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Space–time complexity and multifractal predictability

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Abstract

Time complexity is associated with sensitive dependence on initial conditions and severe intrinsic predictability limits, in particular, the 'butterfly effect' paradigm: an exponential error growth and a corresponding characteristic predictability time. This was believed to be the universal long-time asymptotic predictability limit of complex systems. However, systems that are complex both in space and time (e.g. turbulence and geophysics) have rather different predictability limits: a limited uncertainty on initial and/or boundary conditions over a given subrange of time and space scales, grows across the scales and there is no characteristic predictability time. The relative symmetry between time and space yields scaling (i.e., power-law) decays of predictability. Furthermore, intermittency plays a fundamental role; the loss of information occurs by intermittent puffs. Therefore, contrary to the prediction of homogeneous turbulence theory its description should depend on an infinite hierarchy of exponents, not on a unique one. However, we show that for a large class of space–time multifractal processes this hierarchy is defined in a straightforward manner. We point out a few initial consequences of this result.

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1. Lessons from complexity in time

Lorenz's deceptively simple 3-component model [1] demonstrated not only the pertinence of Poincaré's criticism [2] of Laplace's infinite predictability limit [3], but also

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the ubiquity and practical significance of sensitive dependence on initial conditions. Its success helped to diminish the interest in finding "approximate solutions" and underscored the need to get fundamental insights on the qualitative behavior of these systems. The corresponding dynamical system approach brought a wealth of striking results for (finite) nonlinear (ordinary) differential systems in a d-dimensional embedding space E_d:

$$\dot{\underline{X}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\underline{X} = \underline{F}(\underline{X}, t)\ . \tag{1}$$

This is also true for the simpler discrete iteration maps:

$$\underline{X}_{n+1} = \underline{G}(\underline{X}_n)\ , \tag{2}$$

which can be motivated either as a discrete approximation of a differential system for $\underline{X}_n = \underline{X}(n\,\delta t)$; for any choice of a small time increment $\delta t, \underline{G}(\underline{x}) = \underline{x} + \delta t\underline{F}(\underline{x})$ or as a Poincaré map of the discrete series of intersections of the trajectory $X(t)$ with a hyper-plane (of dimension $d-1$) transverse to it. Of particular importance was the discovery of an exponential error growth

$$|\delta\underline{X}(t)| \approx \mathrm{e}^{\mu t}|\delta\underline{X}(0)| \tag{3}$$

for the amplitude of the infinitesimal separation $\delta\underline{X}(t)$ of a pair of points ($\underline{X}_1(t)$, $\underline{X}_2(t) = \underline{X}_1(t) + \delta\underline{X}(t)$), with a (finite) Lyapunov exponent μ. Fundamentally, this Multiplicative Erodic Theorem (MET) derives from the straightforward fact that the pair separation is multiplicatively modulated by the derivative of the vector field $\underline{F}$ (or the map $\underline{G}$) at the point $\underline{X}(t)$. Whereas the d-dimensional case involve the noncommutative algebra of matrices [4,5], the one-dimensional case of discrete maps is pedagogically straightforward, since

$$\frac{1}{n\delta t_0}\,\mathrm{Log}[|\delta X_n|/|\delta X_0|] \approx \frac{1}{n\delta t_0}\sum_{n'=0,n-1}\mathrm{Log}(|D_{X_{n'}}G|)\ . \tag{4}$$

Let us assume that the process defined by Eq. (2) is ergodic with respect to a given probability measure $\mathrm{d}P$, i.e., the time average of the right hand side of Eq. (4) P-almost surely converges to the corresponding ensemble average (denoted by square brackets $\langle . \rangle$) and yields Eq. (3) as long as μ, defined by

$$\mu = \langle \mathrm{Log}(|D_X G|)\rangle\ , \tag{5}$$

is finite, an assumption usually taken for granted. In order to get more insights at intermediate times, consider the time evolution of the probability that the dynamics will visit a given neighborhood. Let us recall [6] that for any well-posed finite d-dimensional differential system (Eq. (1)), an ergodic measure exists and is regular with respect to the Lebesque measure of the embedding space $(\mathrm{d}X_1\mathrm{d}X_2,\ldots,\mathrm{d}X_d)$, i.e., it has almost everywhere a well-defined density $\rho(\underline{X}, t)$ which is the solution of the Liouville equation [7], i.e., a continuity equation in the phase space:

$$\frac{\partial}{\partial t}\rho(\underline{X}, t) + \sum_{i=1}^{d}\frac{\partial}{\partial X_i}[\dot{X}_i(t)\rho(\underline{X}, t)] = 0\ . \tag{6}$$

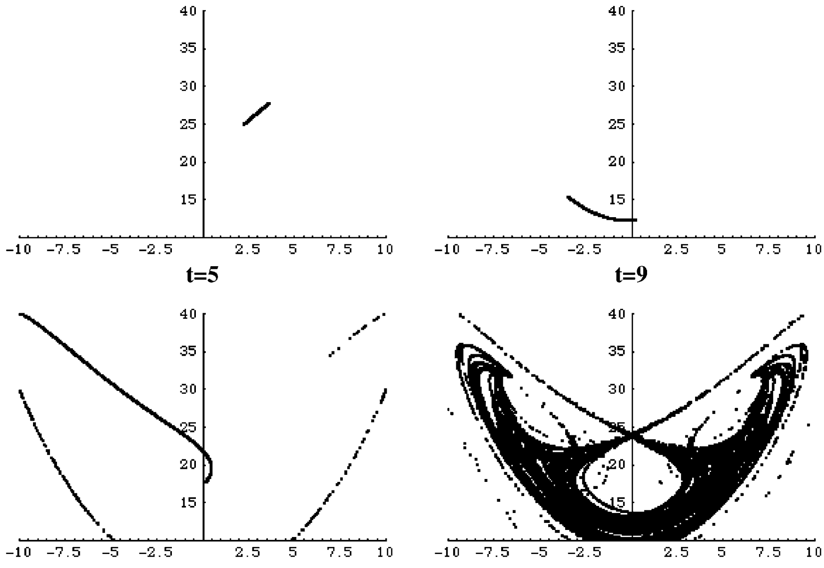

Fig. 1. Evolution of the empirical pdf of a system complex in time (here the x–z projection of the Lorenz model) simulated with the help of 100 000 points initially uniformly distributed ($\sigma = 0.027$) in the neighborhood of $x_0, y_0, z_0 = (6.27, 13.9, 19.5)$. The pdf first diffuses in a quasi-linear manner ($t = 5, 9$), then in a very nonlinear manner ($t = 10$), quickly converging ($t = 15$) to an invariant measure of the finite strange attractor.

This equation has attracted attention in turbulence theory, as well in meteorology [8]. It has been presented [9] as the most general framework to describe in a probabilistic manner the time-dependent behavior of an ensemble of solutions of a numerical weather forecast model started from different initial conditions or from similar models, the so-called Ensemble Prediction System (EPS) [10–12]. The Liouville equation together with MET has often been used to argue in favor of the following route to unpredictability (illustrated by Fig. 1): an initial uncertainty occupies a small fraction of the phase space region, which first grows linearly, then nonlinearly and in a finite time spreads over a strange attractor. We will show that a rather different scenario occurs in systems that are complex in time and space.

2. Preliminary elements of complexity in space

Complexity in space rather corresponds to "the gap dividing simple chaotic systems and fully developed turbulence" [13]. This is usually introduced with the help of nonlinear partial differential systems:

$$\frac{\partial}{\partial t}\underline{u}(\underline{x},t) = \underline{F}(\underline{u}(\underline{x},t), \nabla\underline{u}(\underline{x},t), \Delta\underline{u}(\underline{x},t), \ldots) \,. \tag{7}$$

A significant example is furnished by the Navier–Stokes equations

$$\frac{\partial\underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{\underline{\nabla}p}{\rho} + \nu\nabla^2\underline{u} + \underline{f};\ \frac{\partial}{\partial t}\rho + \underline{\nabla}(\underline{u}\rho) = 0 \tag{8}$$

where $\underline{u}$ is the velocity, t the time, p the pressure, ρ the fluid density, [1] ν the viscosity, $\underline{f}$ the forcing density (external forcing, gravity), as well as the associated advection-diffusion equations for a scalar field θ (f_θ is the forcing density for the scalar, κ is the diffusivity):

$$\frac{\partial\theta}{\partial t} + (\underline{u}\cdot\underline{\nabla})\theta = \kappa\nabla^2\theta + f_\theta \ . \tag{9}$$

Contrary to the (academic) passive case (ρ, f, independent of θ), active scalar fields θ are for applications that are as important as the velocity field $\underline{u}$. Examples include convection, where ρ sensitively depends on θ, either the temperature (atmosphere or lithosphere) or salinity (oceans).

Such systems theoretically require consideration of (functional) infinite dimensional spaces; the attendant difficulties include the nonequivalence of norms. For instance, there have only been limited extensions of the MET for compact operators [14], but not for arbitrarily bounded operators. This requires us to consider variational extension ($\partial X \to \delta X(\underline{x}, t)$) of the Liouville equation (Eq. (6)): the once celebrated Hopf equation [15] corresponds in fact to its Fourier transform.

One way to avoid the functional problem is to project the partial differential system onto a finite d-dimensional space of 'resolved scales', and to proceed to a 'subgrid parametrization' of the others. This already requires one to be very concerned with the question of noisy perturbations due to subgrid scales. If–as is commonly assumed –these perturbations are a gaussian white noise $f(t)$ of intensity ε, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{X} = \underline{F}(\underline{X}, t) + \underline{f}(t); \langle f_i(t) f_j(t')\rangle = \varepsilon\delta_{i,j}\delta(t-t') \ , \tag{10}$$

then the Fokker–Planck equation [16] generalizes the Liouville equation with the introduction of a Laplace diffusive operator

$$\frac{\partial}{\partial t}\rho(\underline{X}, t) + \sum_{i=1}^{d}\frac{\partial}{\partial X_i}[\dot{X}_i(t)\rho(\underline{X}, t)] - \varepsilon\Delta_X\rho(\underline{X}, t) = 0 \ . \tag{11}$$

However, we will argue (Section 6) that these perturbations are strongly nongaussian. This requires major changes. For instance if the perturbations are stable Lévy white noise, a broad "fractional" generalization of the Fokker–Planck equation should be considered. The latter involves fractional derivatives (e.g. Ref. [17] and references herein), in particular fractional powers of the Laplace operator, as well as some exotic convective–diffusive operators. Another major difficulty is that these perturbations are also presumably colored rather than white noises. The fundamental question is: do we converge to the functional problem when we enlarge the dimension of the projection?

3. Scaling

Although the preliminary mathematical properties of the Navier–Stokes equations (e.g. existence and uniqueness of their solutions) correspond to one of the well-known

[1] Hereafter ρ no longer denotes a probability density, as in a previous section.

Hilbert problems [18] that remain fully open, these equations have a (more or less formal) scale symmetry. This has been known for some time under the rubric of self-similarity, e.g. Ref. [19], but with unnecessary limitations. These equations remain formally invariant under any (affine) contraction of the time–space (of scale ratios λ, λ^{1-H}).

$$\underline{x} \to \underline{x}/\lambda, \quad \lambda \to t/\lambda^{1-H} . \tag{12}$$

By suitably renormalizing the dependent variables

$$\underline{u} \to \underline{u}/\lambda^{H}, \quad \theta \to \theta/\lambda^{H'}, \quad \rho \to \rho/\lambda^{H''} ,$$

$$\underline{v} \to \underline{v}/\lambda^{1+H}, \quad K \to K/\lambda^{1+H}, \quad p \to p/\lambda^{H''-1} ,$$

$$\underline{f} \to \underline{f}/\lambda^{2H-1}, \quad f_\theta \to f_\theta/\lambda^{H+H'-1} . \tag{13}$$

One may either consider the asymptotic case of fully developed turbulence (with an infinite Reynolds number ($\mathrm{Re} \to \infty$) or vanishing viscosity ($v \to 0$)) for the incompressible Navier-Stokes equations [20], or more generally nonzero eddy viscosity (resp. eddy diffusivity), rather than a molecular one [21]. To get more physical insights, the notion of "eddy turnover time" $\tau(\ell)$ is helpful. This is the characteristic time, if it exists, for structures of scale ℓ to "turnover" within a velocity shear $\delta u(\ell)$:

$$\tau(\ell) \propto \ell/\delta u(\ell) . \tag{14}$$

Since the characteristic time of destruction of structure of this scale ℓ can be only proportional to the eddy turnover time [e.g. [22]], one finds for instance that the rate of transfer of energy to scales smaller than ℓ is

$$\varepsilon(\ell) \propto \delta u(\ell)^2/\tau(\ell) \propto \delta u(\ell)^3/\ell . \tag{15}$$

Therefore,

$$\delta u(\ell) \propto \varepsilon(\ell)^{1/3}\ell^{1/3} \quad \tau(\ell) \propto \varepsilon(\ell)^{-1/3}\ell^{2/3} . \tag{16}$$

On assuming that $\varepsilon(\ell)$ does not fluctuate too much (i.e., $< \varepsilon^q > \approx \langle\varepsilon\rangle^q$), is ergodic and that its spatial average $\bar{\varepsilon}(\ell)$ is scale independent ($\bar{\varepsilon}(\ell) = \bar{\varepsilon}$), the Kolmgorov–Obukhov scaling law [23,24]) for velocity shear is obtained

$$\langle\delta u(\ell)^2\rangle \propto \bar{\varepsilon}^{2/3}\ell^{2/3}, \quad E(k) \propto \bar{\varepsilon}^{2/3}k^{-5/3} , \tag{17}$$

where $E(k)$ is the energy spectrum at the wave number k. This yields the following Lyapunov exponent and the characteristic space scale ℓ_e reached by the error at time t:

$$\mu(\ell) \propto 1/\bar{\tau}(\ell) \propto \bar{\varepsilon}^{1/3}\ell^{-2/3}, \quad \ell_e(t) \propto \bar{\varepsilon}^{1/2}t^{3/2} . \tag{18}$$

This shows that, contrary to the usual assumption, the Lyapunov exponent μ diverges at small scales, unless they are homogeneous.

4. Energetics and spectral analysis of the error growth

Thompson [25] and Lorenz [26] set up a rather general framework to study the error growth of a complex time–space field, namely the solutions of meteorological models or Navier–Stokes equations, by considering the two solutions $\underline{u}^1(\underline{x},t)$ and $\underline{u}^2(\underline{x},t)$ as initially identical, but for a perturbation $\delta\underline{u}(\underline{x},0)=\underline{u}^2(\underline{x},0)-\underline{u}^1(\underline{x},0)$, confined to (infinitesimal) small spatial scales. When the nonlinear interactions preserve the kinetic energy (e.g. the incompressible Navier-Stokes equations), it is convenient and important (although not sufficient as discussed below) to consider the correlated (kinetic) energy (per unit of mass)

$$e^c(\underline{x},t)=\tfrac{1}{2}\,\underline{u}^2(\underline{x},t).\underline{u}^1(\underline{x},t)\,, \tag{19}$$

and the corresponding decorrelated energy

$$e^{\Delta}(\underline{x},t)=\tfrac{1}{2}\,(\delta\underline{u}(\underline{x},t))^2=\tfrac{1}{2}(\underline{u}^2(\underline{x},t)-\underline{u}^1(\underline{x},t))^2 \tag{20}$$

as well as the total energy and the energy of each solution

$$e^T(\underline{x},t)=e^1(\underline{x},t)+e^2(\underline{x},t);\qquad e^n(\underline{x},t)=\tfrac{1}{2}(\underline{u}^n(\underline{x},t))^2\,. \tag{21}$$

This implies the straightforward, but nevertheless important relation

$$e^T(\underline{x},t)=e^c(\underline{x},t)+e^{\Delta}(\underline{x},t)\,, \tag{22}$$

which shows that if the total energy is statistically stationary, there is a flux of correlated energy $e^c(\underline{x},t)$ to decorrelated energy $e^{\Delta}(\underline{x},t)$. The same property holds for the corresponding energy spectra ($E^T(k,t)=E^c(k,t)+E^{\Delta}(k,t)$), since the latter correspond to linear decomposition of the former with respect to the wave numbers (e.g. $\langle e^n(\underline{x},t)\rangle=\int_0^\infty \mathrm{d}k\,E^n(k,t)$). Therefore, the decorrelated energy spectrum $E^{\Delta}(k,t)$ steadily increases in magnitude from large to small wave numbers, to reach the total energy spectrum $E^T(k,t)\approx k^{-5/3}$. A critical wave number $k_e(t)\approx 1/\ell_c(t)\approx t^{-3/2}$ (Eq. (18)) of the transition from dominant correlation to dominant decorrelation can be defined by $E^c(k_c(t),t)=E^{\Delta}(k_c(t),t)$.

Thompson [25] studied the initial error growth with the help of the initial time derivatives, whereas Lorenz [26] proceeded to a time integration with the help of a quasi-normal closure, [27] and [28,29] refined the latter with the help of the Eddy-Damped Quasi-Normal Model [30] and the Test Field Model [31].

5. How many spatial scales are involved?

The choice by [26] of a 3D energy spectrum up to synoptic scales for the statistical quasi-normal closure of a 2D deterministic large scales models has been often questioned [e.g. Ref. [32]] for being at odds with the prevailing standard model of atmospheric dynamics [33,34], which considers small-scale motions as quasi-three dimensional (quasi-3D), and large-scale motions as quasi-two dimensional (quasi-2D), i.e., rather regular. In order to stop the former from destabilizing the latter, the two regimes were supposedly separated by a ‘meso-scale’ gap. As a consequence only the micro-scales (say below 10 km) would be involved in the error growth.

However, there is no evidence [35,36] of this gap along this horizontal anywhere near 10 km in the GASP experiment (airplane measurements), but on the contrary a Kolmogorov–Obukhov scaling extended to at least hundreds of kilometers. This scaling was found in very different climatological and meteorological regimes, including tropical and cyclonic conditions [37]. Nor did various radiosonde analyses [38–41] also find a meso-scale gap along the vertical and furthermore that the vertical spectrum of the horizontal wind follows Bogliano–Obhukhov (BO) $k^{-11/5}$ scaling throughout the troposphere. Combining the vertical BO $k^{-\beta_v}$, $\beta_v \approx \frac{11}{5}$ with the horizontal Kolmorogov $k^{-\beta_h}$, $\beta_h \approx \frac{5}{3}$, a 'unified scaling' model [40] of turbulent stratified atmosphere was proposed from planetary scales down to dissipation scale, whose effective "elliptical dimension" is neither 3 nor 2, but $D_{el} = 2 + (\beta_h - 1)/(\beta_v - 1)$, hence $D_{el} = \frac{23}{9} = 2.555$. For recent empirical confirmation of the relevance of this model, see Refs. [42,43], lidar data of passive scalar yields $D_{el} = 2.55 \pm 0.02$. Therefore, all atmospheric scales—from planetary to viscous ones, ratio $\Lambda \approx 10^9$—are presumably involved in the error growth process.

6. The limitations of the spectral approach

Lilly [44] criticized the statistical framework of the predictability analysis of Lorenz, Leith and Kraichnan, pointing out that a consequence of the quasi-normal framework of their closure schemes is that their analyses are local as well as global. This does not agree with the observation that various atmospheric structures (e.g. rotating thunderstorms [45]) maintain a stable identity much longer than their turnover time. Lilly cited the results of [46] showing that the probability distributions of log potential temperature gradient ($x = \delta \log\theta$) and wind shear ($x = \delta u$) have power law probability tails, i.e.:

$$Pr(x \geqslant s) \approx s^{-q_D} \quad s >> 1 \tag{23}$$

he then argued that the error statistics should also be power law, so that they will be much more variable and extreme than those assumed by quasi-normal closures. Indeed, the power-law exponent q_D of the probability tails (Eq. (23)), which has been taken as a basic feature of self-organized criticality [74]; is a critical order of divergence of statistical moments. For instance, empirical estimates in the vertical are $q_D = 5$, 3.3, 1 for the horizontal wind, buoyancy force and Richardson number [40]. This means that the (theoretical) statistical moment of order $q \geqslant q_D$ are infinite, whereas their empirical estimates on finite samples, although finite continue to grow (i.e., diverge) with the number of samples [46]. As an immediate consequence, second-order moments, such as energy spectra, are inadequate characterizing the variability of the process. Unfortunately, quasi-normal closures close the infinite hierarchy of moments with the help of second-order moments.

7. Intermittency, cascades and multifractals

This criticism echoed a fundamental debate on the relevance of homogeneity. Landau questioned this assumption [47] as early as 1942 [48]; there is no homogeneity at on a large scale, and this influences the average estimates (in particular when using an ergodic assumption). Batchelor and Townsend [49,50] empirically confirmed that not only does the "activity" of turbulence induce inhomogeneity, but the activity itself is very inhomogeneously distributed. There are "puffs" of active turbulence inside puffs of (active) turbulence. The word "intermittency" has been used for this inhomogeneity that became theoretically understood with the help of cascade models. The general idea is that [51] in turbulence the successive cascade steps define independent fractions of the flux F transmitted to smaller scales and that a cascade is scaling. To make this more precise, denote an intermediate scale ratio ("resolution") $\lambda = \frac{L}{\ell}$ where L is the outer scale and ℓ the scale corresponding to scale ratio λ and the total scale ratio $\Lambda = \frac{L}{\ell'} = \lambda\lambda'$. Scaling means that the cascade from λ to Λ corresponds to a cascade from ratio 1 to λ' contracted by T_λ of scale ratio λ; $T_\lambda(f(\underline{x})) = f(T_\lambda(\underline{x}))$; in the self-similar (isotropic) case $T_\lambda(\underline{x}) = \frac{\underline{x}}{\lambda}$. Together, these two properties imply that the flux is a multiplicative group ($\stackrel{d}{=}$ means equality in probability distribution)

$$F_{\Lambda=\lambda\lambda'} \stackrel{d}{=} F_\lambda . T_\lambda(F_{\lambda'}) \quad (any\ \lambda, \lambda' \geqslant 1)\,. \tag{24}$$

This yields a similar group property for the statistical moments, therefore the following scaling law:

$$\langle F^q_{\lambda\lambda'} \rangle = \lambda'^{K(q)} \langle F^q_\lambda \rangle \quad (any\ q, any\ \lambda, \lambda' \geqslant 1)\,, \tag{25}$$

where the "scaling moment function" $K(q)$ is convex and is in fact, as discussed below, the cumulant generating function of the generator of the group (Eq. (24)). With the help of the Mellin transform [52,53] one obtains the corresponding scaling law for the probability distribution of the event $\{F_\lambda \geqslant \lambda^\gamma\}$

$$\Pr\{F_\lambda \geqslant \lambda^\gamma\} \propto \lambda^{-c(\gamma)}\,. \tag{26}$$

The arbitrary exponent γ, which defines a given level of activity or intensity at all resolution λ, is called a "singularity" (more precisely a "singularity order") since $\gamma > 0$ defines the power-law divergence of F_λ with $\lambda \to \infty$. The scaling exponent $c(\gamma)$ of the probability is a "statistical codimension" [54] also called the "Cramer function" [55,56]. For large λ's, the Mellin transform reduces to the celebrated Legendre transform [57] for the corresponding exponent functions

$$K(q) = \max_\gamma \{q\gamma - c(\gamma)\} \qquad c(\gamma) = \max_q \{q\gamma - K(q)\}\,. \tag{27}$$

These are the main properties common to all multifractal formalisms, i.e., an infinite hierarchy of statistical exponents, e.g. Refs. [46,58–60], and the infinite hierarchy of singularities, e.g. Refs. [57,61,62]. However, there are substantial differences. For example, there is an upper bound for singularities of "geometrical" multifractals [57,61], where each singularity is assumed to be supported by a well-defined geometrical (fractal) set, and for singularities of "microcanonical" multifractal processes [63–65] that conserve fluxes scale by scale on each realization. This upper bound, acknowledged as artificial by [20], does not exist [66] for canonical multifractals, which respects only conservation on ensemble averages of fluxes. Therefore, it is only for embedding dimensions $D > c(\gamma)$ that the event $\{F_\lambda \geqslant \lambda^\gamma\}$ almost surely corresponds to a fractal set of dimensions: $D(\gamma) = D - c(\gamma)$. This is no longer the case for higher singularities (with

$c(\gamma) \geqslant D$): they correspond to extreme events almost surely absent on an individual realization. In general, there exists a finite critical singularity γ_D such that the codimension $c(\gamma)$ becomes linear, i.e., the corresponding probability distribution becomes a power-law (Eq. (23)), for conservative processes: $K(q_D) = q_D(D-1)$.

Multiplicative processes, in particular when continuous in scale, can readily be obtained from a white noise, their "sub-generator". Indeed, they can be obtained by the exponential of an additive process $\Gamma_\lambda(\underline{x})$, the "generator" of the flux ($\underline{x} = (x, y, z)$ for a 3D spatial process, $\underline{x} = (x, y, z, t)$ for a time–space process):

$$F_\lambda = e^{\Gamma_\lambda} . \tag{28}$$

In order to respect the scaling property of the statistical moments (Eq. (25)). the generator must have a logarithmic divergence with the resolution ($\Gamma_\lambda \approx \mathrm{Log}(\lambda);\ \lambda \to \infty$). This is readily achieved by applying an adequate linear transform to a (infinitely divisible) white noise (limited to the resolution λ), the sub-generator $\gamma_{0,\lambda}$. To obtain universal cascades (satisfying multiplicative central limit theorem [54,67]), the sub-generator should be a Levy stable noise of Levy stability index α and the linear transform is a fractional integration of order h, i.e., a convolution (denoted by $*$) by a power-law function g of dimension $D_h = d - h$ (d being the dimension of the embedding space):

$$\Gamma_\lambda = g * \gamma_{0,\lambda}, \quad g \propto |\underline{x}|^{-D_h}, \ D_h = \frac{d}{\alpha} . \tag{29}$$

In general, the basic fluxes are conservative, i.e., for any scale ratio $\langle F_\lambda \rangle = \langle F_1 \rangle$, whereas they are related to nonconservative fields u with the help of scaling laws

$$|\delta u| \approx F^a |\underline{\delta x}|^H \tag{30}$$

the standard example being the Kolmogorov law (Eq. (17)) for the velocity shear amplitude $|\delta u|$, with $F = \varepsilon$, and $a = H = \frac{1}{3}$. A straightforward model of nonconservative fields is obtained by performing another fractional integration

$$u_\lambda = G^* F_\lambda^a, \quad G(\underline{x}) \propto |\underline{x}|^{-d-H} . \tag{31}$$

This Fractionally Integrated Fluxes (FIF) model was originally motivated on purely scaling considerations. However, it has strong connections with the dynamics. This is first achieved by taking into account the (scaling) anisotropy between space and time with the help of a generalized scale notion [68,69] as well as by implementing the causality condition to all the fractional integrations [70]. Both can be implemented by considering a scaling (retarded) Green's function (or propagator) g of a linear (fractional) anisotropic differential operator $g^{-1} (g^{-1} * g = \delta)$, e.g.

$$G_R^{-1} = \frac{\partial}{\partial t} + (-\Delta)^{(1-H_t)/2} , \tag{32}$$

where $H_t \neq 0$ measures the scaling anisotropy between time and space. The generator and the nonconservative field are solutions of this type of equation.

$$g^{-1} * \Gamma_\lambda = \gamma_{0,\lambda}, \quad G^{-1} * u_\lambda = F_\lambda^a, \quad F_\lambda = e^{\Gamma_\lambda} .$$

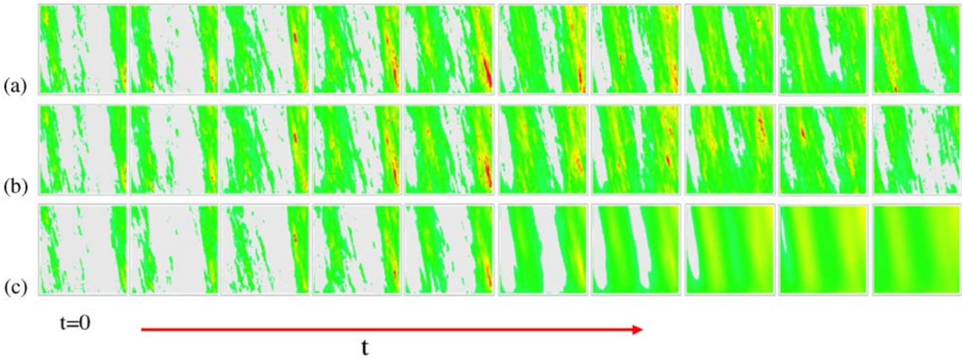

Fig. 2. The top two rows (a–b) show successive snap shots ($t = i \times \tau$, $i = 3, 6, 9..27$, τ being the eddy turn over time of the smallest structures) of two simulations (256×256 in space) that are identical until time $t = 0$, when their fluxes at small scales become independent step by step due to the sudden independence of the sub-generators at that time. Most of the difference between the two realizations is concentrated in a few "hot spots". The bottom row (c) shows a forecast based on the "memory" of the evolution up to $t = 0$ of (a), i.e., it has the same stochastic subgenerator until time $t = 0$, then defined in a deterministic manner to preserve the mean of the flux. Note the more rapid disappearance of small-scale structures. Parameters are $\alpha = 1.5$, $C_1 = 0.2$, $H = 0.1$ (close to those of rain), and color scale is a logarithmic. The anisotropy of space–time is characterized by $H_t = 2/3$.

8. Multifractal predictability

To generalize the approach followed in the spectral analysis of predictability (Section 4), we consider the time evolution of a pair of multifractal fields $u^1(\underline{x}, t)$ and $u^2(\underline{x}, t)$ of common resolution Λ. They have sub-generators $\gamma^n_{0,\Lambda}$, which are identical up to the time t_0, after which they become independent. It was checked [70,71] that the spectral analyses of these multifractal simulations agree with homogeneous turbulence results (Section 4), as well as the presence of bursts of violent fluctuations that cannot be accounted for with the help of second-order statistical moments, e.g. energy spectra. Indeed, although the energetics of the upscale cascade of errors remain basically the same, they do not constrain the largest fluctuations of the errors as much as before. Similar bursts are observable in time and space on the difference $\delta u(x, t)$ of the pair of fields $u^1(\underline{x}, t)$ and $u^2(\underline{x}, t)$ displayed in Fig. 2a and b.

The sudden independence of the sub-generators at t_0 only introduces—due to the (causal) fractional integration—a slow logarithmic growth of relative independence between the corresponding generators and therefore the two fluxes ε^1, ε^2 only up to a scale resolution $\lambda(t) \leqslant \Lambda$. Due to the (scaling) anisotropy between time and space (i.e., $H_t \neq 0$)

$$t \leqslant t_0 : \lambda(t) = \Lambda; t > t_0 : \lambda(t) \approx \mathrm{Min}[\Lambda, (T/(t - t_0))^{1/(1-H_t)}] , \tag{33}$$

where T is the outer time scale. As a consequence, one obtains [71] for the (normalized) covariance of order q

$$C^{(q)}(\varepsilon^1_\Lambda, \varepsilon^2_\Lambda) = \langle(\varepsilon^1_\Lambda \varepsilon^2_\Lambda)^q\rangle / \langle(\varepsilon^1_\Lambda)^q\rangle\langle(\varepsilon^2_\Lambda)^q\rangle \propto \lambda(t)^{K(q,2)};$$
$$K(q, 2) = K(2q) - 2K(q) , \tag{34}$$

where $K(q)$ is the scaling moment function of the fluxes $\varepsilon^1, \varepsilon^2$. The nonlinearity of $K(q,2)$ is a consequence of that of $K(q)$ and shows that the joint field $\varepsilon^1_\Lambda \varepsilon^2_\Lambda$ is multifractal, but the distinctive feature is that the relevant range of scale ratios $[1, \lambda(t)]$ also has a power-law decay (Eq. (33)), instead of being fixed at Λ (as for $\varepsilon^1_\Lambda = \varepsilon^2_\Lambda$). With the help of the Legendre transform, the scaling of the probability distribution of the joint field singularities is easily obtained:

$$\Pr(\varepsilon^1_\Lambda \varepsilon^2_\Lambda \geqslant \lambda(t)^\gamma \langle \varepsilon^1_\Lambda \rangle \langle \varepsilon^2_\Lambda \rangle) \approx \lambda(t)^{-c(\gamma,2)};$$
$$c(\gamma,2) = \max_q \{q\gamma - K(q,2)\} . \tag{35}$$

Similar relations hold for the correlation of the field u^1, u^2 obtained by fractionally integrating the fluxes $\varepsilon^1, \varepsilon^2$. It is important to appreciate that these power laws are valid for all time scales, not only for large time scales. This is in a sharp contrast with the "Liouville+MET" scenario discussed in Section 1, in particular with respect to Fig. 1 and its series of distinct periods. Let us also emphasize that these laws are purely determined by the multifractality of the fields ε^i (e.g. their scaling moment function), either theoretically or empirically, which may be known, either theoretically or empirically.

Let us point out that these laws could be used to quantify the performance of forecast procedures. Indeed, the decay law of the (normalized) covariance $C^{(q)}(\varepsilon^F_\Lambda, \varepsilon^0_\Lambda)$ of order q of the forecast field ε^F_Λ and of the observed field ε^0_Λ should be as close as possible to the theoretical $C^{(q)}(\varepsilon^1_\Lambda, \varepsilon^2_\Lambda)$ (Eq. (34)). Statistical biases introduced by a given type of forecast procedure can also be readily assessed. Indeed, due to its white noise property, the sub-generator $\gamma(x,t)$ can be split into two independent components $\gamma^-(x,t)$ and $\gamma^+(x,t)$ corresponding, respectively, to the past (i.e., "real" history up to time t_0) and the forecast period (after t_0). The corresponding generators $\Gamma^-(x,t)$ and $\Gamma^+(x,t)$ are, respectively, dominant in the evolution of the field at scales larger and smaller than $L/\lambda(t)$; therefore the scaling of the forecast field will be:

$$t \geqslant t_0 : \langle (\varepsilon^F_\Lambda(\underline{x},t)^q) \rangle \approx \lambda(t)^{K^-(q)} (\Lambda/\lambda(t))^{K^+(q)} . \tag{36}$$

When the scaling function of the forecast procedure $K^+(q)$ sensitively differs from the past $K^-(q)$ (assumed to be identical to $K(q)$), important statistical bias are introduced. This is illustrated by Fig. 2c, where $\gamma^+(x,t)$ was defined in a deterministic manner to preserve the mean of the flux, whereas $\gamma^-(x,t)$ is identical to that of Fig. 2a and b. $K^+(q)$ is then linear, instead of being nonlinear like $K(q)$. Fig. 2c in comparison with Fig. 2a and b displays the drastic loss of all extreme events ($q \geqslant 1$) related to smoothing due to the time decay of $\lambda(t)^{K(q)}$, which is no longer compensated by the second factor of the r.h.s. of Eq. (36). This could explain the recent empirical evidence that stochastic parametrizations do better than deterministic ones [72,73].

9. Conclusion

We critically discussed the predictability concepts that emerged from the study of systems that are complex in time. We argued that complexity in space implies strong limitations on the applicability of the Multiplicative Ergodic Theorem (MET) and of the Liouville equation. Both statistical closure models and the phenomenology of homogeneous turbulence indicate that predictability decay laws are not exponential: they are algebraic. Unfortunately, the quasi-normal framework of these models prevents them from dealing with intermittency; the "bursts" of the energy fluxes through scales. We showed that multifractals offer a very convenient framework to quantify the predictability of space–time complex systems. This should help us to find alternative modeling strategies approaching the intrinsic predictability limits.

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