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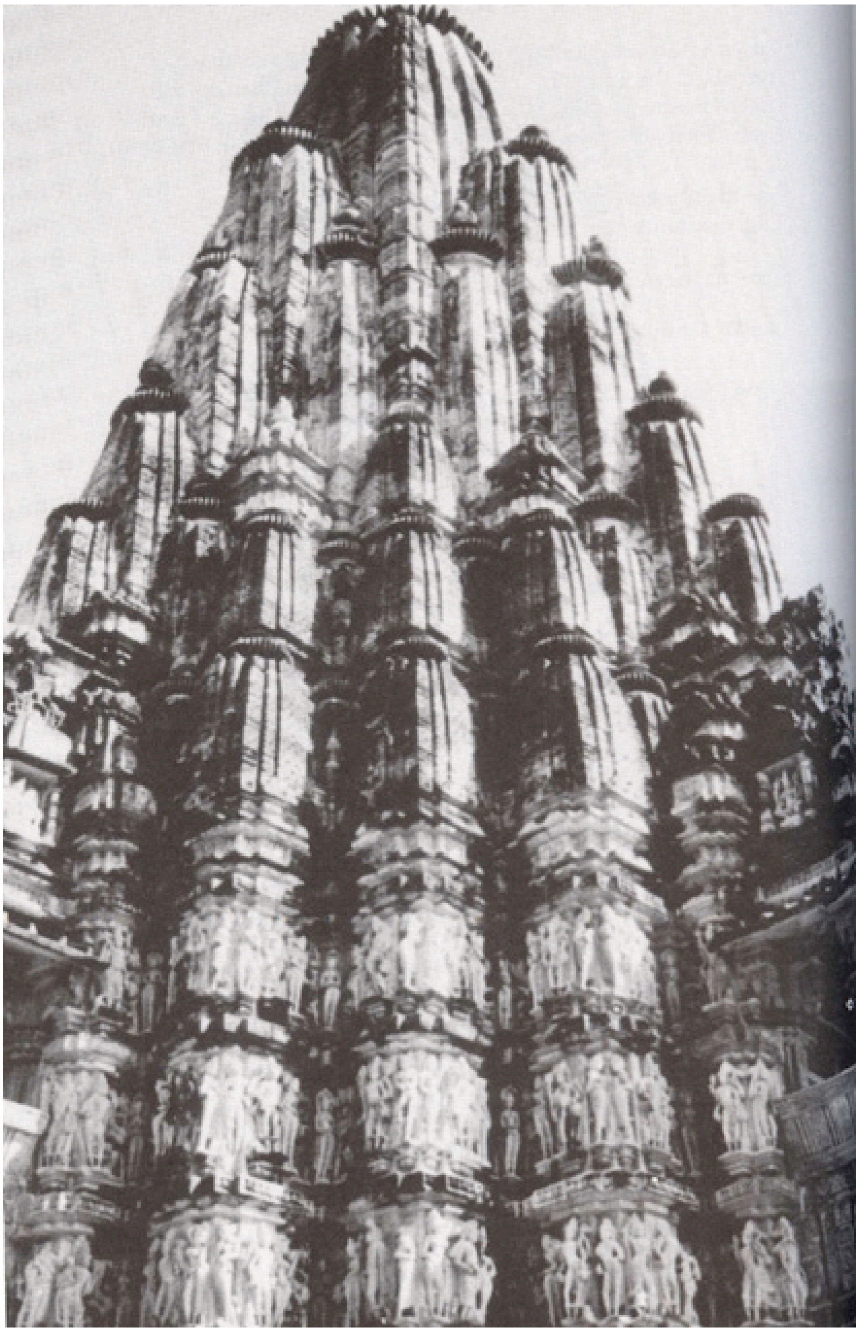

Stochastic Chaos, Symmetry, and Scale Invariance

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Introduction: Chaos vs. Cosmos

Perhaps the oldest and most basic philosophical question is the origin of order from disorder; indeed the terms of the ancient Greek dichotomy *chaos/cosmos* are still with us. Although this problem is posed in all areas of life, it is undoubtedly in physics where it has reached its fullest and most precise expression, concomitant with the recent "chaos revolution," which has changed our outlook in so many areas. In the revolution's wake, the physics notion of *deterministic chaos is* invading territory as distant as literature and criticism.

While the validity of this attempted conceptual transplant can be criticized, the relevance of the notion to its original domain of application, physics, is rarely discussed. In this essay, we criticize the underlying philosophy of deterministic chaos as well as many of its significant claims to applications. We then show the need to develop a rather different framework which we call *stochastic chaos*. While we believe this alternative to be quite broad, potentially encompassing much of our atmospheric and geophysical environment, we do not claim exclusivity. Rather we view stochastic chaos as complementary to deterministic chaos with the former being necessary in systems involving many interacting components and the latter being useful when only a few corre-

Fig. 1 The temple at Khajuraho showing a small part of the hierarchical structure within structure architecture executed following an essentially self-similar algorithm in accord with the Hindu cosmos of smaller and smaller structures within structures.

sponding degrees of freedom are important. Both model types belong in the physicist's toolbox. While we are primarily concerned with providing a critique within physics, we have endeavored to do so in a widely accessible way.[1]

The utility of stochastic chaos lies primarily in its ability to exploit a (nonclassical) symmetry principle called "scale invariance," associated with fractals and multifractals. We will argue that the ubiquity of fractals in nature is an indication of the wide scope for applying stochastic chaos models. The basic idea of scale invariance is that small parts of a scale invariant object are similar to the whole, thus its relevance to art and architecture. For example, Trivedi argued that Indian temples from at least the tenth century onward were explicitly constructed according to fractal mathematical algorithms (fig. 1). Similarly, Briggs and Peat suggested that certain examples of Celtic art exemplify a deliberate fractal type construction.[2] The connection between art and fractals is occasionally explicit, for example, the photo album, as in "An Eye for Fractals" by McGuire.[3] More recently, a shared interest in scale invariant clouds lead us to collaborate with sculptor M. Chin on a projection piece at the StoreFront for Art and Architecture, New York City, entitled "Degrees of Paradise." This piece featured an evolving multifractal cloud simulation displayed on 16 monitors suspended from the ceiling.[4]

Order vs. Disorder and the Scientific World View

Although God(s) were traditionally invoked to explain order, their role was drastically diminished with the advent of the Newtonian revolution. By the middle of the nineteenth century, these laws had become highly abstract (thanks notably to Laplace, Hamilton, and Lagrange), while the corresponding scientific world view had become determinism. In this regard probably the most extreme views have been attributed to Laplace (1886), who went so far as to postulate a purely deterministic universe in which "if a sufficiently vast intelligence exists" it could solve the equations of motion of all the constituent particles of the Universe. In Laplace's universe, such a divine calculator could determine the past and future from the present in an abstract high dimensional "phase space."

Unfortunately, due to man's imperfect knowledge we are saddled with measurement errors that clearly involve notions of chance. This lead Laplace (following Voltaire) to identify chance and probability with ignorance. Much later, in 1870, probability was explicitly introduced into the formulation of physical laws by James Clerk Maxwell. This is the basic idea of classical Statistical Mechanics: that the unobserved or unknown degrees of freedom ("details"), are the source of "random" behavior such as fluctuations about a mean temperature. Although highly partial information is the rule, macroscopic objects are typically described by parameters such as temperature, pressure, and density. Most of the degrees of freedom, such as the positions and velocities of the constituent particles, are unimportant and can be reduced to various averages using statistics. Hence, the dichotomy of objective deterministic interactions of a large number of degrees of freedom coexisting with randomness arising from our subjective ignorance of the details.

Starting with Gibbs and Boltzman, this identification of statistics with ignorance evolved somewhat to the more objective identification of statistics with the irrelevance of most of the details;[5] however, this did not alter the deeply held prejudice that statistics were a poorman's substitute for determinism. A corollary to this was the hierarchical classification of scientific theories; fundamental theories being deterministic, the less fundamental involving randomness or ignorance.

Since then—even in spite of the Quantum revolution—this prejudice has become fairly entrenched even though a number of developments (especially in deterministic and more recently, in stochastic chaos) have occurred which make it obsolete. Unfortunately there has not been an adequate conceptual reassessment. In this essay, we outline these developments, and propose an alternative framework for chaos that we believe overcomes the limitations of strict determinism: *stochastic chaos*.

The Deterministic Chaos Revolution: The Butterfly Effect

The rigid "Newtonian" or "mechanical determinism" of Laplace runs into trouble as soon as one attempts to solve the equations of motion for anything but the most simple systems: as recognized by Poincare, three particles are already sufficient.[6] However, the general property of nonlinear systems of having "sensitive dependence" on initial conditions only became widely known in the 1970s. Better known as the "butterfly effect," this term denotes me general property of nonlinear systems to amplify small per-turbations—such as the possible large-scale consequences of the flapping of a butterfly's wings on the atmospheric circulation. Even if Laplace's calculator had both *almost* perfect information at an initial instant (i.e. only infinitesimally small perturbations are present) and infinite precision in its numerics, the predicted future would not in fact be predictable. On the contrary, finite random-like "chaotic" behavior would result. With fluids, this led to the idea that such "turbulence" arose through a short (rather than infinite) series of instabilities, contrary to the pioneering idea of Landau.[7] However—and this crucial point has often been overlooked—even if only three instabilities are necessary, the asymptotic state of "fully developed fluid turbulence" still depends on an infinite number of degrees of freedom!

In addition to the butterfly effect, two additional key developments were necessary to make the "chaos" revolution. The first was the reduction of the scope of study to systems with a small number of degrees of freedom. The second was the discovery that under very general circumstances that quantitatively the same behavior could result—the celebrated Feigenbaum con-stant. This universality finally allowed for quantitative empirical tests of the theory. By the early 1980s these developments had led to what could properly be called the "chaos revolution."

Later Developments and Problems

The basic outlook provoked by the developments in chaos (that random-like behavior is "normal" and not pathological) is valid irrespective of the number of degrees of freedom of the system in question. The success of systems with a small number of degrees of freedom led to some bold prognostications, such as "junk your old equations and look for guidance in clouds' repeating patterns."[9] This fervor was unfortunately accompanied by a drastic restriction of the scope of chaos to meaning precisely deterministic systems with few degrees of freedom. The restriction, coupled with the development of new empirical techniques, led to a major focus on applications and a number of curious, if not absurd, claims.

It is perhaps easiest to understand these aberrations by considering the example of the climate system (fig. 2). Numerical modeling the climate has always been one of the great scientific challenges, if only because of the large (practically infinite) number of degrees of freedom assumed to be involved. However, when new chaos tools were applied to the data; it was even claimed[10] that only four degrees of freedom were required to specify the state of the climate." Attempts were even made to prove objectively from analysis of data that, in spite of appearances, random-like signals were in fact deterministic in origin.

These attempts were flawed at several levels, the most important of which is philosophical: the supposition that nature is (ontologically) either deterministic or random. In reality, the best that any empirical analysis could demonstrate was that specific deterministic models fit the data better (or worse) than specific stochastic ones.

The Alternative for Large Numbers of Degrees of Freedom Systems: Stochastic Chaos

We have argued that by the mid 1980s, the ancient idea of chaos had come to take on a very narrow and restrictive meaning, essentially characterizing deterministic systems with small numbers of degrees of freedom. The philosophy underlying its use as a model for complex geophysical, astrophysical, ecological, or sociological systems—each involving nonlinearly interacting spatial structures or fields—has two related aspects, each of which we argue are untenable. The first is the illogical inference that because deterministic systems can have random-like behavior, that random-like systems are best modeled as not random after all. The second is that the spatial structures which apparently involve huge variability and many degrees of free-dom spanning wide ranges of scale, can in fact be effectively reduced to a small finite number. In short, at a philosophical level, deterministic chaos is an attempt to resurrect Newtonian determinism.

In order to overcome the limitations of deterministic modeling, we note that the axiomatization of probability theory early in this century clarified the objective status of probabilities and made the idea that statistics is somehow an expression of ignorance rather outdated. We follow general usage in denoting such objective randomness as "stochastic." The fundamental characteristic of stochastic theories—models which distinguishes them from their deterministic counterparts is that they are defined on probability spaces (usually infinite dimensional), whereas their deterministic counterparts are only manageable on low dimensional spaces.

The stochastic chaos alternative for nonlinear dynamics with many degrees of freedom is now easy to state: contrary to Einstein's injunction that "God does not play dice," we seek to determine "how God plays dice" with large numbers of interacting components.[12]

Objections to Stochastic Chaos

Before continuing, we should briefly consider various objections to the use of stochastic theories/models.

Fundamental theories should be deterministic : Ever since statistical mechanics forced physicists to embrace stochasticity in a major physical theory, there has been an attempt to discount the significance of this fact by claiming that at least fundamental theories should be deterministic. However, since Quantum Mechanics itself admits a completely consistent stochastic interpretation,[13] it is hard to avoid the conclusion that stochastic the-ories can also be fundamental.

Causality requires determinism: Another reason for clinging to determinism is the common misconception that causality is identical to determinism, or equivalendy, that indeterminism implies a degree of acausality. This view has already been criticized by one of the founders of quantum mechanics. Max Born.[14] At a formal level, causality is nothing more than a specific type of objective determination or necessity, and, as emphasized by Bunge,[15] it by no means exhausts the category of physical determination that includes other kinds of lawful production/interconnection, including statistical determination.

Structures are evidence of determinism: Another common prejudice is the idea that the phenomenological identification of structures is a kind of signature of determinism, while the presence of variability without "interesting" structures is a symptom of noise. The inadequacy of this view of randomness is brought home by the (still little-known fact) that stochastic models can (in principle) explain the same phenomena; the key is a special kind of "stochastic chaos" involving a scale invariant symmetry principle in which a basic (stochastic) cascade mechanism repeats scale after scale after scale, from large to small scales, eventually building up enormous variability. The result, multi-fractal fields, is the subject of much of the remainder of this essay. The main point is that unlike classical noises, such stochastic processes specifically have extreme events called "singularities," which are strong enough to create structures. There is insufficient "self-averaging," the result is far from a featureless white noise.

Physical Arguments for Stochastic Chaos:
The Example of Turbulence

We have already noted that since stochastic processes are defined on infinite dimensional probability spaces, stochastic models are *a priori* the simplest whenever the number of degrees of freedom is large. In particular, we argue that stochastic chaos is particularly advantageous with respect to classical approaches when a nonclassical symmetry is present: scale invariance.

Consider the example of fluid turbulence. The basic dynamical, and deterministic ("Navier-Stokes"), equations of fluid motion have been known for nearly 150 years, yet the fundamental problem remains whole: how to reconcile the (violent) nonclassical turbulent statistics/structures with the equations. If only because of this relative lack of progress, turbulence must be counted among the most difficult problems in physics. The main difficulty is the presence of a very strong type of inhomogeneity called "intermittency." Not only does the "activity" of turbulence induce inhomogeneity, but the activity itself is inhomogeneously distributed. There are "puffs" of (active) turbulence inside "puffs" of (active) turbulence.[16] It should now be no surprise that the cascade paradigm provides a convenient framework to study this phenomenology, yielding concrete models and interesting conjectures. In particular, it is now increasingly clear that a general outcome of stochastic cascades are multi-fractals, as shown below.

Scale Invariance Symmetries and Cascades:
Cascades and Multifractals

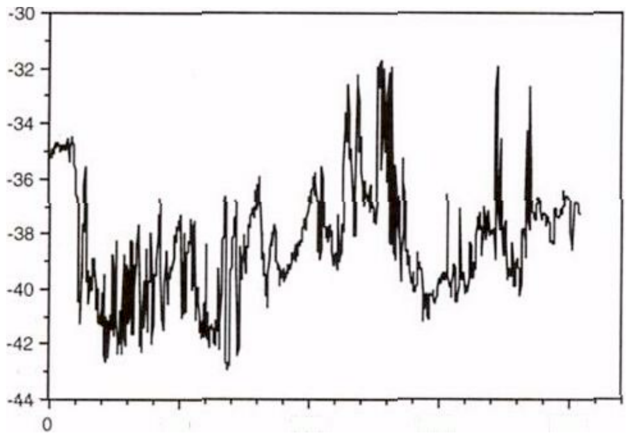
Scale invariant cascades have served as a paradigm of turbulence, at least since Richardson's celebrated poem in his 1922 book, *Weather Forecasting by Numerical Process:*

> *Big whorls have little whorls that feed on their velocity, and little whorls have smaller whorls, and so on to viscosity (in the molecular sense).. .*

Initially this cascade was nothing more than a conceptual scheme for explaining the transfer of energy from the planetary scales (where the pole/equation temperature gradient forces the large-scale circulation), down to the small scales (roughly one millimeter) where it is dissipated by viscosity. However, a key feature of the atmospheric circulation (and more generally of turbulence) is that it is far from being homogeneous; it is intermittent in both time and space. In order to model this intermittency, many explicit multiplicative cascade models were developed. The simplest ("beta model") is obtained by making the simplistic assumption that at each cascade step, the turbulence is either dead or alive. Since the same mechanism (the "coin tossing" to decide whether the daughter eddy is kept alive or is killed) is repeated unchanged, scale after scale, the process is scale invariant; in the small scale limit the "active" regions form a geometrical fractal set of points.

Ignoring for the moment the artificiality of the straight construction lines and the factor of two break-up of eddies into subeddies, we can now take a step towards realism by introducing a slight modification: we continue to flip coins, but now we multiplicatively "boost" or "decrease" the energy flux density rather than boosting or killing the eddies (the "alpha model"). The result is a multifractal field with an infinite number of levels of activity, depending on the sequence of boosts and decreases: the sets of points exceeding a given intensity form geometric fractal sets, except that the fractal dimension (i.e. the sparseness) of each fractal set decreases as the intensity level increases.

The "alpha model" is an example of what physicists call "toy models": they are simple without being simplistic; their generic consequences are quite nontrivial. In this case, the alpha model



50 100 150 200 Time (kyr BP)

Fig. 2 High resolution (2oo year average; δO18 record (parts per thousand of oxygen 18 compared to to oxygen 16/,Coming from the recent GRIP Greenland ice core; these values are believed to be roughly proportional to the past temperature. Sharp fluctuations occuring on small time scales are clearly visble; note that the sudden warming associated with the end of the last ice age (about 10.000 years ago, far left) is by no means ther only sudden transition. Schmitt et al shows that this series is scale invariant having statistics near the theoretically predicted universal multifractal type (Schmitt et al, 1995).

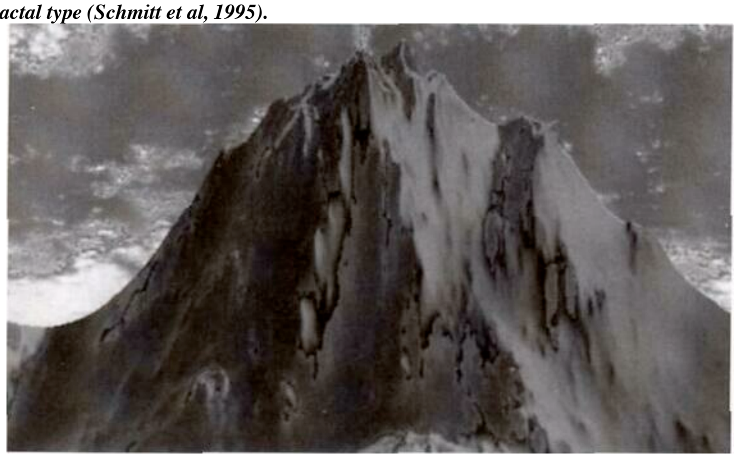


Fig. 3 Multifractal cloud and mountain from the exposition" "Strange Attractors Signs of Chaos," September 14 -; November 26. 1989. at the museum for contemporary art, New York (G. Sarma. J. Wilson).

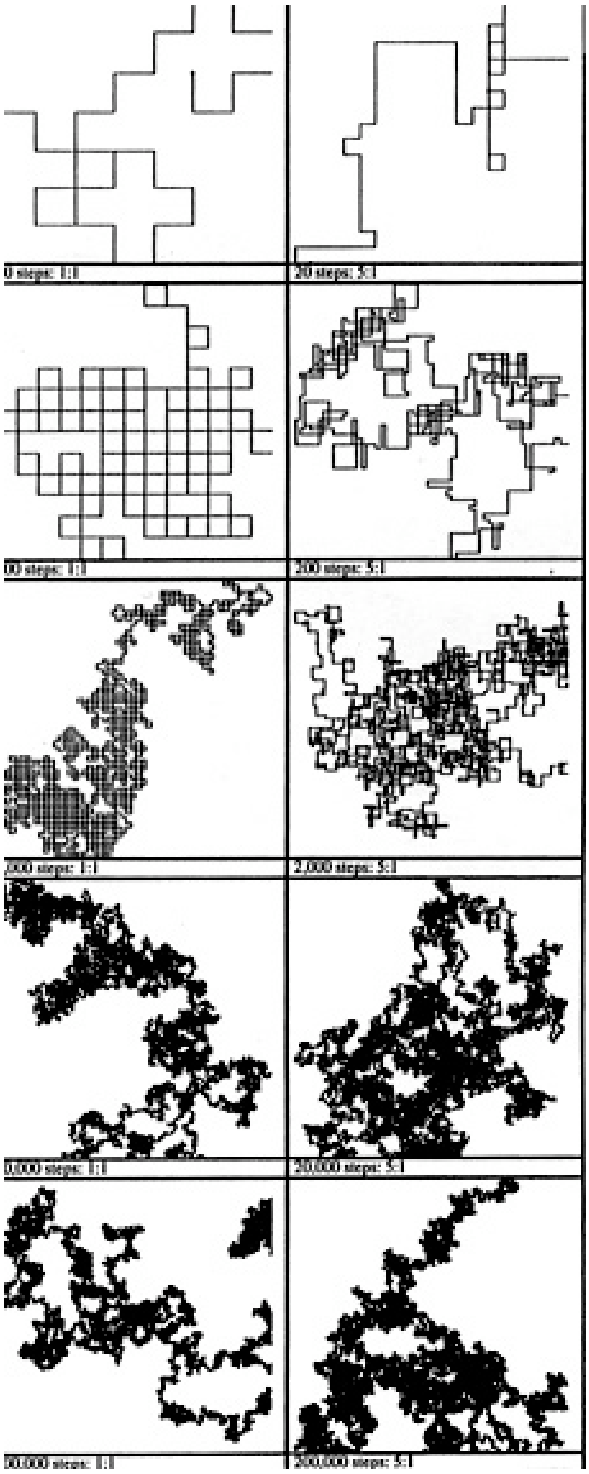
has all the essential ingredients of the more sophisticated models needed for realism. Specifically, the "continuous cascade" process eliminates the discrete (factor 2) scale ratio—and hence the ugly straight-line artifacts as well as the boost/decrease dichotomy. The most important point about continuous cascades is that they generically yield "universal multifractals," special multifractals which occur irrespective of the details of the basic dynamical mechanism and depend on only three basic parameters, just like random walks discussed earlier (fig. 4).[17]

The Multifractal (Stochastic) Butterfly Effect and Self-Organized Criticality

The significance of sensitive dependence on initial conditions, the "butterfly effect," is that if the system is sufficiently unsta-ble, then a small disturbance can grow, totally modifying the future state of the system. In the atmosphere this would mean that the sequence of weather events (which would include Texas tornadoes in Lorenz's metaphor) would be different, although presumably not the climate (which is a kind of ill-defined "average weather"). In our stochastic multifractal cascade model, we may identify an analogous "stochastic butterfly effect" by studying the small scale limit of the cascade and by determining under which conditions the small scale can dominate the large.

Contrary to "additive" stochastic chaos, such as Brownian motion, the turbulent activity around a given point is not changed by smaller and smaller amounts as cascades proceed to smaller and smaller scales; rather, it is modulated by random factors. It turns out that these lead to extreme events that are governed by nonstandard statistics characterized by algebraic (rather than exponential) probability tails, exactly the same type as those associated with avalanches and "self-organized critical phenomena."[18] However, contrary to Bak's original "sand pile model," which is a system nearly at thermodynamic equilibri-

Fi'g. 4 Universality of random variables illustrated by two random walks. Steps are chosen randomly to be up, down, left, or right with equal probability. On the left of each pair of columns, the steps are all of equal length, whereas on the right of each pair of columns they are occasionally five times longer (the lengths have been normalized so that the variances are the same). For a small number of steps, the walks are very dijferent, but for a large number, they tend to the same limit and look similar.


0 steps: 1:1
20 steps: 5:1
00 steps: 1:1
200 steps: 5:1
000 steps: 1:1
2,000 steps: 5:1
0,000 steps: 1:1
20,000 steps: 5:1
00,000 steps: 1:1
200,000 steps: 5:1

um, one has to consider a nonclassical self-organized criticality[19] in a stochastic framework. This is the multifractal/cascade version of the butterfly effect: most of the time, the flapping of the wings will lead to nothing special; the perturbation will be small compared to the existing large-scale weather structures. However, in all probability, the overall effect of the small-scale dynamics (which includes those small enough to be perturbed by a butterfly's flapping) will occasionally dominate the effect of the large-scale dynamics. This specific cascade prediction has been verified empirically in a dozen or so geophysical fields.

Nonclassical (Anisotropic) Zooms and Generalized Scale Invariance

Up to this point, we have considered scale invariance intuitively using the example of cascade processes, in which a simple mechanism is repeated scale by scale (coin tossing and multiplicative modulation). This mechanism is the same in all directions (isotropic); the resulting fractals and multifractals are therefore "self-similar" in the sense that a small piece when blown up using a standard isotropic "zoom" statistically resembles the whole. With the minor exception of "self-affinity," which involves squashing along a coordinate axis, self-similarity is the very special case discussed by Mandelbrot,[20] and in most of the fractal/multifractal literature. However, no natural system is exactly isotropic; many physical mechanisms exist that can introduce preferred directions, the most obvious being gravity. Gravity, for example, leads to a differentially stratified atmosphere: ocean and earth interior. Sources of anisotropy that can lead to differential rotation are the Coriolis force (due to the earth's rotation) or stresses (in fluids or rock) induced by external boundary conditions. Contrary to the conventional wisdom equating scale invariance with self-similarity, and hence with isotropy, scale invariance still survives, although the notion of scale undergoes a profound change. The resulting formalism of Generalized Scale Invariance (GSI) involves essentially two ingredients.[21] The first is the definition of a unit (reference) scale; the second is a family of scale-changing operators (i.e., rules) that describe how the unit scale is blown up or down. The fundamental restriction is that the rule should only involve ratios of scales so that there is no absolute notion of size (the reference scale is arbitrary). Mathematically, this implies that the scale changes form a mathematical group with a corresponding group generator. Perhaps the key factor to note is that the great differences in the appearance of the shape at different scales does not imply the existence of a characteristic scale whatsoever. Any member of the family could be taken as the unit shape and mapped onto any of the others simply by blowing up or down anisotropically by an appropriate ratio. Physically, in GSI the distribution of conserved (turbulent) fluxes determines the notion of size; this relation between scale and dynamics is analogous to that of General Relativity between the distribution of matter and energy and the metric.

The most interesting application of this is in the modeling and analysis of multifractals, for example clouds with various scale invariant generators (fig. 3). Changing the generator has the effect of changing cloud or mountain morphology/type. With GSI, we may no longer infer that such phenomenological differences in appearance necessarily correspond to differences in dynamics.

Symmetries and Dynamics

Having argued that even for a mathematically well defined deterministic problem such as hydrodynamic turbulence (the Navier-Stokes equations), the basic obstacle is an adequate treatment of the scale invariance symmetry: the "puffs within puffs." On the other hand, with practically no ingredients beyond this symmetry, stochastic cascades provide immediate insights: universal multifractals in which all the statistics are characterized by only three fundamental exponents. Just as a complete description of the dynamical equations is theoretically sufficient to specify the evolution of a system, a complete knowledge of the relevant symmetries will also suffice. This idea, applied to the turbulent cascade approach suggests that the latter would be equivalent to the usual deterministic approach if the remaining symmetries (i.e., other than scale invariance) of the fluid equations were known.

The Emergence of Stochastic Chaos from Determinism

Physics is an evolving hierarchy of interlocking theories. Statistical mechanics, at the same time, has accustomed us to the fact that methods and concepts necessary to understand large numbers of degrees of freedom (atoms/molecules in the latter) are qualititatively different from those that describe only a few. While we expect no contradiction between the theories that describe systems of a few and of many degrees of freedom (e.g., between classical mechanics and classical statistical mechanics), the idea that stochasticity is a significant "emergent quality" is not new.

In the frontier between low and high numbers of degrees of freedom, randomness emerges from determinism. Although hydrodynamic turbulence is an apparently mathematically well-defined deterministic problem, and few doubt the mathematical correctness of the equations (or their physical usefulness when only a few degrees of freedom are excited), they have been singularly unhelpful in elucidating the nature of turbulence with many degrees of freedom. A promising recent approach is a deterministic cascade model, the "Scaling Gyroscopes Cascade,"[22] which confirms with many new insights the general intuition that the chaos of turbulence is related to an infinite dimensional space (the infinite number of tops in SGC); it is also related to the complex interplay between determinism and randomness rather their simple opposition.

The Consequences: How Bright, How Hot, How Windy, is the Coast of Brittany?

We have argued that nonlinear scale invariant dynamics generically leads to a special type of stochastic chaos (universal multifractals) and that the existence of fractal structures is evidence for such multifractality. The basic feature of such scale invariant systems is that all their usual properties depend on the scale at which they are measured or observed, the exceptions being precisely the exponents such as the fractal dimensions which are scale invariant. Probably the first explicit recognition of the resolution dependence problem was by Perrin on the question of the tangent of the coast of Brittany;[23] the mathematically equivalent problem of the length was addressed in the 1950s by

Steinhaus for the left bank of the river Vistula. Richardson quantified this effect empirically by measuring the scaling exponents of the coast of Britain and of several frontiers using the "Richardson dividers" method.^ Shortly afterwards, in the 1967 celebrated paper, "How Long is the Coast of Britain?," Mandelbrot interpreted Richardson's scaling exponent in terms of fractal dimensions.[25] By the i98os, it had become generally accepted that the length of a coastline was primarily a function of the resolution of the map.

Turning to the example of the atmosphere, the relevant multifractal question is "how hot, how windy, how wet, how bright is the coast of Brittany?"[26] There is now growing evidence (especially of the multifractal nature of cloud brightness fields as measured by satellite) supporting widespread multifractality. Hence, the answer depends on the space-time resolution at which the temperature, wind speed, and rain rate is measured.

Limits to Predictability and Stochastic Forecasting

There are major differences between the way in which deterministic and stochastic systems are forecast. We have already outlined the usual approach for forecasting global weather, which involves solving the highly nonlinear governing equations starting from very limited initial data, an approach limited by the computer's inability to model structures smaller than several hundred kilometers. The scale invariance symmetry is broken: over the remaining factor of 100 million or so the atmosphere is assumed homogeneous. In contrast, stochastic forecasts respect the scale invariance symmetry and (statistically) take into account interactions over the entire range of about ten billion in scale.

Considering next the nature of the stochastic forecast, we find that it is conceptually quite different from its deterministic counterpart. The deterministic forecast attempts to predict the smallest detail far ahead in the future. In contrast, a stochastic forecast could be directy related to the probability of an event occurring, as well as the statistical reliability of that probability. While conventional deterministic forecasts have been the object of 40 years or more of scientific development, the corresponding stochastic forecasts are still in their infancy. They are nevertheless promising.[27]

Conclusions

The idea of chaos can be traced back to antiquity, in art, at least to Da Vinci. Ever since Newton provided the prototypical model of deterministic, regular, nonchaotic motion for the solar system, chaos has been a recurring theme in physics. Laplace and others elevated this Newtonian determinism to a pedestal: all physical law should aspire to its form, randomness and chance being disdainfully ascribed to ignorance (Voltaire's "chance is nothing"). However, as physical theory evolved to encompass more phenomena, physics was led to the introduction of physical laws featuring intrinsic randomness, starting with Maxwell's distribution of molecular velocities. Initially, this randomness was regarded as an expression of ignorance; in the twentieth century, however, probability theory was axiomatized to make clear that randomness can be objective, or "stochastic." While it seems only natural to model chaotic, randomlike behavior with stochastic models, this obvious step has faced resistance due to strong residual prejudices in favor of the Newtonian world view. With the discovery in the 1960s that even simple nonlinear deterministic systems with as few as three degrees of freedom (interacting components) could have randomlike chaotic behavior, unreconstructed Newtonian determinism experienced a revival; such "deterministic chaos" promised to provide a deterministic explanation of randomness, finally placing it within the Newtonian orbit. The study of nonregular, nonsmooth behavior became suddenly respectable; chaotic behavior was discerned in field after field and by the end of the 1970s it was rapidly becoming the norm. This fundamental change in world view is the enduring kernel of the "chaos revolution."

An unfortunate effect of the revolution was the restriction of the notion of chaos to systems with deterministic evolution laws; further developments led to a further important limitation. Initially, nonlinear systems with few degrees of freedom were studied as simplified caricatures of systems with many degree of freedom, especially fluid turbulence. However, following the explosion of developments in these simple systems, deterministic chaos became correspondingly restricted. By the 1980s the caricature was often mistaken for the reality, leading to a number of unfortunate attempts to explain many complex random-like systems (including the weather and climate) with only a handful of degrees of freedom.

In this essay we have argued that when many interacting components are present, as is typically the case in turbulent or turbulent-like systems involving nonlinearly interacting spatial structures/fields, a more appropriate paradigm is "stochastic chaos," objective random models involving probability spaces and an infinite number of degrees of freedom. Indeed, following the example of statistical mechanics, stochastic approaches are natural to use in high dimensional systems. Stochasticity is then a quality "emergent" from low dimensional deterministic theories as the latter are extrapolated to many degrees of freedom.

To date, the primary utility of stochastic chaos is the facility with which it enables us to exploit a nonclassical symmetry: scale invariance. Scale invariance is much richer than is usually supposed, providing for example, a potentially unifying paradigm for geophysics (and possibly astrophysics). Probably the most familiar examples of scale invariant objects are geometric fractal sets; however, fields such as the temperature, wind, and cloud brightness/density are much more interesting. These are multifractals, which are generically produced in cascade processes. Such cascades involve a dynamical generator which repeats scale after scale from large to small structures (in the atmosphere, eddies). In this way, it builds up tremendous (non-classical) variability, which need not be self-similar (isotropic, the same in all directions). Such anisotropic scale invariance requires the formalism of Generalized Scale Invariance (GSI) to define new (anisotropic) ways of"zooming"/"blowing up" structures. In GSI the system's dynamics determines the very notion of size. Although this has not yet occurred explicitly, we may expect that these new notions of size and scale will find both scientific and artistic expression.

Twenty years ago, the dominant scientific view was that most interesting physical systems were "smooth," "regular," and predictable. The *deterministic chaos* revolution has made such a view seem strangely antiquated; chaotic, unpredictable variability is now the norm. In spite of this undeniable change in viewpoint, there have been few decisive applications. We believe that this is because most interesting systems have many degrees of freedom; therefore, stochastic chaos combined with the scale invariance symmetry ("multifractals") may allow the chaos revolution to take a step forward by bringing large numbers of degrees of freedom systems into its purview.

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