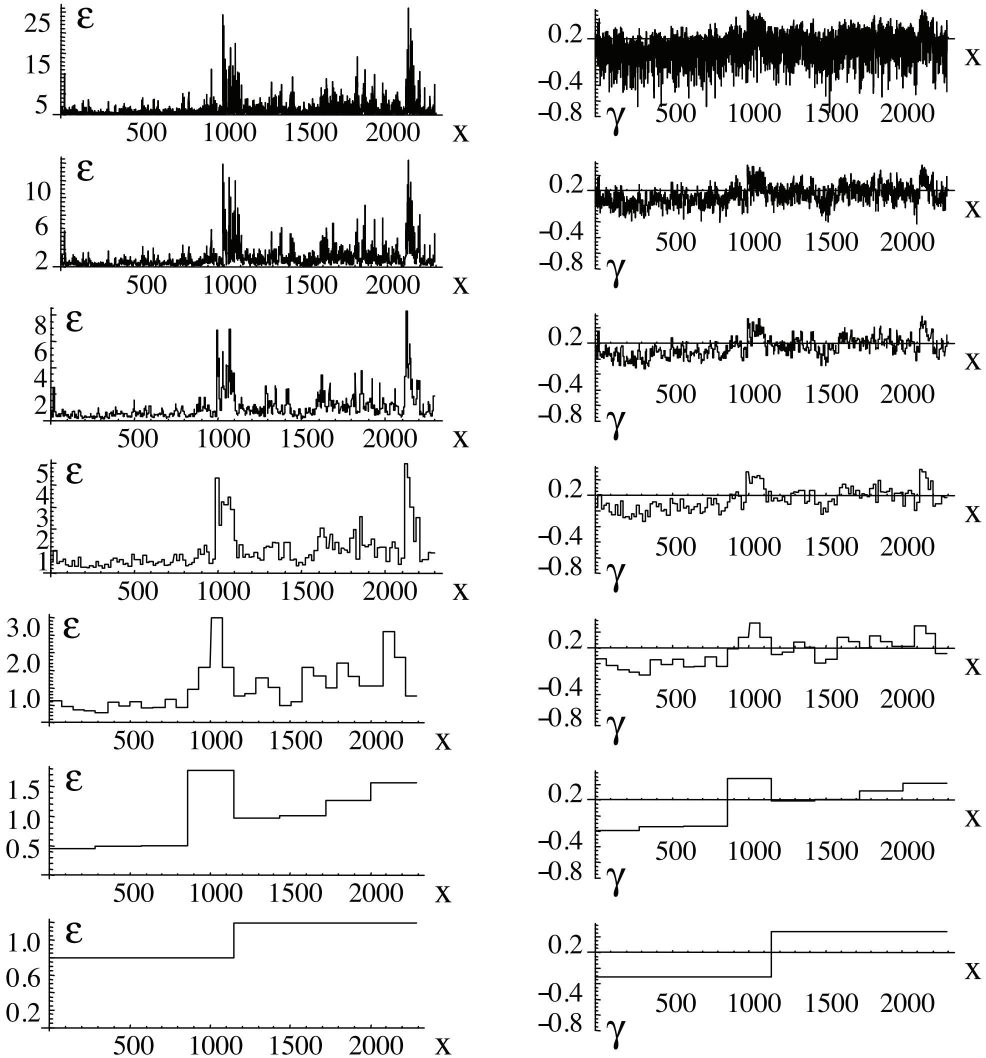


A schematic illustration of a multifractal field analyzed over a scale ratio λ, with two scaling thresholds λ^{γ_1} and λ^{γ_2}. , corresponding to two orders of singularity : $\gamma_2 > \gamma_1$.

Removing the sale dependency of the flux: γ

Example with aircraft data:

Notice that the range of γ's is nearly constant



$$\gamma = \frac{\log \varepsilon_\lambda}{\log \lambda}$$

The Codimension Multifractal Formalism

Codimension of Singularities $c(\gamma)$ and its relation to $K(q)$

We now derive the basic connection between $c(\gamma)$ and the moment scaling exponent $K(q)$. To relate the two; write the expression for the moments in terms of the probability density of the singularities:

$$p(\gamma) = \left|\frac{d\Pr}{d\gamma}\right| \sim c'(\gamma)(\log\lambda)\lambda^{-c(\gamma)} \sim \lambda^{-c(\gamma)}$$

Relation probability density and distribution:

$$\Pr(\gamma' > \gamma) = \int_{\gamma}^{\infty} p(\gamma')d\gamma'$$

(where we have absorbed the $c'(\gamma)\log\lambda$ factor into the "~" symbol since it is slowly varying, subexponential). This yields:

$$\langle\varepsilon_\lambda^q\rangle = \int d\Pr(\varepsilon_\lambda)\varepsilon_\lambda^q \approx \int d\gamma\lambda^{-c(\gamma)}\lambda^{q\gamma}$$

$$\Pr(\varepsilon_\lambda > \lambda^\gamma) = \int_{\lambda^\gamma}^{\infty} p(\varepsilon_\lambda)d\varepsilon_\lambda$$

where we have used $\varepsilon_\lambda = \lambda^\gamma$ (this is just a change of variables ε_λ for γ, λ is a fixed parameter). Hence:

$$\langle\varepsilon_\lambda^q\rangle = \lambda^{K(q)} = e^{K(q)\log\lambda} \approx \int_{-\infty}^{\infty} d\gamma e^{\xi f(\gamma)}; \quad \xi = \log\lambda; \quad f(\gamma) = q\gamma - c(\gamma); \quad \lambda >> 1$$

Legendre transform

We see that our problem is to obtain an asymptotic expansion of an integral with integrand of the form exp(ξ $f(\gamma)$) where ξ = logλ is a large parameter and $f(\gamma) = q\gamma - c(\gamma)$. These expansions can be conveniently preformed using the mathematical technique of "steepest descents" e.g. (Bleistein and Handelsman, 1986) which shows the integral is dominated by the singularity γ which yields the maximum value of the exponent) so that as long as $\xi = \log\lambda >> 1$:

$$\int_{-\infty}^{\infty} e^{\xi f(\gamma)}\, d\gamma \approx e^{\xi \max_{\gamma}(f(\gamma))}$$

so that:

$$\langle \varepsilon_{\lambda}^{q} \rangle = e^{\xi K(q)} \approx e^{\xi \max_{\gamma}(q\gamma - c(\gamma))}; \quad \xi = \log\lambda$$

hence:

$$\boxed{K(q) = \max_{\gamma}(q\gamma - c(\gamma))}$$

Legendre transform

This relation between $K(q)$ and $c(\gamma)$ is called a "Legendre transform" (Parisi and Frisch, 1985).

Inverse Legendre transform: $c(\gamma)$

We can also invert the relation to obtain $c(\gamma)$ from $K(q)$; just as the inverse Laplace transform used to obtain $K(q)$ from $c(\gamma)$ is another Laplace transform so the inverse Legendre transform is just another Legendre transform. To show this, consider the twice iterated Legendre transform $F(q)$ of $K(q)$:

$$F(q) = max_{\gamma}\{q\gamma - (max_{q'}\{q'\gamma - K(q')\})\} = max_{\gamma;q'}\{\gamma(q-q') + K(q')\}$$

Taking $\partial F/\partial\gamma = 0 \Rightarrow q = q'$ so that we see that $F(q) = K(q)$. This shows that a Legendre transform is equal to its inverse, hence we conclude:

$$\boxed{c(\gamma) = \max_{q}\left(q\gamma - K(q)\right)}$$

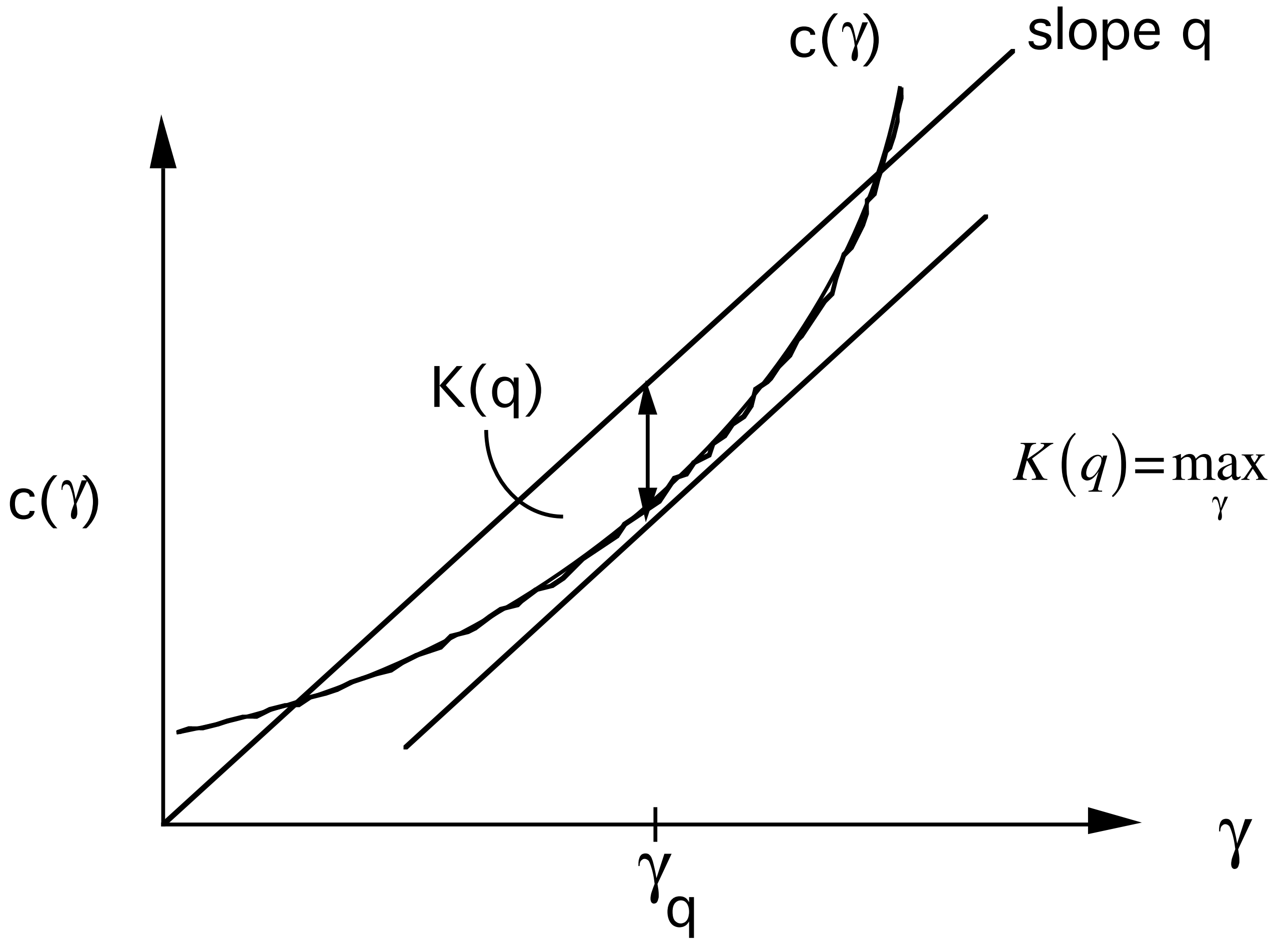
Legendre transform

The γ which for a given q maximizes $q\gamma$ - $c(\gamma)$ is γ_q and is the solution of $c'(\gamma_q) = q$. Similarly, the value of q which for given γ maximizes $q\gamma$- $K(q)$ is q_γ so that:

$$\boxed{\begin{aligned} q_\gamma &= c'(\gamma) \\ \gamma_q &= K'(q) \end{aligned}}$$

This is a one-to-one correspondence between moments and orders of singularities.

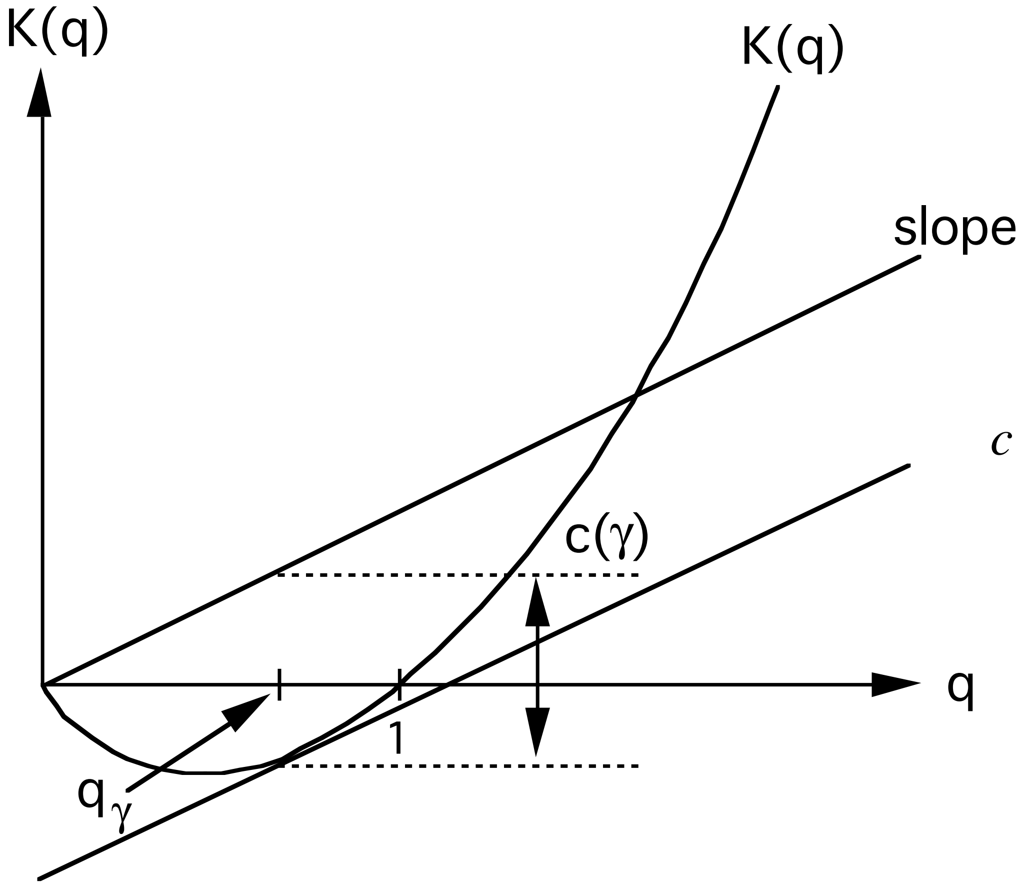
Graphical Legendre transform



$$K(q) = \max_{\gamma}\left(q\gamma - c(\gamma)\right)$$

$c(\gamma)$ versus showing the tangent line $c'(\gamma_q) = q$ with the corresponding chord . Note that the equation is the same as $\gamma_q = K'(q)$.

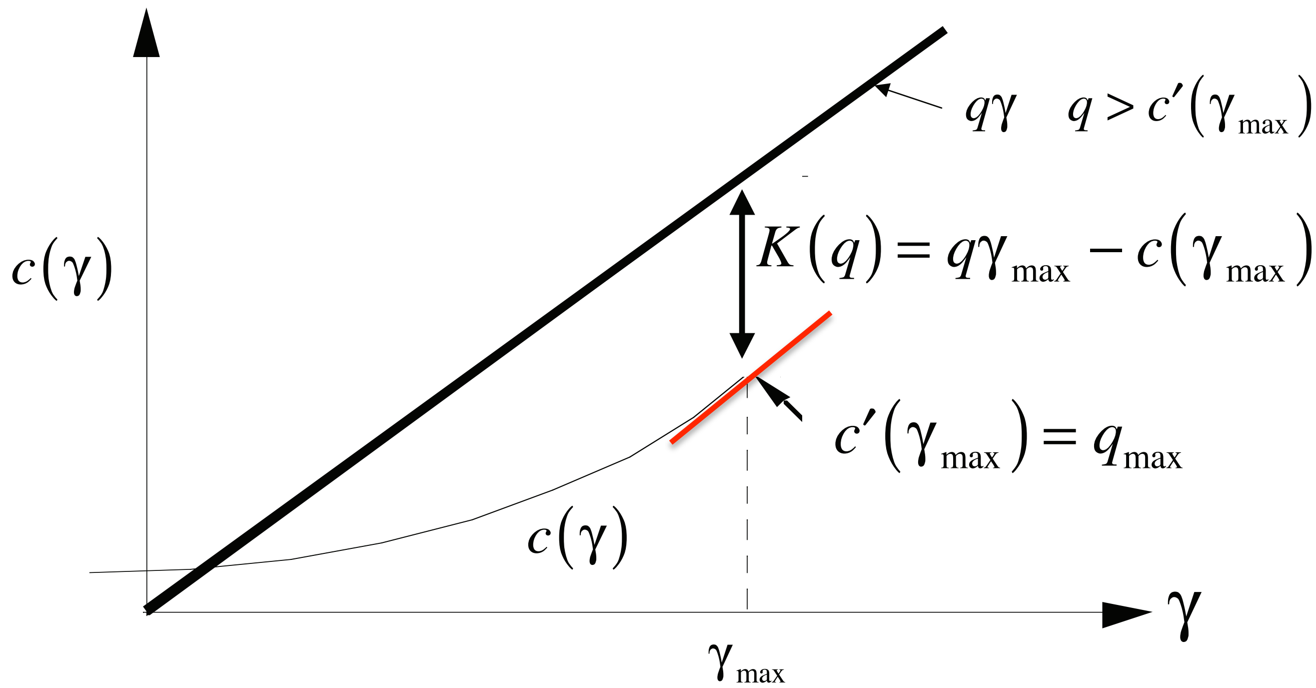
Graphical Legendre transform



$$c(\gamma) = \max_q (q\gamma - K(q))$$

$K(q)$ versus q showing the tangent line $K'(q_g) = \gamma$ with the corresponding chord .

Leendre transform bounded singularities



Using the Legendre transformation to derive $K(q)$ via a Legendre transformation when the maximum order of singularity present (γ_{max}; corresponding moment $q_{max} = c'(\gamma_{max})$) When $q>q_{max}$ the Legendre transform will have a maximum value for $\gamma = \gamma_{\max}$ as shown, this implies $K(q)$ is linear for $q>q_{max}$.

Note that if γ is bounded by γ_{max} (for example in microcanonical cascades, $\gamma \leq d$ or for the α model; $\gamma \leq \gamma_+$) there is a $q_{max} = c'(\gamma_{max})$ such that for $q>q_{max}$, $K(q) = q\gamma_{max} - c(\gamma_{max})$, i.e. $K(q)$ becomes linear in q.

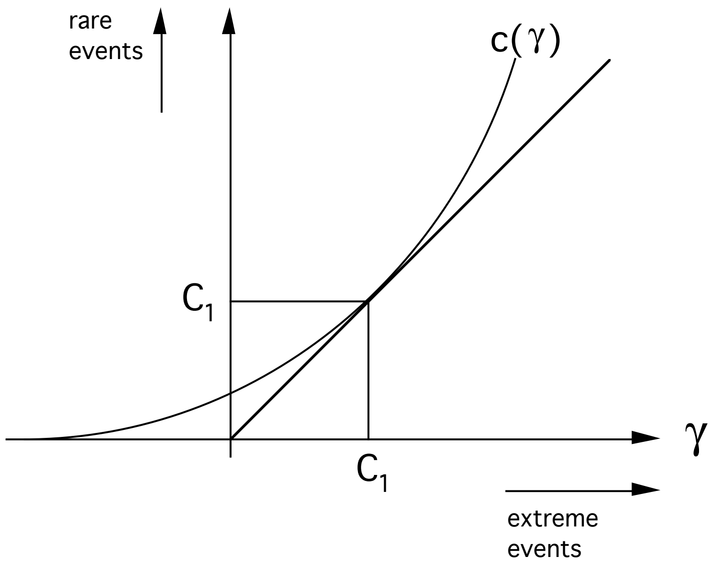
Properties of codimension functions

$c(\gamma)$ is the statistical scaling exponent characterizing how its probability changes with scale.

1) The first obvious property is that due to its very definition $c(\gamma)$ is an increasing function of γ: $c'(\gamma)>0$.

2) Another fundamental property which follows directly from the Legendre relation with $K(q)$, is that $c(\gamma)$ must be convex: $c''(\gamma)>0$.

The special properties of the singularity of the mean, C_1



Many properties of the codimension function can be illustrated graphically.
For example, consider the mean, q =1.

1) First, applying $K'(q) = \gamma$ we find $K'(1) = \gamma_1$ where is the singularity giving the dominant contribution to the mean (the q = 1 moment). We have already defined $C_1 = K'(1)$, so that this implies $C_1 = \gamma_1$; the Legendre relation thus justifies the name "codimension of the mean" for C_1.

2) Also at q = 1 we have $K(1) = 0$ (due to the scale by scale conservation of the flux) so that $C_1 = c(C_1)$ (this is a fixed point relation). C_1 is thus simultaneously the codimension of the mean of the process and the order of singularity giving the dominant contribution to the mean.

3) Finally, applying $c'(\gamma) = q$ we obtain $c'(C_1) = 1$ so that the curve $c(\gamma)$ is also tangent to the line $x = y$ (the bisectrix). If the process is observed on a space of dimension d, it must satisfy $d \geq C_1$, otherwise, following the above, the mean will be so sparse that the process will (almost surely) be zero everywhere; it will be "degenerate". We will see that when $C_1 > d$ that the ensemble mean of the spatial averages (the dressed mean) cannot converge.

Multifractality near $\gamma = C_1$

Finally, since $c(\gamma)$ is convex with fixed point C_1, it is possible to define the degree of multifractality (α) by the (local) rate of change of slope at C_1, (the singularity corresponding to the mean) its radius of curvature $R_c(C_1)$ is:

$$R_c(C_1) = \frac{\left(1 + c'(C_1)\right)^{3/2}}{c''(C_1)}$$

Using the general relation $c'(C_1)=1$ we obtain $R_c(C_1) = 2^{3/2} / c''(C_1)$ hence we can locally (near the mean $\gamma = C_1$) define a curvature parameter α from either of the equivalent relations:

$$\alpha = \frac{2^{3/2} R_c(C_1)}{C_1} = \frac{1}{C_1 c''(C_1)}$$

These local ($q = 1$, $\gamma = C_1$) definitions of α are equivalent to the definition via moments $\alpha = K''(1)/K'(1)$.

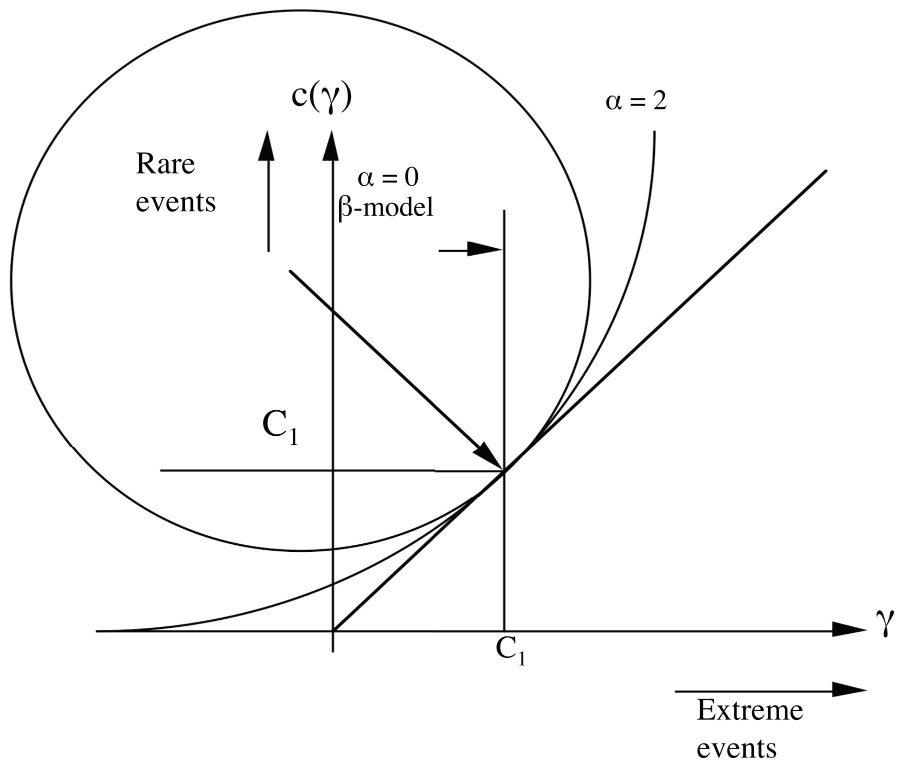
Demonstration

$$q = c'(\gamma) \Rightarrow c''(\gamma) = \frac{dq}{d\gamma}$$

$$\gamma = K'(q) \Rightarrow K''(q) = \frac{d\gamma}{dq}$$

$$K''(q) = \left(\frac{dq}{d\gamma}\right)^{-1} = \frac{1}{c''(\gamma)}$$

$$K''(1) = C_1\alpha = \frac{1}{c''(C_1)}$$



A schematic illustration showing how the $c(\gamma)$ curve can be locally characterized near the mean singularity C_1.

The sampling dimension, sampling singularity and second-order multifractal phase transitions

Sampling dimension

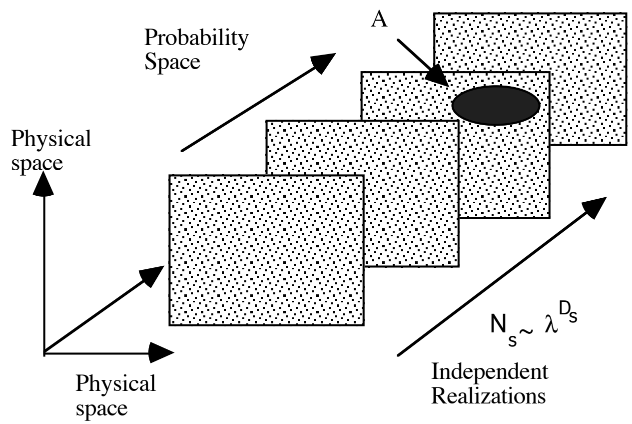
$$D_s = \frac{\log N_s}{\log \lambda}$$

"effective dimension" of a sample of N_s samples, each scale ratio λ

Sampling singularity

$$\gamma_s = \frac{\log \varepsilon_{\lambda,s}}{\log \lambda}$$

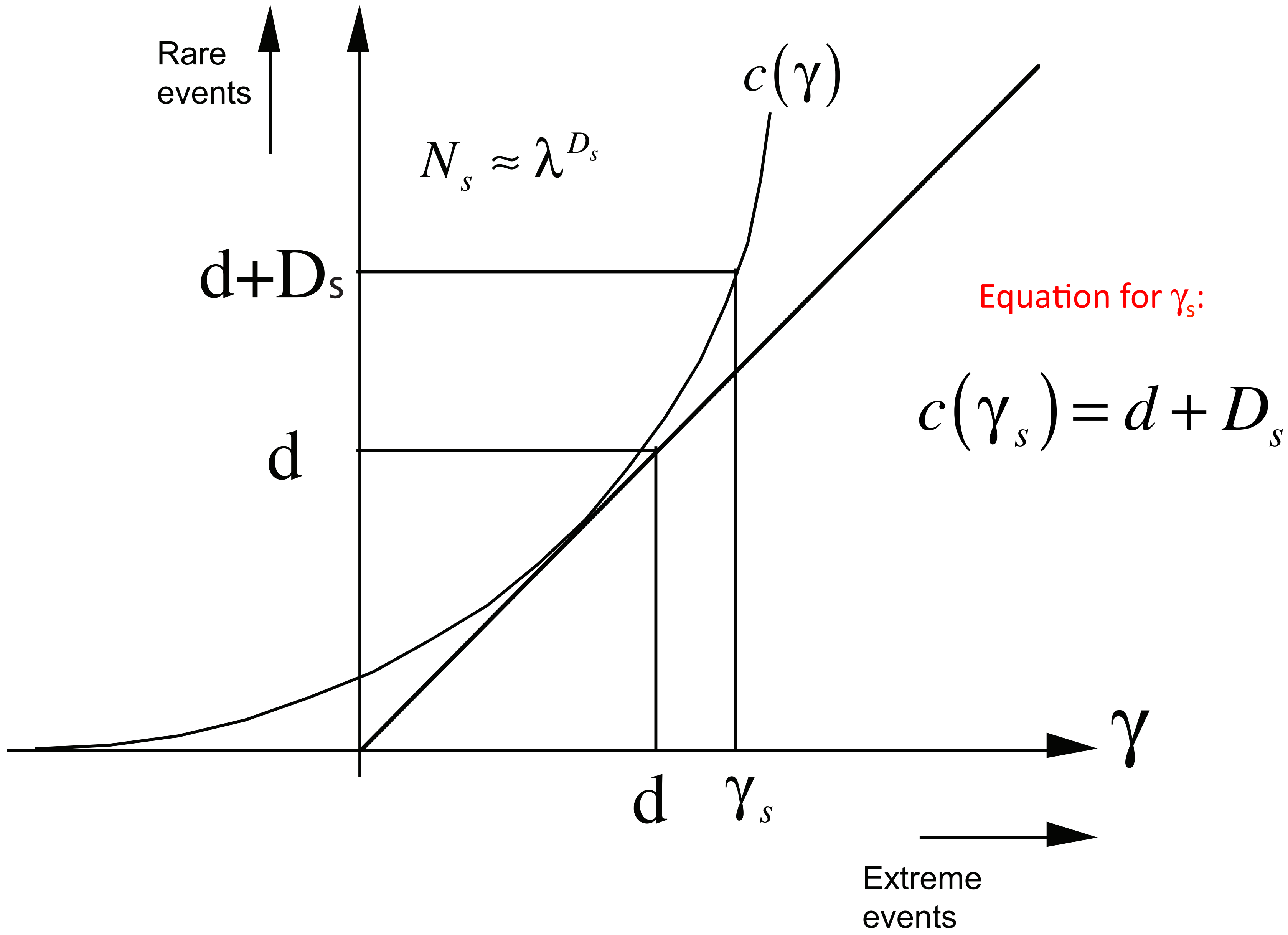
largest singularity γ_s, largest value $\varepsilon_{\lambda,s}$



In order to relate γ_s and D_s, consider a collection of satellite images ($d = 2$). Our question is thus: what is the rarest event with the most extreme γ_s that we may expect to see on a single picture? On a large enough collection of pictures? The answer to, these questions is straightforward: there are a total of λ^{d+D_s} pixels in the sample; hence the rarest event has a probability $\approx \lambda^{-(d+D_s)}$. However the probability of finding γ_s is simply $\lambda^{-c(\gamma_s)}$ so that we obtain the following implicit equation for γ_s:

$$c(\gamma_s) = d + D_s = \Delta_s$$

$\Delta_s = d+D_s$ is the corresponding (overall) effective dimension of our sample. More extreme singularities would have codimensions greater than this effective dimension $c > \Delta_s$ and are almost surely not present in our sample.



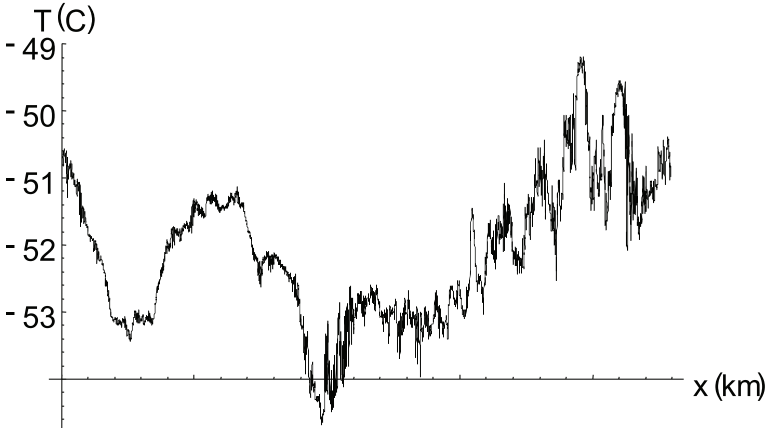
Schematic illustration of sampling dimension and how it imposes a maximum order of singularities γ_s.

Example of D_s, γ_s estimate

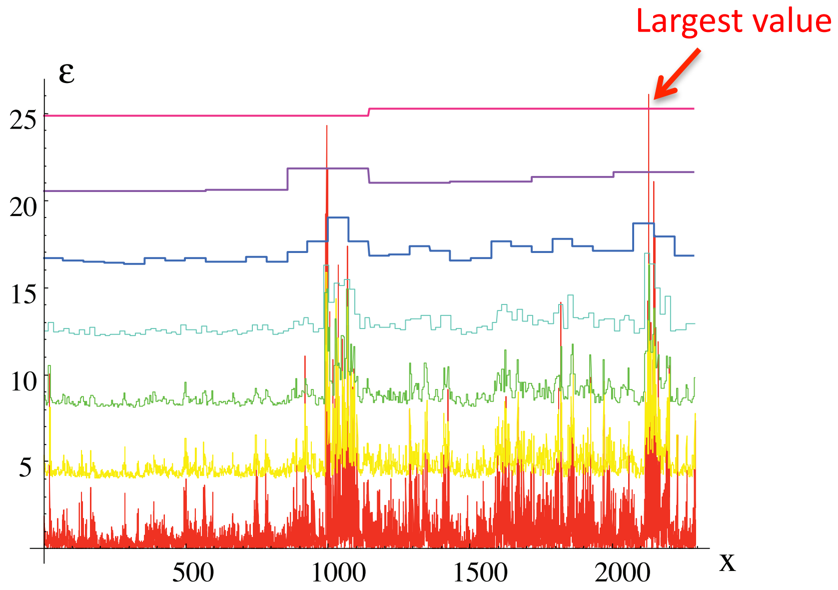
24 flight legs 400 points (1120 *km)* long, with d =1 section:

$D_s = \log 24/\log 4000 = 0.364$

$\Delta_s = D + D_s = 1.364$



Aircraft: 280m resolution, 200 mb (≈12 km)



We can use the aircraft data shown above to estimate the largest singularity that we should expect over a transect 2^{13} points long.

The largest normalized flux value is $\approx \varepsilon_{\lambda,s} = 26.5$

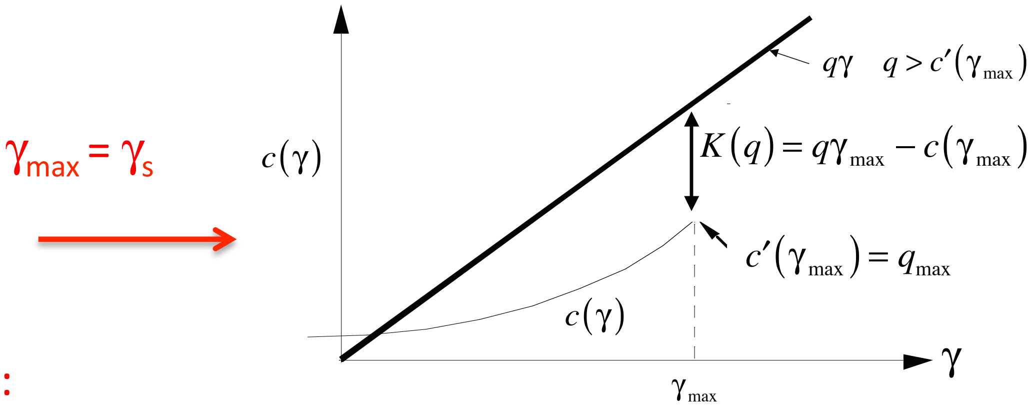
Hence:

$$\gamma_s = \log\varepsilon/\log\lambda = \log(26.5)/\log(2^{13}) = 0.364$$

Using the estimated multifractal parameters $\alpha = 1.8$, $C_1 = 0.06$ (these are mean C_1, α values for that the solution of $c(\gamma_s) = \Delta_s = 1.364$ is $\gamma_s = 0.396$ which is very close to the observed maximum.

Second order phase transition

Here, $\gamma_{max} = \gamma_s$



The effect of finite sample size:

Let us now calculate the moment exponent $K_s(q)$ for a process with N_s realizations. To do this, we calculate the Legendre transform of $c(\gamma)$ but with the restriction $\gamma \leq \gamma_s$; this is the same type of restriction as discussed earlier (take $\gamma_{max} = \gamma_s$):

$$K_s(q) = \gamma_s(q - q_s) + K(q_s)\ , \quad q \geq q_s$$

$$K_s(q) = K(q)\ , \quad q \leq q_s$$

hence at $q = q_s$ there is a jump/discontinuity in the second derivative of K:

$$\Delta K_s'' = -K''(q_s)$$

due to the existence of formal analogies between multifractal processes and classical thermodynamics, this is termed a "second order multifractal phase transition"

Direct empirical estimation of $c(\gamma)$: the probability distribution multiple scaling (PDMS) technique

We saw how to empirically verify the cascade structure and characterize the statistics using the moments, and how to determine their scaling exponent, $K(q)$. In this chapter, we saw how – via a Legendre transform of $K(q)$ - this information can be used to estimate $c(\gamma)$. However, it is of interest to be able to estimate $c(\gamma)$ directly; to do this, we start from the fundamental defining equation, take logs of both sides and rewrite it as follows:

$$Log(\Pr_\lambda(\varepsilon_\lambda > \lambda^\gamma) = -c(\gamma)Log(\lambda) + o(1/Log(\lambda))O(\gamma)$$

$o(1/Log(\lambda))O(\gamma)$ corresponds to the logarithm of the slowly varying factors that are hidden in the "≈" sign and the subscript "λ" on the probability has been added to underline the resolution dependence of the cumulative histograms. For each order of singularity γ, this equation expresses the linearity of log probability with the log of the resolution. The singularity itself must be estimated from the fluxes by:

$$\gamma = \frac{\log(\varepsilon_\lambda)}{\log \lambda}$$

Comments

We now see that things are a little less straightforward than when estimating $K(q)$.

First, the term "$o(1/Log(\lambda))O(\gamma)$ may not be so negligible, in particular for moderate λ's, so that using the simple approximation $c(\gamma) \approx -\log \Pr_\lambda / \log \lambda$ may not be sufficiently accurate.

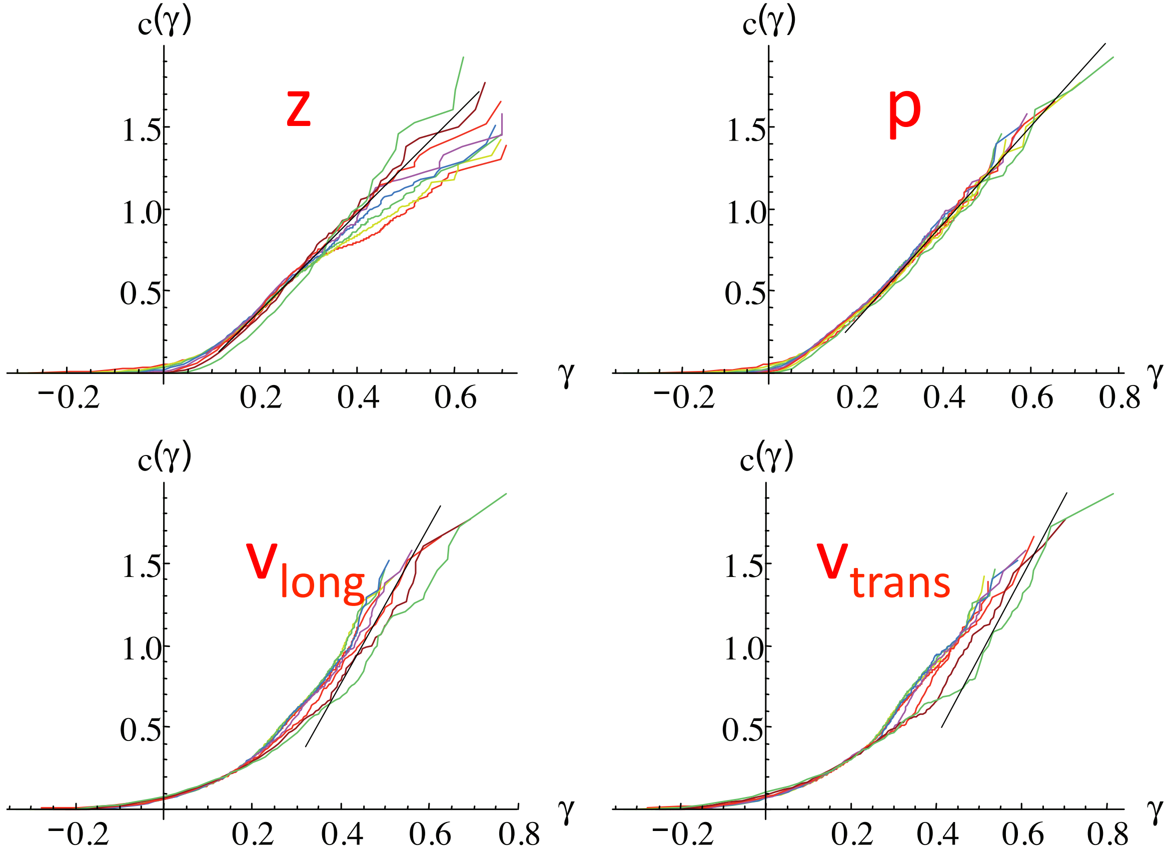
Second, we assumed that ε_λ is normalized such that $<\varepsilon_\lambda> =1$; if it is not, it is can be normalized by dividing by the ensemble mean: $\varepsilon_\lambda \to \varepsilon_\lambda/<\varepsilon_\lambda>$. However from small samples, there may be factors of the order 2 in uncertainty over this so that even the estimate of γ may involve some uncertainty.

In comparison, if one wants to estimate $K(q)$, one needn't worry about either of these issues since (even for the un-normalized ε_λ) the linear relation $\log\langle\varepsilon_\lambda^q\rangle = K(q)\log\lambda$ is exact (at least in the framework of the pure multiplicative cascades): $K(q)$ is simply the slope of the $\log\langle\varepsilon_\lambda^q\rangle$ versus logλ graph (and if the normalization is accurate, the outer scale itself can be estimated from the points where the lines cross). The relative simplicity of the moment method explains why in practice it is the most commonly used. $c(\gamma)$ can then be estimated from $K(q)$ by Legendre transform (either numerically or using a universal multifractal parametrization).

Probability Distribution Multiple Scaling technique

PDMS examples

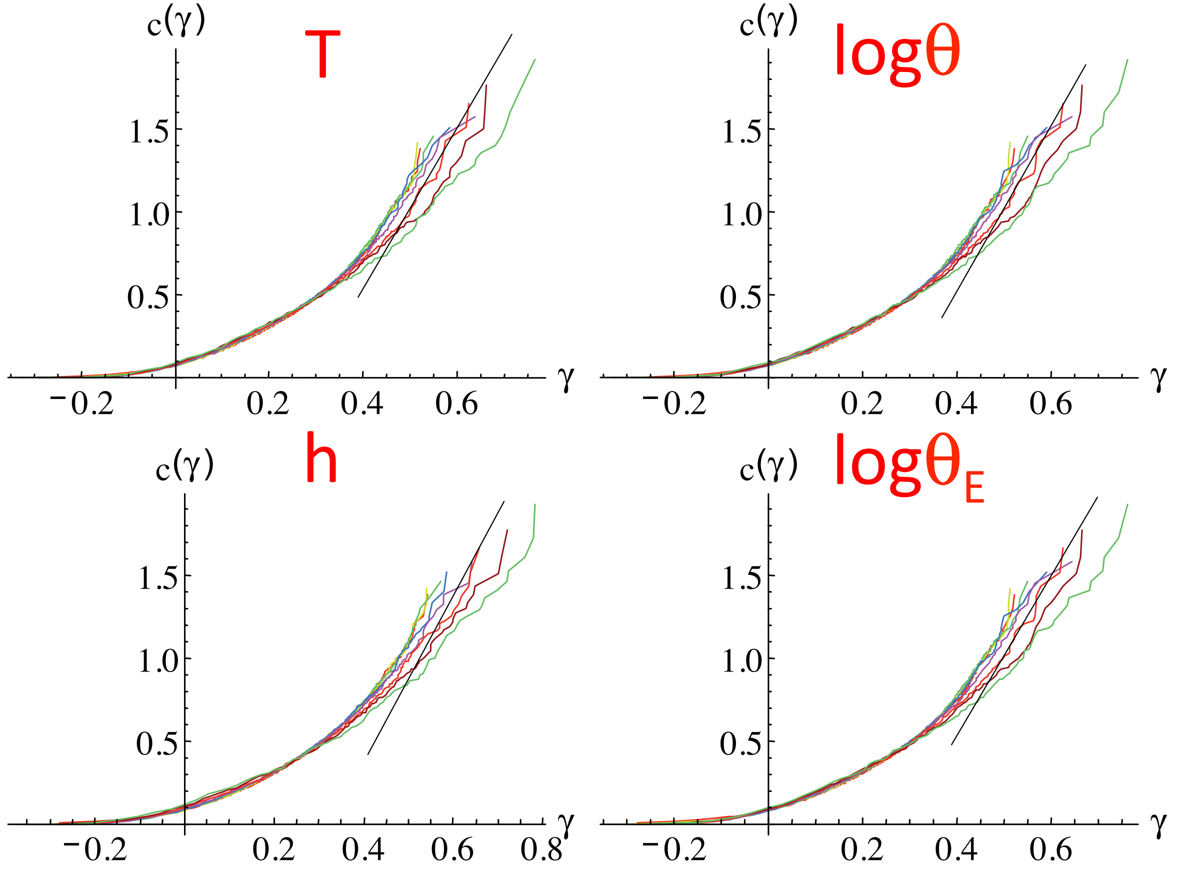
Aircraft at 200mb: 24 flight legs, each 4000 points long, 280 *m* resolution (i.e. 1120 *km*), dynamic variables



$c(\gamma) \approx -\log Pr/\log\lambda$

$c(\gamma)$ estimated from the PDMS method $c(\gamma) \approx -\log Pr/\log\lambda$ described in the text are shown for resolution degraded by factors of 2 from 280 *m* to ≈36 *km* (longest to shortest curves). For reference, lines of slope 3 (top row) and 5 (bottom row) are given corresponding to power law probability distributions with the given exponents.

Thermodyanmic variab les



The reference lines all have slopes of 5

Codimensions of Universal multifractals, cascades

When discussing the moment characterization of the cascades, we have already noted that the two parameters C_1, α are of fundamental significance. C_1 characterizes the order and codimension of the mean singularities of the corresponding conservative flux, it is the local trend of the normalized $K(q)$ near the mean; $K(q) = C_1 (q\text{-}1)$ is the best monofractal "β model approximation near the mean" ($q \approx 1$). Finally, $\alpha = K''(1)/K'(1)$ characterizes the curvature near the mean. The curvature parameter α can also be defined directly from the probability exponent $c(\gamma)$ by using the local radius of curvature $R_c(C_1)$ of $c(\gamma)$ at the point $\gamma = C_1$, i.e. the corresponding singularity. Finally, for the observed field f, there is a third exponent H which characterizes the deviation from conservation of the mean fluctuation $\Delta f \approx \langle\varepsilon\rangle \Delta x^H \approx \Delta x^H$ since $\langle\varepsilon\rangle$ =constant, it is a "fluctuation" exponent.

$$K(q) = \frac{C_1}{\alpha - 1}\left(q^\alpha - q\right); \qquad \alpha \neq 1$$

Universal multifractal K(q)

$$K(q) = C_1 q Log(q); \quad \alpha = 1$$

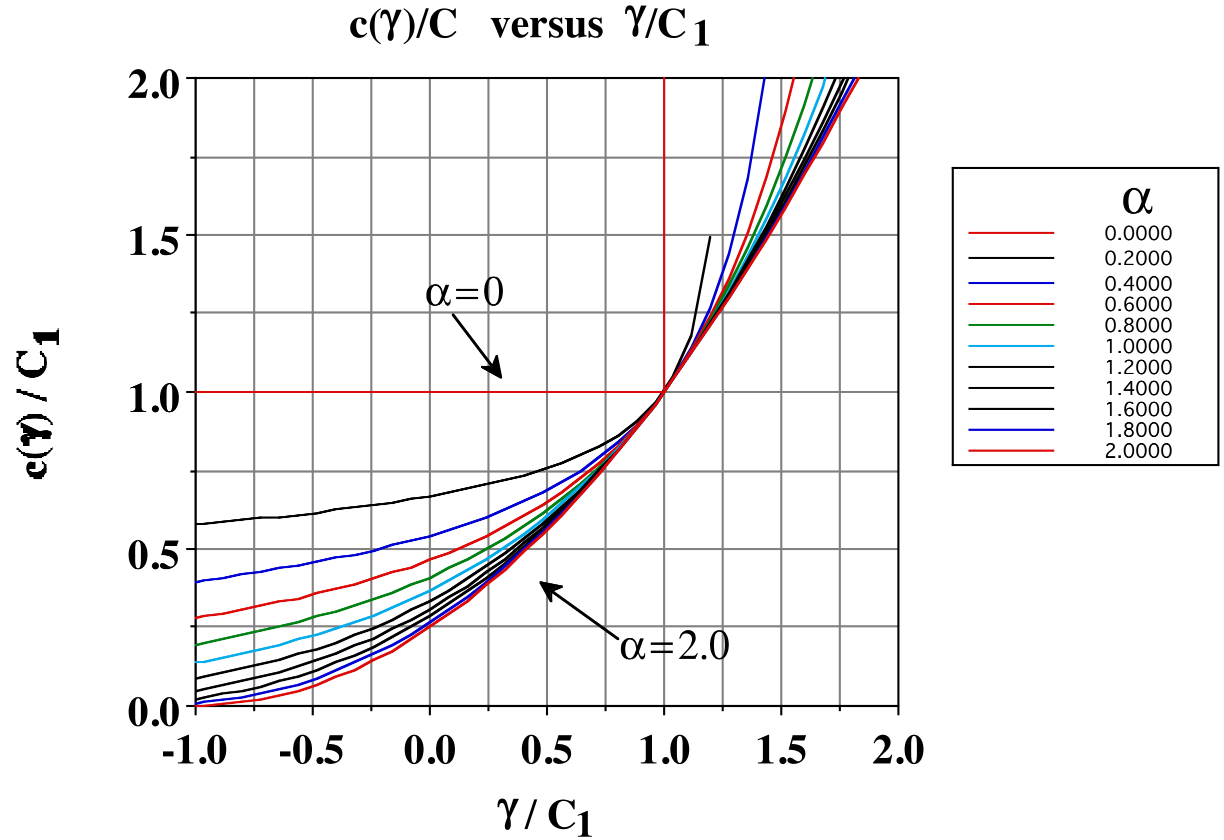
Valid for $0 \le \alpha \le 2$; however, K diverges for all $q < 0$ except in the special ("log-normal") case $\alpha = 2$.
To obtain the corresponding $c(\gamma)$, one can simply take the Legendre transformation to obtain Add: to obtain the α=1 case, just take limit as α->1.

$$c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'}; \quad \alpha \neq 1; \quad 1/\alpha' + 1/\alpha = 1$$

Universal multifractal c(γ)

$$c(\gamma) = C_1 e^{\left(\frac{\gamma}{C_1} - 1\right)}; \quad \alpha = 1$$

Universal c(γ)



Universal $c(\gamma)$ vs γ, for different $\alpha = 0$ to 2 by increment $\Delta\alpha = 0.2$.

Note that since α' changes sign at $\alpha = 1$, for $\alpha<1$, there is a maximum order of singularity $\gamma_{max} = C_1/(1-\alpha)$ so that the cascade singularities are "bounded", whereas for $\alpha>1$, there is on the contrary a minimum order $\gamma_{min} = -C_1/(\alpha-1)$ below which the prefactors dominate ($c(\gamma) = 0$ for $\gamma<\gamma_{min}$) but the singularities are unbounded.

α<1, α>1 cases: bounded, unbounded singularities

$$\alpha' = \frac{\alpha}{\alpha-1}$$

$2 \ge \alpha > 1; \quad \alpha' > 2$

$$c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'}; \quad \gamma > \frac{-C_1}{\alpha-1}$$

$$c(\gamma) = 0; \quad \gamma \le \frac{-C_1}{\alpha-1}$$

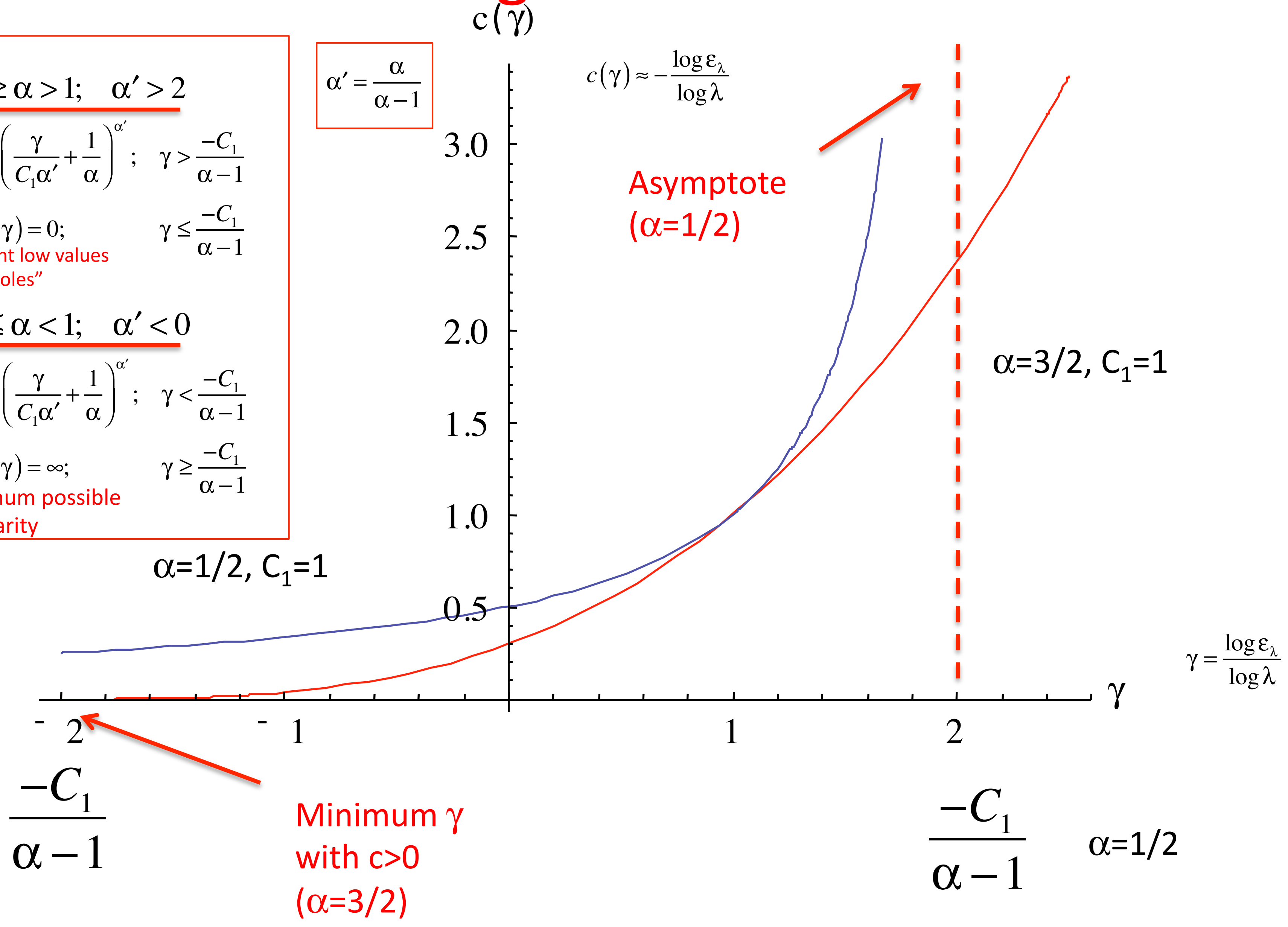
Frequent low values
"Levy holes"

$0 \le \alpha < 1; \quad \alpha' < 0$

$$c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'}; \quad \gamma < \frac{-C_1}{\alpha-1}$$

$$c(\gamma) = \infty; \quad \gamma \ge \frac{-C_1}{\alpha-1}$$

Maximum possible
singularity



Other multifractal models: Log-Poisson

By taking a different limit of the α model, one obtains "Log-Poisson" cascades which have the following form:

$$\gamma_+ = c\left(1-\lambda_0^{-\gamma_-}\right)$$

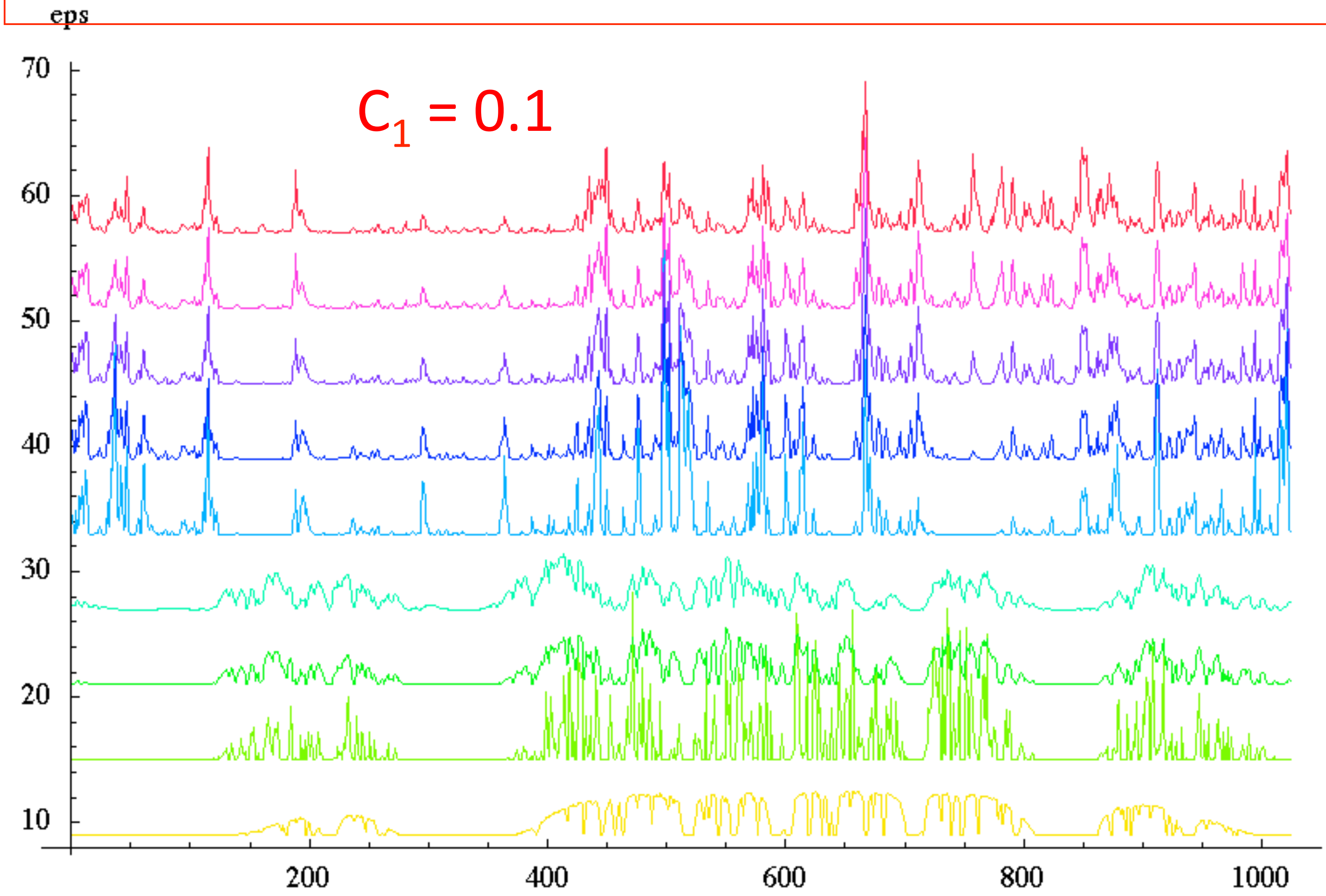
$$K(q) = q\gamma_+ - c + \left(1-\frac{\gamma_+}{c}\right)^q c$$

$$c(\gamma) = c\left(1-\frac{\gamma_+-\gamma}{c\gamma_-}\left(1-\log\frac{\gamma_+-\gamma}{c\gamma_-}\right)\right); \quad \gamma \le \gamma_+$$

$$c(\gamma) = \infty; \quad \gamma > \gamma_+$$

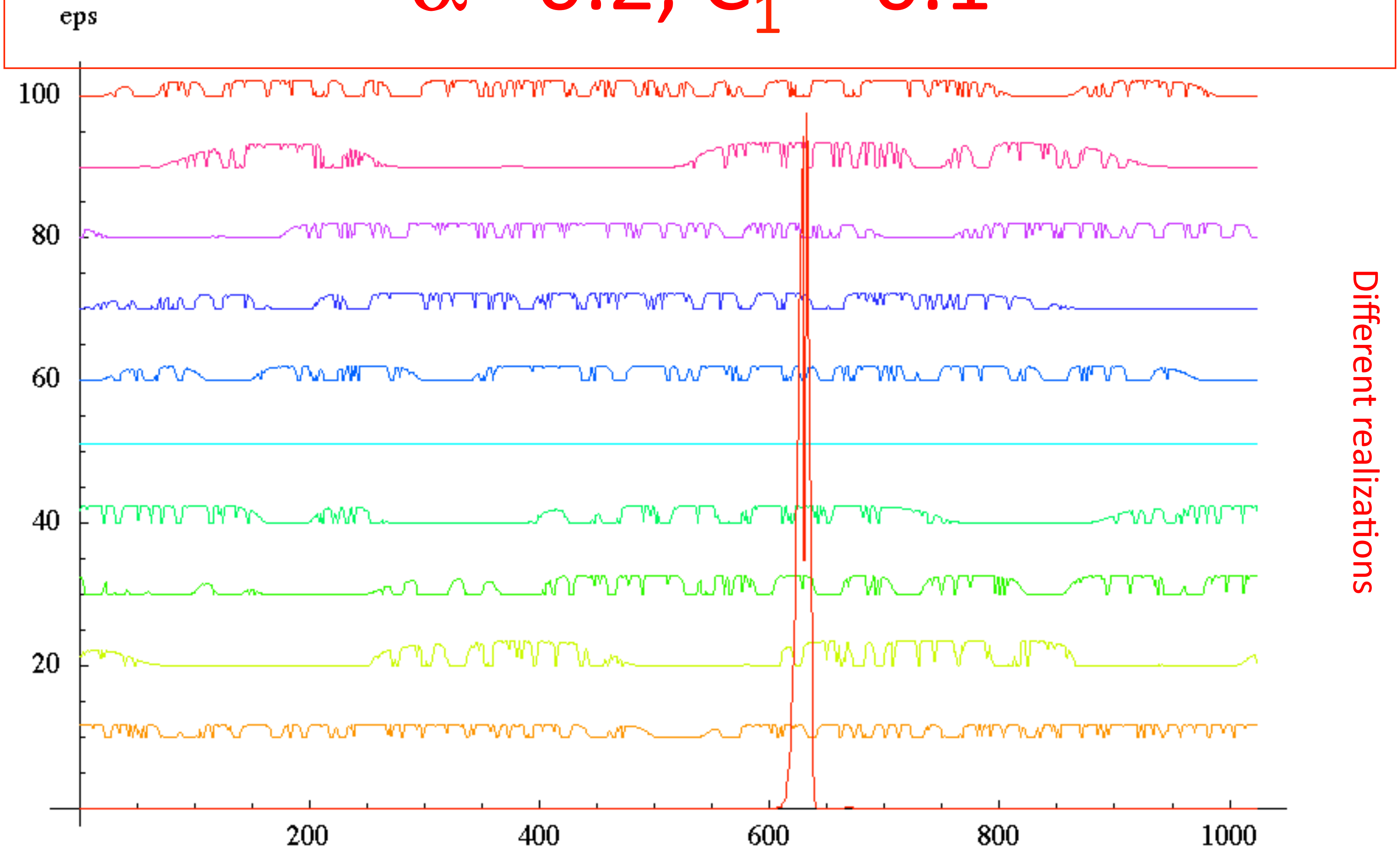
where $\lambda_0>1$ is the cascade ratio for a single step, $c>0$, $\gamma_+>0$ and the relation between γ_+ and γ_- (top line) is from the conservation requirement of the α model. Clearly, γ_+ is the highest order singularity and c is the corresponding codimension so that the log-Poisson cascade has instrinsically a maximum singularity that it can produce.

Examples



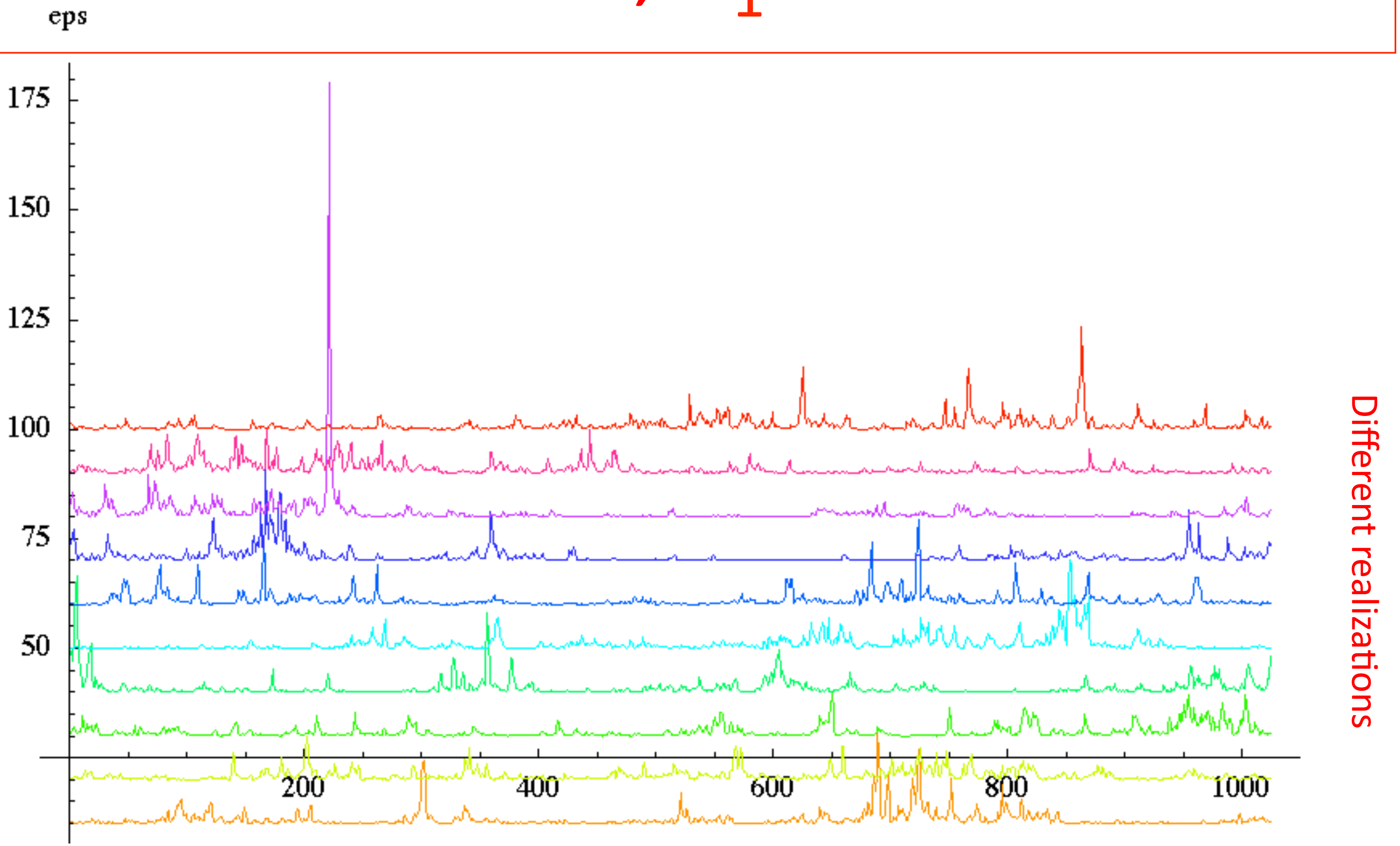
Multifractal simulations C_1=0.1 and α = 0.3, 0.5, …1.9 from bottom to top, offset for clarity (same random seed).

α =0.2, C_1 = 0.1

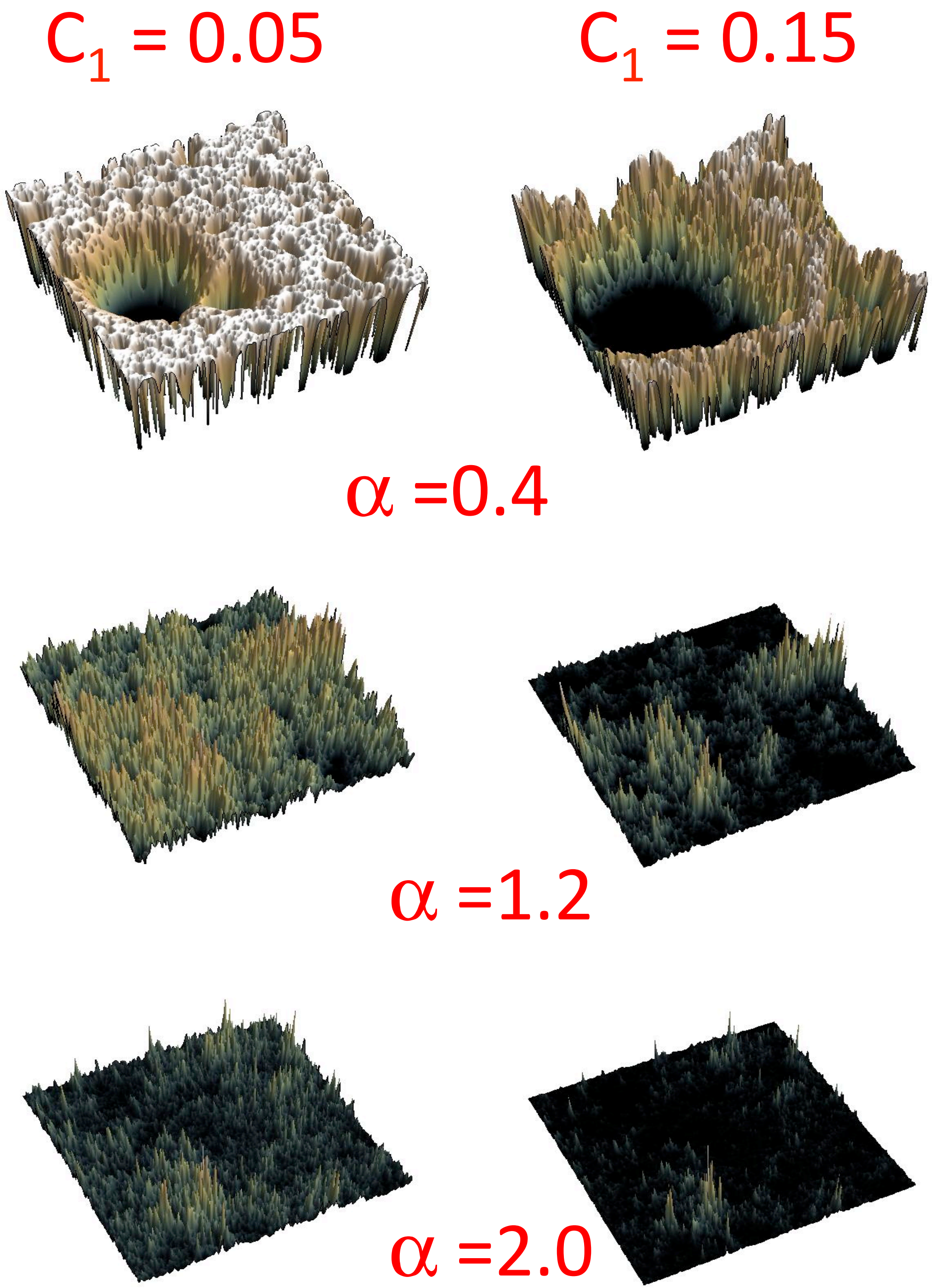


This shows 11 independent realizations of $\alpha = 0.2$, $C_1 = 0.1$ indicating the huge realization to realization variability : the bottom realization is not an outlier! no to so impressive .... with the only exception of a big spike !

α =1.9, C_1 = 0.1



Ten independent realizations of $\alpha = 1.9$, $C_1 = 0.1$, again notice the large realization to realization variability.



This shows isotropic realizations in two dimensions with α = 0.4, 1.2, 2, (top to bottom) and C_1 = 0.05, 0.15 (left to right). The random seed is the same so as to make clear the change in structures as the parameters are changed. The low α simulations are dominated by frequent very low values; the "Lévy holes". The vertical scales are not the same. misleading, we need to find something else..

It's too late to change the name… and if so, to what?