The Fractional MacroEvolution Model:

A simple quantitative scaling macroevolution model

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Abstract:

Scaling fluctuation analyses of the marine animal diversity, extinction and origination rates based on the Paleobiology Database occurrence data have opened new perspectives on macroevolution, supporting the hypothesis that the environment (climate proxies) and life (extinction and origination rates) are scaling over the "megaclimate" biogeological regime (from ≈ 1 Myr to at least 400 Myrs). In the emerging picture, biodiversity is a scaling "cross-over" phenomenon being dominated by the environment at short time scales and by life at long times scales with a cross-over at ≈40Myrs. These findings provide the empirical basis for constructing the Fractional MacroEvolution Model (FMEM), a simple stochastic model combining destabilizing and stabilizing tendencies in macroevolutionary dynamics, driven by two scaling processes: temperature and turnover rates.

Macroevolution models are typically deterministic (albeit sometimes perturbed by random noises), and based on integer ordered differential equations. In contrast, the FMEM is stochastic and based on fractional ordered equations. Stochastic models are natural for systems with large numbers of degrees of freedom and fractional equations naturally give rise to scaling processes.

The basic FMEM drivers are fractional Brownian motions (temperature, T) and fractional Gaussian noises (turnover rates E_+) and the responses (solutions), are fractionally integrated fractional Relaxation processes (diversity (D), extinction (E), origination (O) and $E_- = O - E$). We discuss the impulse response (itself a model for impulse perturbation, e. g. bolide impacts) and derive the full statistical properties including cross covariances. By numerically solving the model, we verified the mathematical analysis and compared both uniformly and irregularly sampled model outputs to paleobiology series.

1. Introduction

Several centuries of paleontological research revealed that the evolution of Life on Earth is characterized by high temporal complexity characterized by periods of sluggish and predictable evolution (Jablonski, 1986; Casey et al., 2021) with mass extinctions characterized by selectivity that is low or different in kind than in "background intervals" (Raup, 1992a ; Raup, 1994 ; Payne & Finnegan, 2007). There are also mass evolutionary radiations which sometimes are contemporaneous with mass extinctions (Cuthill et al., 2020). Moreover apparently the factors and modes of macroevolution vary with time—e.g. Cambrian explosion or Ediacaran-Cambrian radiation and post-Cambrian evolution (Gould, 1990 ; Erwin, 2011 ; Mitchell et al., 2019); environment (Kiessling et al., 2010; Jablonski et al., 2006; Miller & Foote, 2009 ; Boyle et al., 2013; Spiridonov et al., 2015; Tomašových et al., 2015); and timescales (Crampton et al., 2018 ; Van Dam et al., 2006 ; Spiridonov et al., 2017b; Beaufort et al., 2022). Moreover macroevolution is strongly influenced by Earth system —geological, climatic, and paleoceanographic—factors (Marshall et al., 1982; Lieberman & Eldredge, 1996; Lieberman, 2003 ; Saupe et al., 2019; Halliday et al., 2020; Carrillo et al., 2020), but also by biotic interactions, which can translate into patterns which are apparent on extremely long time scales of tens to hundreds of millions of years (Vermeij, 1977 ; Jablonski, 2008 ; Erwin, 2012; Vermeij, 2019). Also, there are questions on the role of general stochasticity and path dependence/memory in evolutionary dynamics (Schopf, 1979; Hoffman, 1987; Erwin, 2011 ; Erwin, 2016 ; Gould, 2001; Gould, 2002 ; Cornette & Lieberman, 2004). The question is: can we reconcile in a single simple model this multitude of hierarchically organized and causally heterogenous processes producing macroevolutionary dynamics, while maintaining simplicity and conceptual clarity? Here we argue that we can.

The development of large, high temporal resolution databases – both of past climate indicators (Veizer et al., 1999 ; Song et al., 2019 ; Grossman & Joachimski, 2022) and of paleobiological information such as Paleobiology Database (Alroy et al., 2001 ; Alroy et al.,

2008) or NOW (Jernvall & Fortelius, 2002; Žliobaitė et al., 2017; Žliobaitė, 2022), is transforming our understanding of macroevolution. Time series are frequently long enough that they be studied systematically - not just as chronologies to be compared with other chronologies - but as functions of temporal *scale*, i.e. the behaviour of their fluctuations as functions of duration (or equivalently, their behaviour as functions of frequency). A regime over which fluctuations ΔT are scaling i.e. of the form $\Delta T(\Delta t) \propto \Delta t^H$ where Δt is duration - "lag", scale, and H is an exponent - can be used to objectively define dynamical regimes (this scaling relationship holds in a statistical sense discussed below). This is because over such a regime, long duration fluctuations at scale $\lambda\,\Delta t$ ($\lambda > 1$) are related to the shorter duration fluctuations by: $\Delta T(\lambda \Delta t) = \lambda^H \Delta T(\Delta t)$ i.e. the fluctuations at different time scales differ only in their amplitudes. In addition, we can already distinguish the qualitatively different types of regime by the sign of the exponent H. $H>0$ implies that fluctuations increase with scale whereas $H<0$) implies that they decrease.

An important consequence for our understanding of deep time biogeodynamics - here understood as joint Earth-Life systems - is the robustness of the "megaclimate" regime of positively scaling (a short hand for $H>0$) with time scale temperature fluctuations meaning that at longer time scales climates become more and more distinct, first (Lovejoy 2013), (Lovejoy 2015) on the basis of long paleotemperature data from ocean core stacks (Veizer et al. 2000), (Zachos et al. 2001). Megaclimate is the hypothesis that there is a unique (presumably highly nonlinear) biogeological dynamical regime that operates over time scales spanning the range $\approx$ 1 Myr to (at least) several hundred Myrs. This would be the consequence of a unique (albeit complex, nonlinear) underlying dynamic that is valid over this wide range of scales; presumably it involves a scaling (hence hierarchical) mechanism that operates from long to short durations. A consequence is the existence of a statistical scaling regimes (notably of paleo temperatures), empirically verified throughout the Phanerozoic. While its inner scale appears to be fairly robust at around 1 Myr, its outer scale (the longest duration over which it is valid) is not known although it appears to be at least 300 Myrs. The megaclimate regime implies that the underlying biology - climate dynamics are essentially the same over these time scales: i.e. that the statistics are stationary (although they may *appear* to be nonstationary at shorter time scales).

The hypothesis that biology and the climate are linked, and that climate is crucial and defining variable in ecological and evolutionary turnovers (Vrba, 1985 ; Vrba, 1993 ; Eldredge, 2003; Lieberman et al., 2007; Hannisdal & Peters, 2011; Mayhew et al., 2012 ; Crampton et al., 2016; Spiridonov et al., 2016; Spiridonov et al., 2017a; Spiridonov et al., 2020a; Spiridonov et al., 2020b; Mathes et al., 2021), is hardly controversial - after all - the paleoclimate indicators themselves are often based on stable oxygen isotopic analyses of $CaCO_3$ from ancient foraminifera (O'Brien et al., 2017) or sometimes for more recent periods estimated directly from occurrences and abundances of taxa using modern analog techniques (Dowsett & Robinson, 1998; Green, 2006). However, the scope and utility of the megaclimate notion would increase if it could be backed up by direct analysis of paleobiological series, particularly of extinction and origination rates. This has now been done. A recent paper (Spiridonov and Lovejoy 2022), hereafter SL) found that genus - level extinction and origination rates exhibited scaling statistics over roughly the same range as the paleo temperatures confirming that the megaclimate includes these key macroevolutionary parameters.

The shortest scale of SL's paleobiological time series was closer to ≈ 3 Myrs (average stage resolution was 5.9 Myrs) which correspond to the durations of shortest Paleobiology Database stages – a standard shortest time resolution for Phanerozoic scale global biodiversity analyses (e.g. (Alroy et al., 2008; Alroy, 2010b)). Systematic reviews and multiple case studies revealed that even variously defined (molecular, morphological, phylogenetic, and taxic) evolutionary rates universally exhibit negative time scaling behavior (Gingerich, 1993; Gingerich, 2001; Gingerich, 2009; Roopnarine, 2003; Harmon et al., 2021; Spiridonov & Lovejoy, 2022), which suggests the universality of the temporal scaling - hence hierarchical - evolutionary dynamics. Although an inner megaclimate scale of ≈ 1 Myrs was also proposed in (Lovejoy 2013), (Lovejoy 2015) and is discussed at length in the nonspecialist book (Lovejoy 2019). The scaling, and thus by implication dominance of time symmetric hierarchical interactions, was also detected on multimillion year time scales in sedimentation rates/stratigraphic architecture (Sadler, 1981), sea level (Spiridonov and Lovejoy, 2022), and dynamics of continental fragmentation (Spiridonov et al., 2022), which shows universality of the pattern in major Earth systems as well. Therefore, the time scaling patterns of evolution and megaclimate overlap at the very wide range of temporal scales (from $\approx 10^6$ to $> 4\times10^8$ yrs), which motivates the development of quantitative models which explicitly tackle and integrate together these time scale symmetries.

If macroevolution and climate respect wide range scaling, then it may be possible to resolve a longstanding debate in macroevolution. In terms first posed by (Van Valen 1973), we may ask: are evolutionary processes dominated by external factors - especially climate, the "Court Jester" (Barnosky, 2001; Benton, 2009) - or is life itself - the "Red Queen" (Van Valen, 1973 ; Finnegan et al., 2008) - the determining process. SL proposed a scaling resolution of the debate in which at scales below a critical transition time τ of ≈40 Myrs, the climate process is dominant, but there is a "cross-over" beyond which life (self-regulating by means of geodispersal and competition) are dominant. SL thus quantitatively concluded that at long enough time scales the Red Queen ultimately overcomes the Court Jester. The scaling processes of the Earth system here are playing double role (thus Geo-Red Queen theory) – climate fluctuations growing with time scale cause perturbations in diversity to grow in their size, but at the same time, at longer and longer time scales fluctuating climates and plate tectonics cause the mixing and competitive matching of biota, thus effectively globally synchronizing it. Later results in a described "cross-over" when ustable and wandering diversity regime changes to longer time scale fluctuation canceling or stabilizing regime (*Spiridonov and Lovejoy*, 2022).

Physicists use the term "cross-over", as a short-hand to describe analogous phenomena involving processes that are subdominant over one scale range but eventually become dominant at longer scales. However, such transitions are typically modelled by Markov processes so that the autocorrelations are exponential so that at the critical time scale, the transition between two regimes is fairly sharp. In SL, on the contrary, in keeping with the basic megaclimate scaling dynamics, the cross-over was postulated to be a the consequence of the interaction of two scaling processes i.e. the transition is a slow, power law one. An analogous scaling cross-over phenomenon was found in phytoplankton where the competing scaling processes were phytoplankton growth (with turbulence) and a predator-prey process of zooplankton grazing (Lovejoy et al. 2001).

SL argued that while both macro evolution and climate respect wide range statistical scaling, that their quantitative and qualitative differences are significant and this was the key to macroevolution power law cross-overs. While temperature (T) fluctuations vary with time scale Δt as $\Delta T(\Delta t) \approx \Delta t^{H_T}$ with $H_T \approx 0.25$, the corresponding laws for extinction (E) and origination (O) have H_E , $H_O \approx -0.25$. When $H > 0$, fluctuations grow with scale so that the corresponding series tend to "wander" without any tendency to return to a well-defined value, they appear "unstable". On the contrary, when $H < 0$, successive fluctuations tend to have opposite signs so that they increasingly cancel over longer and longer time scales, they fluctuate around a long term value, they appear stable.

To deepen our understanding, it is necessary to build a quantitative model of the interaction of climate and life. In recognition of the strongly nonlinear nature of evolutionary dynamics, there have developed numerous deterministic chaos models such as predator - prey models (e.g. (Huisman and Weissing 1999), (Caraballoa et al. 2016)). Although extensions with some stochastic forcing exist (e.g. (Vakulenko et al. 2018)), in the latter, the stochasticity is a perturbing noise on an otherwise deterministic system. In paleontology the model of exponential (unconstrained) proportional growth of diversity was historically popular (Stanley, 1979; Benton, 1995), or expanded for possible acceleration due to niche construction effects (second-order positive feedback) - a hyperbolic model (Markov and Korotayev, 2007). These simple models of expansion were contrasted by single or coupled logistic models of resource constrained competitive macroevolutionary dynamics, sometimes also supplemented with random perturbations which account for effects of mass extinctions (Sepkoski 1984; 1996); or implicitly hierarchical, and also competition constrained Gompertz models (Brayard et al., 2009), However, such models assume that only a few degrees of freedom are important (typically fewer than 10) whereas the true number is likely to be astronomical. It is therefore logical to model the process in a stochastic framework (involving infinite dimensional probability spaces), where the primary dynamics are stochastic using the scaling symmetry as a dynamical constraint. Therefore, there is growing recognition of stochastic models as essential tools for understanding macroevolutionary dynamics. Actually some of the first models that tried to explain complexities of macroevolutionary dynamics were stochastic Markovian birth and death models (Raup, 1985 ; Raup & Valentine, 1983 ; Gould et al., 1977 ; Raup, 1992a ; Nee, 2006). Several recent applications of linear stochastic differential equations were used in causal inference of macroevolutionary drivers and competitive interactions between clades (Reitan & Liow, 2017 ; Liow et al., 2015 ; Lidgard et al., 2021).

Beyond the realism of implicitly involving larger numbers of degrees of freedom, stochastic models have the advantage that they may be linear even though the corresponding deterministic model may be highly nonlinear. Also, by the simple expedient of using fractional ordered differential equations rather than the classical integer ordered ones, stochastic models can readily handle scaling which is rarely explicitly considered in macroevolutionary analyses. This is because fractional equations have impulse response functions (Green's functions) and hence solutions that are based on scaling (power laws) rather than the exponential Green's functions associated with integer ordered differential equations.

In this paper, we therefore build a simple model for biodiversity (*D*) that can reproduce and explain SL's findings. The model is parsimonious: has only two scaling drivers - the climate and life – and by construction - it reproduces the observed scaling cross-over at 40Myrs. Although the model has two basic exponents (climate and life) and two correlation coefficients, it satisfactorily reproduces the fluctuation statistics of *D*, *T*, *E*, *O* as well as the turnover ($E_+ = O + E$) and difference $E_- = O - E$ over the range ≈ 3 Myrs to several hundred Myrs (the longest scales available). Beyond this, it explains the 15 pairs of (scale by scale fluctuation correlations) over the same observed range. The data are from SL paper-they represent stage level time series of Phanerozoic marine animal genera O and E (second-for-third origination and extinction proportion (Kocsis et al., 2019 ; Alroy, 2015) not-standardized for the duration of stages), sample standardized using shareholder quorum method (Alroy, 2010a) D of Phanerozoic marine animals based on Paleobiology Database data (https://paleobiodb.org/). While paleotemperatures (T) are also the same as in the SL paper, obtained from (Song et al., 2019)

As a final comment, we should note that the basic – simplest - stochastic "cross-over" process is the fractionally integrated fractional relaxation noise (ffRn process) whose properties were only fully elucidated very recently (Lovejoy 2022) in the context of long term weather forecasts (Del Rio Amador and Lovejoy 2021) and climate projections (Procyk et al. 2022). The new model has conceptual commonalities with the environmental "stress model" of M. Newman that attempted to replicate the scaling statistics of extinction intensities of marine biota (Newman, 1997; Newman & Palmer, 2003). The model presented here is more sophisticated since it ties the principal macroevolutionary variables — O, and E — to a principal geophysical scaling process — the megaclimate — in producing realistic multi time scale global dynamics of marine animal biodiversity, while keeping its conceptual simplicity in transparently using a few crucial parameters of time scaling and correlations. The model also explicitly hierarchical through scaling relations – having a desirable feature of a unified evolutionary theory (Eldredge, 1985 ; Eldredge, 1989 ; Gould, 2002 ; Lieberman et al., 2007).

2. The model:

2.1. The equations:

2.1.1 The basic diversity equation

The SL picture is one where the extra-biological factors ("the climate") are scaling and drive biodiversity from ≈ 1Myr to ≈ 40 Myrs, where the cross-over occurs followed by the domination of biotic-regulation at the longer time scales, which also enabled by global homogenization of biota at long time scales by continental drift and changes in climate zones (Geo-Red Queen dynamics). Based on this picture, we propose the following Fractional Macro Evolution Model (FMEM). At first we describe the model, we then comment on it.

The basic diversity equation is:

$$\tau^h \frac{d^h\left(D - s_T T\right)}{dt^h} + D = s_E E_+ ; \qquad E_+ = O + E \qquad (1)$$

τ is the cross-over time scale (≈40 Myrs) and $E_+ = E + O$ is the turnover rate. Whereas D, E_+ are already nondimensional, T must be nondimensionalized, for example by the standard deviation of its increments at some convenient reference scale, say 1 Myr. s_T, s_E are constants that are determined by the coupling between T and D (s_T) and E_+ and D (s_E).

2.1.2 The drivers:

The basic drivers are the climate (T) and life (E_+), themselves driven by Gaussian white noises γ_T, γ_E:

$$\tau^{\alpha+h}\frac{d^{\alpha+h}T}{dt^{\alpha+h}}=\gamma_T$$
$$\tau^{\alpha}\frac{d^{\alpha}E_+}{dt^{\alpha}}=\gamma_E \qquad (2)$$

α is the basic biology (extinction and origination rate) exponent ($\alpha \approx 0.25$ as deduced from SL's analysis) and h is the exponent difference (contrast) between the temperature and biology, from SL's analysis $h = 0.75 - \alpha \approx 0.5$. Combined with the diversity equation (eq. 1), these determine D. The derivatives are fractional, in this paper we use the semi-infinite "Weyl" fractional derivatives. For the arbitrary function $W(t)$, the ζ ordered Weyl fractional derivative is defined as:

$$\frac{d^{\zeta}W}{dt^{\zeta}}=\frac{1}{\Gamma(1-\zeta)}\frac{d}{dt}\int_{-\infty}^{t}(t-s)^{-h}W(s)ds;\quad 0<\zeta<1 \qquad (3)$$

Since fractional derivatives (and their inverse, fractional integrals) are – as in eq. 3 - generally convolutions, different fractional operators are defined on different ranges of integration for the convolutions. Weyl derivative are particularly simple to deal with since they are simply power law filters in Fourier space, see below (see e.g. (Miller and Ross 1993), (Podlubny 1999) for more information on fractional equations).

The γ's are Gaussian white noises, they are proportional to "unit" white noises γ. Unit white noises have the properties:

$$\langle\gamma(t_1)\gamma(t_2)\rangle=\delta(t_1-t_2);\qquad \langle\gamma^2\rangle=1;\quad \langle\gamma\rangle=0 \qquad (4)$$

where the angle brackets indicate ensemble (statistical) averaging. Eq. 2 therefore implies that T, E_+ are fractional integrals of white noises. Depending on the value of the exponents), these are fractional Gaussian noises (fGns) and fractional Brownian motions (fBms), (Mandelbrot and Van Ness 1968) (see the later discussion on the small and large scale limits).

2.1.3 Closing the model, the Diagnostic Equation:

The preceding equations 1, 2 determine D, E_+, T. However, in order for the model to determine E and O, we need a final equation for E_-:

$$E_- = \tau_D \frac{dD}{dt}; \qquad E_- = O - E \tag{5}$$

This is just the differential form of the usual discrete - time definition of diversity: $D_{j+1} = D_j\left(1 + O_j - E_j\right)$ where j is a time index. τ_D is the discretization time, it is the basic resolution of the series. Equation 5 plays no role in the dynamics, conventionally, it is the definition of D. Mathematically, eq. 5 is thus a ”diagnostic equation” because it simply allows us to close (complete) the model by determining O, E:

$$\begin{aligned} O &= \left(E_+ + E_-\right)/2 \\ E &= \left(E_+ - E_-\right)/2 \end{aligned} \tag{6}$$

2.2 Discussion:

2.2.1 Diversity as a Fractionally Integrated Fractional Relaxation (ffRn) process

The diversity model was written in a nonstandard way (eq. 1) because in this form, it’s basic behaviour is transparent. When h>0, the fractional term is the highest order derivative, at high frequencies it therefore dominates the zero$^{\text{th}}$ order (D) term so that at short lags $\Delta t<\tau$, diversity fluctuations $\Delta D \propto \Delta T$ so that D follows the temperature. However at low frequencies ($\Delta t > \tau$), the zero$^{\text{th}}$ order term dominates and we have instead $\Delta D \propto \Delta E_+$. By inspection, the model therefore reproduces the cross-over at lag τ, and the crossover will be scaling due to the scaling of T, E_+ (eq. 2). The mathematical structure of the model is clearer if we substitute the driver in terms of their own Gaussian forcings γ_T, γ_E (eq. 2), rewriting eq. 1 as:

$$\tau^h \frac{d^h D}{dt^h} + D = \tau^{-\alpha} \frac{d^{-\alpha}}{dt^{-\alpha}}\left(s_T\gamma_T + s_E\gamma_E\right) \qquad (7)$$

($d^{-\alpha} / dt^{-\alpha}$ is a fractional integral order α: for Weyl derivative and integrals it is the inverse of the α order derivative d^{α} / dt^{α}).

The linear combination of white noises $s_T\gamma_T + s_E\gamma_E$ is also a white noise. The D equation, is thus an order h fractional relaxation equation forced by an order α fractionally integrated white noise, i.e. it is a "fractionally integrated fractional relaxation" process (ffRn, (Lovejoy 2022)). The basic "unit" ffRn process $U_{h,\alpha}(t)$ satisfies:

$$\left(\frac{d^{h+\alpha}}{dt^{h+\alpha}} + \frac{d^{\alpha}}{dt^{\alpha}}\right)U_{\alpha,h} = \gamma \qquad (8)$$

Where γ is the unit white noise defined above and we have used the fact that for Weyl fractional derivatives fractional differentiation and integration commute. If time is rescaled ($t \rightarrow t/\tau$), we see (from eq. 7) that D is proportional to $U_{\alpha,h}$. We note that if $h = 1$, the D equation (eq. 1) would be a classical relaxation equation and if forced by a white noise (i.e. if $\alpha = 0$), D would be a classical Ornstein-Uhlenbeck (OU) process. OU processes are currently conventional approaches to the modeling and analysis of microevolutionary as well as macroevolutionary dynamics (Khabbazian et al., 2016; Bartoszek et al., 2017 ; Liow et al., 2022).

2.2.2 Deterministic behaviour: Impulse response functions

The D process - the solution to eq. 7 - is the response of the operator $\left(\frac{d^{h+\alpha}}{dt^{h+\alpha}}+\frac{d^{\alpha}}{dt^{\alpha}}\right)$ to a white noise forcing. The general behaviour of responses to linear operators is determined by their impulse response (Green's) functions $G_{\alpha,h}$ that satisfy:

$$\left(\frac{d^{h+\alpha}}{dt^{h+\alpha}}+\frac{d^{\alpha}}{dt^{\alpha}}\right)G_{\alpha,h}=\delta(t) \tag{9}$$

(Lovejoy 2022)), where δ(t) is the Dirac ("delta") function. $G_{\alpha,h}$ can be expressed in terms of "generalized exponentials" or Mittag-Leffler functions $e_{h,h+\alpha}$ as:

$$G_{\alpha,h}(t)=\begin{cases} & ;\quad \alpha\geq 0;\quad 0\leq h\leq 2 \\ t^{h-1+\alpha}e_{h,h+\alpha}\left(-t^{h}\right)=t^{\alpha-1}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^{nh}}{\Gamma(\alpha+nh)}; & t\geq 0 \\ 0; & t<0 \end{cases} \tag{10}$$

$$e_{a,b}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(an+b)}$$

(Γ is the gamma function). At small t, the leading order term is therefore $G_{\alpha,h}(t)\approx\frac{t^{\alpha-1+h}}{\Gamma(\alpha+h)}$.

The large t (asymptotic) expansion is:

$$G_{\alpha,h}(t)=t^{\alpha-1}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{\Gamma(\alpha-nh)}t^{-nh};\quad t>>1 \tag{11}$$

(Podlubny 1999). Whereas the small t expansion is $t^{\alpha-1}$ times terms of positive powers of h, the large t expansion is in terms of $t^{\alpha-1}$ times terms in negative powers of h, with leading term $G_{\alpha,h}(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$. Unless $h=0$, $G_{\alpha,h}(t)$ therefore transitions between two different power laws.

The special case $h=0$ that applies to the temperature and turnover forcings (eq. 2), corresponds to the pure power law $G_{\alpha,0}(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$. $G_{\alpha,h}$ has the property that if it is (fractionally) integrated ζ times, the result is just $G_{\alpha+\zeta,h}$. As explained in appendix A, $G_{\alpha,h}$ is useful for numerical simulations.

At a typical highest resolution of global datasets of 1Myr time scales, for example a bolide strike (During et al., 2022; Alvarez et al., 1980), supernova or gamma ray burst (Fields et al., 2020), or even much slower hyperthermal event such as PETM (McInerney & Wing, 2011 ; Gingerich, 2006) or Cenomanian-Turonian Event (Eaton et al., 1997 ; Meyers et al., 2012 ; Venckutė-Aleksienė et al., 2018) is effectively an impulse, so that $G_{\alpha,h}(t)$ could be considered as the response to a short time scale stressor such as meteorite or asteroid impact or extensive volcanic eruption episode. This impulse response property is desirable, since the global stratigraphic stages and substages are defined based on the episodes of turnover, which implies that at the measurement scales of million years or more, most of turnover is intermittent—near instantaneous or impulse-like (Foote, 2005 ; Foote, 1994). Figs. 1, 2 show the impulse response functions for the empirical parameters estimated in SL ($\alpha \approx 0.25$, $h \approx 0.5$). Since the equations are linear, these impulse responses will be superposed onto the stochastic white noise driven responses. We could remark that the power law decay of the impulse responses in much slower than that of conventionally assumed exponential decays. This means that our model predicts that there are long term impacts of bolide catastrophic events. This is in accord – for example - with the findings of (Krug et al., 2009; Krug & Jablonski, 2012), that the K-Pg mass extinction caused by the effects of Chixulub asteroid impact changed long-term origination rates and their spatial distribution, that persists today, 66 million years after the event, in accord with this long memory feature of the FMEM model.

2.3. Solving the model

Fractional derivatives are generally convolutions (with power laws, eq. 3) and therefore according to the range of integration of the convolution the fractional derivatives and integrals will be different. Different convolution ranges therefore correspond to different definitions of fractional derivatives. Most often (e.g. the Riemann-Liouville and Caputo fractional derivatives), the latter are taken from time = 0 to t in which case the initial conditions are important and dealing with them is technically somewhat complex. In these cases, the main tool is the Laplace transform.

Here however, we consider statistically stationary white noise forcing that starts at time $= -\infty$. In this case, we can use the "Weyl" fractional derivative (a convolution from $-\infty$ to t, eq. 3) whose Fourier transform ("F.T.") is particularly simple:

$$\frac{d^h}{dt^h} \overset{F.T.}{\leftrightarrow} (i\omega)^h \tag{12}$$

If we Fourier transform (denoted with a tilde), equations 1, 2, we can write the model in matrix form as:

$$\tilde{\underline{S}}(\omega) = (i\omega\tau)^{-\alpha}\, \tilde{\underline{\underline{F}}}(\omega) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s_T & s_E \end{pmatrix} \begin{pmatrix} \tilde{\gamma}_T \\ \tilde{\gamma}_E \end{pmatrix}$$

$$\tilde{\underline{S}}(\omega) = \begin{pmatrix} \tilde{T} \\ \widetilde{E_+} \\ \tilde{D} \end{pmatrix}$$

$$\underline{\underline{F}}(\omega) = \begin{pmatrix} (i\omega\tau)^{-h} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{1+(i\omega\tau)^h} \end{pmatrix} \tag{13}$$

(the single underline indicates a vector, the double underline, a matrix).

As noted above, the D forcing is a linear combination of white noises (eq. 7), so that the sum on the RHS of eqs. 7, 13 is a correlated white noise. However, from the data (see fig. 4), we see that E_+, T are themselves correlated. We therefore rewrite the model in terms of two statistically independent ($\langle \gamma_1 \gamma_2 \rangle = 0$) unit ($\langle \gamma_1^2 \rangle = \langle \gamma_2^2 \rangle = 1$) white noise drivers γ_1, γ_2:

$$\begin{pmatrix} \gamma_T \\ \gamma_E \end{pmatrix} = \begin{pmatrix} \sigma_T & 0 \\ 0 & \sigma_E \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \rho_E & \sqrt{1-\rho_E^2} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \tag{14}$$

So that:

$$\sigma_T^2 = \langle \gamma_T^2 \rangle; \quad \sigma_E^2 = \langle \gamma_E^2 \rangle; \quad \rho_E = \frac{\langle \gamma_T \gamma_E \rangle}{\sigma_T \sigma_E}; \quad \langle \gamma_T \rangle = \langle \gamma_E \rangle = 0 \quad . \tag{15}$$

Where σ_T is the standard deviation of γ_T, σ_E of γ_E and ρ_E is the T, E_+ correlation. Eq. 14 is the standard Cholesky decomposition of correlated random variables, noises.

Fourier transforming eq. 14 and using eq. 13, we can write the model as:

$$\underline{\tilde{S}}(\omega) = (i\omega\tau)^{-\alpha} \underline{\underline{F}}(\omega) \underline{\underline{\sigma}} \underline{\underline{\rho}} \underline{\tilde{\gamma}} \tag{16}$$

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_T & 0 & 0 \\ 0 & \sigma_E & 0 \\ 0 & 0 & \sigma_D \end{pmatrix}; \qquad \underline{\underline{\rho}} = \begin{pmatrix} 1 & 0 \\ \rho_E & \sqrt{1-\rho_E^2} \\ \rho_D & \operatorname{sgn}(r)\sqrt{1-\rho_D^2} \end{pmatrix}; \quad \underline{\tilde{\gamma}} = \begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \end{pmatrix}$$

Where the parameters:

$$\sigma_D = s_T \sigma_T \sqrt{1 + 2\rho_E r + r^2}; \quad r = \frac{s_E \sigma_E}{s_T \sigma_T}$$
$$\rho_D = \frac{1 + r\rho_E}{\sqrt{1 + 2r\rho_E + r^2}} \tag{17}$$

depend on both the driver statistics (σ_T, σ_E and ρ_E) and the model parameters s_T, s_E. While σ_D does parametrize the amplitude of the diversity fluctuations, unlike σ_T, σ_E (that must be ≥ 0), it is not a true standard deviation: if s_T<0 it will be negative. Similarly, we will see that ρ_D determines the D, E_+ and D, T correlations but is not itself a correlation coefficient and it depends on the sign of the ratio r.

2.3 Stochastic response to white noise forcing

2.3.1 Scaling processes: fGn, fBm

We are interested in the statistical properties of the solutions $\underset{\sim}{\widetilde{S}}(\omega)$. These can be expressed in terms of fGn, fBm and ffRn (fractionally integrated fractional Relaxation noises) processes. Before discussing the full statistics that includes the cross correlations, let us therefore discuss their statistics.

Let us start with the scaling processes T, E_+ that are of the form:

$$\frac{d^{\alpha_X} X}{dt^{\alpha_X}} = \gamma \overset{F.T.}{\leftrightarrow} (i\omega)^{\alpha_X} \widetilde{X} = \widetilde{\gamma} \qquad (18)$$

For the statistics, we can determine the power spectrum:

$$E_X(\omega) = \left\langle \left|\widetilde{X}\right|^2 \right\rangle = \frac{1}{2\pi}|\omega|^{-\beta_X}; \quad \beta_X = 2\alpha_X \qquad (19)$$

Where β_X is the spectral exponent and we have used the fact that the spectrum of a Gaussian white noise is flat:

$$\left\langle \left|\widetilde{\gamma}(\omega)\right|^2 \right\rangle = \frac{1}{2\pi}\left\langle \gamma^2 \right\rangle = \frac{1}{2\pi} \qquad (20)$$

$E_X(\omega)$ is thus the basic form of the T, E_+ spectra. From the Wiener-Khintchin theorem, the (real space) autocorrelation function $R_X(\Delta t)$ is the inverse transform:

$$R_X(\Delta t) = \left\langle X(t)X(t-\Delta t)\right\rangle \propto \Delta t^{H_X} \overset{F.T.}{\leftrightarrow} \widetilde{R}_X(\omega) = E_X(\omega) = \left\langle \left|\widetilde{X(\omega)}\right|^2 \right\rangle \propto |\omega|^{-\beta_X}; \quad H_X = \frac{\beta_X - 1}{2} = \frac{\alpha_X}{2} \qquad (21)$$

The technical difficulty is that due to a low frequency divergence, the inverse transform of pure power spectra (eq. 19) only converges for $\beta_X < 1$ (i.e. $\alpha_X < ½$, $H_X < 0$); this is the fGn regime appropriate for E_+. Even here, $R_X(\Delta t)$ is infinite for $\Delta t = 0$. Since $R_X(0)$ is the variance, fGn processes are (like the white noise special case $\alpha_X = 0$) generalized functions that must be averaged (integrated) over finite intervals in order to represent physical processes. Averaging to yield a finite resolution process is adequate for $\beta_X > -1$ ($\alpha_X > -1/2$, $H_X > -1$) so that the fGn process is defined for $-1 < \beta_X < 1$ (i.e. $-1/2 < \alpha_X < ½$, $-1 < H_X < 0$). After averaged over a finite resolution τ_r: X_{τ_r} with the result $\left\langle X_{\tau_r}^2 \right\rangle^{1/2} \propto \tau_r^{H_x}$ and since $H_x<0$ the data will be highly sensitive to the resolution τ_r.

When $\alpha_X \geq ½$, the low frequency divergences imply that the $X(t)$ process is nonstationary (the process generally "wanders off to plus or minus infinity). However, for $1 < \beta_X < 3$ (i.e. $½ < \alpha_X < 3/2$, $0 < H_X < 1$; this is the range appropriate for T: $H_T \approx 0.25$, $\beta_T \approx 3/2$), it's increments are (stationary) fGn processes, this regime defines the fBm process. Finally, since all physical scaling processes exist over finite ranges of scale, there will be finite outer (longest) time scale (smallest frequency) so that truncating the spectrum at low frequencies (as for the ffRn processes, see below) leads to an overall stationary process.

When analysing paleo series, it is convenient to analyze the statistics in real space, the main reason being that these are easier to interpret (the difficulty in interpretation is the cause of the quadrillion error in climate temperature spectra that was only recently discovered (Lovejoy 2015)). An additional reason is that paleo series are typically not available at uniform sampling / averaging intervals making the spectrum more difficult to estimate.

We have already noted that the autocorrelation functions are only adequate for $H_X < 0$ ($\alpha_X < 1/2$, $\beta_X < 1$), this is why when $0<H_x<1$, it is conventional to define fluctuations using differences $\Delta X(\Delta t) = X(t- \Delta t) - X(t)$, which are stationary over this range. Differences avoid low frequency divergences but will still have high frequency divergences when $H_X<0$. In order to avoid the problems at both small scale (resolution dependencies) and at large scales (nonstationarity), it is convenient to use Haar fluctuations. Over the interval Δt the Haar fluctuation $\Delta X(\Delta t)$ is defined as the difference between the average of the first and second halves of the interval.

$$\left\langle \Delta X(\Delta t)^2 \right\rangle^{1/2} \propto \Delta t^{H_X} \leftrightarrow E_X(\omega) \propto \omega^{-\beta_X}; \quad \begin{array}{c} -1 < H_X < 1 \\ -1 < \beta_X < 3 \\ \beta_X = 2H_X + 1 \end{array} \tag{22}$$

(valid for Haar fluctuations). Over the indicated range of parameters, the Haar fluctuations are stationary and are independent of the resolution.

Comparing eq. 7 and 2 we find:

$$\begin{aligned} &\left\langle \Delta E_+(\Delta t)^2 \right\rangle^{1/2} \propto \Delta t^{H_E}; \quad H_E = \alpha - \frac{1}{2} \\ &\left\langle \Delta T(\Delta t)^2 \right\rangle^{1/2} \propto \Delta t^{H_T}; \quad H_T = h + \alpha - \frac{1}{2} \end{aligned} \tag{23}$$

2.3.2 Two scaling regimes: fRn, ffRn

From eq. 8, 9, the basic Fourier transforms of ffRn processes and their impulse responses are:

$$\widetilde{U}_{\alpha,h}(\omega) = \frac{\widetilde{\gamma}}{(i\omega)^{\alpha}\left(1+(i\omega)^{h}\right)}; \quad \widetilde{G}_{\alpha,h}(\omega) = \frac{1}{(i\omega)^{\alpha}\left(1+(i\omega)^{h}\right)}; \quad 0 < \alpha < 1/2; \quad 0 < h < 2$$

$$\tag{24}$$

The fractional Relaxation noise (fRn) process is the special case where $\alpha = 0$. The ffRn power spectrum is therefore:

$$E_{\alpha,h}(\omega) = \left\langle \left| \widetilde{U}_{\alpha,h} \right|^2 \right\rangle = \frac{1}{2\pi |\omega|^{2\alpha} \left| 1 + (i\omega)^h \right|^2} \tag{25}$$

$E_{\alpha,h}(\omega)$ is thus the basic form of the D spectrum.

The full statistical properties of ffRn processes (including series expansions) are discussed in (Lovejoy 2022), however for our purposes, the low and high frequency scaling exponents are sufficient. For these, eq. 25, yields:

$$E_{\alpha,h}(\omega) \propto |\omega|^{-\beta}; \qquad \begin{array}{ll} \beta_l = 2\alpha; & \omega << 1 \\ \beta_h = 2(\alpha + h); & \omega >> 1 \end{array} \tag{26}$$

("h" for high frequency, "l" for low frequency). In order to obtain the basic fluctuation statistics, it is sufficient to apply eq. 22 to each regime separately. Indeed, more generally, "Tauberian theorems" (e.g. (Feller 1971)) imply that if the spectrum is a power law over a wide enough range, then the corresponding (second order) real space statistics will also be scaling. Therefore:

$$\left\langle \Delta U_{\alpha,h}(\Delta t)^2 \right\rangle^{1/2} \begin{array}{lll} \propto \Delta t^{H_l}; & H_l = \alpha - \frac{1}{2}; & \Delta t >> 1 \\ \propto \Delta t^{H_h}; & H_h = \alpha + h - \frac{1}{2}; & \Delta t << 1 \end{array} \tag{27}$$

Using the empirical values $\alpha \approx 0.25$, $h \approx 0.5$, we see E_+ is a fractional Gaussian noise and T is an fBm process. Also, we find (c.f. eqs. 7, 27) that $H_D \approx H_T$ ($\Delta t \ll \tau$) and $H_D \approx H_E$ ($\Delta t \gg \tau$).

2.4. The full model statistics: spectra, correlations:

2.4.1 The basic model:

The model is linear and has stationary Gaussian (white noise) forcing (T, E_+), therefore D, E_-, E, O are also Gaussian so that their statistics are determined by spectra and cross-spectra – or equivalently in real space (via the Wiener-Khintchin theorem), by the autocorrelations and cross-correlations:

$$R_{ij}(\Delta t) = \left\langle S_i(t) S_j(t-\Delta t)\right\rangle \overset{F.T.}{\leftrightarrow} \tilde{R}_{ij}(\omega) = \left\langle \tilde{S}_i(\omega)\tilde{S}_j^*(\omega)\right\rangle \qquad (28)$$

(the diagonal terms are the spectra of the components: $\tilde{R}_{ii}(\omega) = E_i(\omega)$). In matrix notation:

$$\begin{aligned}\underline{\underline{\tilde{R}}}(\omega) &= \left\langle \underline{\tilde{S}}\,\underline{\tilde{S}}^{T*}\right\rangle = |\omega\tau|^{-2\alpha}\, \underline{\underline{F}}(\omega)\underline{\underline{\sigma}}\,\underline{\underline{\rho}}\left\langle \underline{\tilde{\gamma}}\,\underline{\tilde{\gamma}}^{T*}\right\rangle \underline{\underline{\rho}}^{T*}\underline{\underline{\sigma}}^{T*}\underline{\underline{F}}(\omega)^{T*} \\ &= \frac{|\omega\tau|^{-2\alpha}}{2\pi}\underline{\underline{F}}(\omega)\underline{\underline{\sigma}}\,\underline{\underline{\rho}}\,\underline{\underline{\rho}}^{T*}\underline{\underline{\sigma}}^{T*}\underline{\underline{F}}(\omega)^{T*}\end{aligned} \qquad (29)$$

Where we have used:

$$\left\langle \underline{\tilde{\gamma}}\cdot\underline{\tilde{\gamma}}^{*T}\right\rangle = \left\langle \begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \end{pmatrix}\begin{pmatrix} \tilde{\gamma}_1 & \tilde{\gamma}_2 \end{pmatrix}\right\rangle = \frac{1}{2\pi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2\pi}\underline{\underline{1}} \qquad (30)$$

The key correlation matrix (from eq. 16) is:

$$\underline{\underline{\rho}}\,\underline{\underline{\rho}}^{T*} = \begin{pmatrix} 1 & \rho_{TE} & \rho_{TD} \\ \rho_{TE} & 1 & \rho_{ED} \\ \rho_{TD} & \rho_{ED} & 1 \end{pmatrix} \qquad (31)$$

Where

$$\rho_{TE} = \rho_E; \quad \rho_{TD} = \rho_D; \quad \rho_{ED} = \rho_E\rho_D + \mathrm{sgn}(r)\sqrt{1-\rho_E^2}\sqrt{1-\rho_D^2} \qquad (32)$$

and

$$\underline{\underline{\sigma}}\,\underline{\underline{\rho}}\,\underline{\underline{\rho}}^{T*}\,\underline{\underline{\sigma}}^{T*} = \begin{pmatrix} \sigma_T^2 & \rho_{TE}\sigma_T\sigma_E & \rho_{TD}\sigma_D\sigma_T \\ \rho_{TE}\sigma_T\sigma_E & \sigma_E^2 & \rho_{DE}\sigma_E\sigma_D \\ \rho_{TD}\sigma_D\sigma_T & \rho_{DE}\sigma_E\sigma_D & \sigma_D^2 \end{pmatrix} \tag{33}$$

2.4.2 Closing the model: the diagnostic equation for E_-:

Before writing down the final spectra, let's close the system with the help of the diagnostic equation that allows us to determine E_- from D (and hence E, O, eq. 6).

The Fourier transform of the diagnostic equation (eq. 5) is:

$$\widetilde{E}_- = \left(\frac{\tau_D}{\tau}\right)(i\omega\tau)\widetilde{D} \tag{34}$$

Therefore the full system is:

$$\begin{pmatrix} \widetilde{T} \\ \widetilde{E}_+ \\ \widetilde{D} \\ \widetilde{E}_- \end{pmatrix} = (i\omega\tau)^{-\alpha} \begin{pmatrix} (i\omega\tau)^{-h} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{1+(i\omega\tau)^h} & 0 \\ 0 & 0 & 0 & \dfrac{i\omega\tau}{1+(i\omega\tau)^h} \end{pmatrix} \begin{pmatrix} \sigma_T & 0 & 0 & 0 \\ 0 & \sigma_{E_+} & 0 & 0 \\ 0 & 0 & \sigma_D & 0 \\ 0 & 0 & 0 & \dfrac{\tau_D}{\tau} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \rho_E & \sqrt{1-\rho_E^2} \\ \rho_D & \mathrm{sgn}(r)\sqrt{1-\rho_D^2} \\ \rho_D & \mathrm{sgn}(r)\sqrt{1-\rho_D^2} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \tag{35}$$

From this we can find E, O:

$$\begin{aligned} \widetilde{E} &= \frac{1}{2}\left(\widetilde{E}_+ - \widetilde{E}_-\right) \\ \widetilde{O} &= \frac{1}{2}\left(\widetilde{E}_+ + \widetilde{E}_-\right) \end{aligned} \tag{36}$$

The explicit formulae for $E_\pm$ are:

$$\begin{aligned} \widetilde{E}_+ &= (i\omega\tau)^{-\alpha}\left[\sigma_{E_+}\left(\rho_E\widetilde{\gamma}_1 + \sqrt{1-\rho_E^2}\,\widetilde{\gamma}_2\right)\right] \\ \widetilde{E}_- &= (i\omega\tau)^{-\alpha}\frac{i\omega\tau_D}{1+(i\omega\tau)^h}\left(\rho_D\widetilde{\gamma}_1 + \mathrm{sgn}(r)\sqrt{1-\rho_D^2}\,\widetilde{\gamma}_2\right) \end{aligned} \tag{37}$$

The overall final statistics are:

$$\underline{\underline{\widetilde{R}}}(\omega)=\begin{pmatrix} \left\langle\left|\widetilde{T}\right|^2\right\rangle & \left\langle\widetilde{T}\widetilde{E_+^*}\right\rangle & \left\langle\widetilde{T}\widetilde{D^*}\right\rangle & \left\langle\widetilde{T}\widetilde{E_-^*}\right\rangle \\ \left\langle\widetilde{E_+}\widetilde{T^*}\right\rangle & \left\langle\left|\widetilde{E_+}\right|^2\right\rangle & \left\langle\widetilde{E_+}\widetilde{D^*}\right\rangle & \left\langle\widetilde{E_+}\widetilde{E_-^*}\right\rangle \\ \left\langle\widetilde{D}\widetilde{T^*}\right\rangle & \left\langle\widetilde{D}\widetilde{E_+^*}\right\rangle & \left\langle\left|\widetilde{D}\right|^2\right\rangle & \left\langle\widetilde{D}\widetilde{E_-^*}\right\rangle \\ \left\langle\widetilde{E_-}\widetilde{T^*}\right\rangle & \left\langle\widetilde{E_-}\widetilde{E_-^*}\right\rangle & \left\langle\widetilde{E_-}\widetilde{D^*}\right\rangle & \left\langle\left|\widetilde{E_-}\right|^2\right\rangle \end{pmatrix}$$

$$=\left|\omega\tau\right|^{-2\alpha}\begin{pmatrix} \left|\omega\tau\right|^{-2h}\sigma_T^2 & \left(i\omega\tau\right)^{-h}\rho_{TE}\sigma_T\sigma_E & \dfrac{\rho_{TD}\sigma_D\sigma_T}{\left(i\omega\tau\right)^h\left(1+\left(-i\omega\tau\right)^h\right)} & \dfrac{\rho_{TD}\tau_D\sigma_T\left(-i\omega\tau\right)^{1-h}}{\tau\left(1+\left(i\omega\tau\right)^h\right)} \\ \left(-i\omega\tau\right)^{-h}\rho_{TE}\sigma_T\sigma_E & \sigma_E^2 & \dfrac{\rho_{ED}\sigma_E\sigma_D}{\left(1+\left(-i\omega\tau\right)^h\right)} & \dfrac{\rho_{ED}\tau_D\sigma_E\left(-i\omega\tau\right)}{\tau\left(1+\left(i\omega\tau\right)^h\right)} \\ \dfrac{\rho_{TD}\sigma_D\sigma_T}{\left(-i\omega\tau\right)^h\left(1+\left(i\omega\tau\right)^h\right)} & \dfrac{\rho_{ED}\sigma_E\sigma_D}{\left(1+\left(i\omega\tau\right)^h\right)} & \dfrac{\sigma_D^2}{\left|1+\left(i\omega\tau\right)^h\right|^2} & \dfrac{\sigma_D\tau_D\left(-i\omega\tau\right)}{\tau\left|1+\left(i\omega\tau\right)^h\right|^2} \\ \dfrac{\rho_{TD}\tau_D\sigma_T\left(i\omega\tau\right)^{1-h}}{\tau\left(1+\left(i\omega\tau\right)^h\right)} & \dfrac{\rho_{ED}\tau_D\sigma_E\left(i\omega\tau\right)}{\tau\left(1+\left(i\omega\tau\right)^h\right)} & \dfrac{\sigma_D\tau_D\left(i\omega\tau\right)}{\tau\left|1+\left(i\omega\tau\right)^h\right|^2} & \dfrac{\tau_D^2\left|\omega\tau\right|^2}{\tau^2\left|1+\left(i\omega\tau\right)^h\right|^2} \end{pmatrix}$$

(38)

Using eqs. 36, 37, the spectra of *E, O* can be determined:

$$\begin{aligned} \left\langle\left|\widetilde{E}\right|^2\right\rangle &= \frac{1}{4}\left(\left\langle\left|\widetilde{E_+}\right|^2\right\rangle+\left\langle\left|\widetilde{E_-}\right|^2\right\rangle-2\left\langle\widetilde{E_+}\widetilde{E_-^*}\right\rangle\right)\approx\frac{1}{4}\left\langle\left|\widetilde{E_+}\right|^2\right\rangle \\ \left\langle\left|\widetilde{O}\right|^2\right\rangle &= \frac{1}{4}\left(\left\langle\left|\widetilde{E_+}\right|^2\right\rangle+\left\langle\left|\widetilde{E_-}\right|^2\right\rangle+2\left\langle\widetilde{E_+}\widetilde{E_-^*}\right\rangle\right)\approx\frac{1}{4}\left\langle\left|\widetilde{E_+}\right|^2\right\rangle \end{aligned} \qquad (39)$$

The far right approximation can be seen from eq. 37 using the fact that τ_D is the resolution of the series so that for the full range of empirically accessible frequencies, we have $\omega\tau_D<1$. In addition, since $\tau>\tau_D$, the factor $\left|\omega\tau_D/\left(1+\left(i\omega\tau\right)^h\right)\right|<<1$.

3. The properties of the model

3.1 Scaling properties

3.1.1 High and low frequency exponents

In order to interpret the statistics (eqs. 38, 39) in real space, it suffices to use the fact that Fourier scaling implies real space scaling and to use the above relations between real space and Fourier scaling exponents (eq. 22). In matrix form, the spectral exponents are therefore:

$$\beta_h = \begin{pmatrix} 2(\alpha+h) & 2\alpha+h & 2(\alpha+h) & 2(\alpha+h)-1 \\ 2\alpha+h & 2\alpha & 2\alpha+h & 2\alpha+h-1 \\ 2(\alpha+h) & 2\alpha+h & 2(\alpha+h) & 2(\alpha+h)-1 \\ 2(\alpha+h)-1 & 2\alpha+h-1 & 2(\alpha+h)-1 & 2(\alpha+h-1) \end{pmatrix} \tag{40}$$

$$\beta_l = \begin{pmatrix} 2(\alpha+h) & 2\alpha+h & 2\alpha+h & 2\alpha+h-1 \\ 2\alpha+h & 2\alpha & 2\alpha & 2\alpha-1 \\ 2\alpha+h & 2\alpha & 2\alpha & 2\alpha-1 \\ 2\alpha+h-1 & 2\alpha-1 & 2\alpha-1 & 2(\alpha-1) \end{pmatrix}$$

(The elements correspond to T, E_+, D, E_- left to right, top to bottom). Using the relationship between H and β (eq. 22), the high and low frequency (here small and large times, t) have exponents:

$$H_h = \begin{pmatrix} \alpha+h-\frac{1}{2} & \alpha+\frac{h-1}{2} & \alpha+h-\frac{1}{2} & \alpha+h-1 \\ \alpha+\frac{h-1}{2} & \alpha-\frac{1}{2} & \alpha+\frac{h-1}{2} & \alpha+\frac{h}{2}-1 \\ \alpha+h-\frac{1}{2} & \alpha+\frac{h-1}{2} & \alpha+h-\frac{1}{2} & \alpha+h-1 \\ \alpha+h-1 & \alpha+\frac{h}{2}-1 & \alpha+h-1 & \alpha+h-\frac{3}{2} \end{pmatrix} \tag{41}$$

While at low frequencies large Δt (i.e. large lags) we have:

$$H_l = \begin{pmatrix} \alpha + h - \frac{1}{2} & \alpha + \frac{h-1}{2} & \alpha + \frac{h-1}{2} & \alpha + \frac{h}{2} - 1 \\ \alpha + \frac{h-1}{2} & \alpha - \frac{1}{2} & \alpha - \frac{1}{2} & \alpha - 1 \\ \alpha + \frac{h-1}{2} & \alpha - \frac{1}{2} & \alpha - \frac{1}{2} & \alpha - 1 \\ \alpha + \frac{h}{2} - 1 & \alpha - 1 & \alpha - 1 & \alpha - \frac{3}{2} \end{pmatrix} \qquad (42)$$

We should add here that since *E, O* are linear combinations of E_+, E_-, their exponents will the maximum of those of E_+, E_-, so that:

$$\begin{aligned} H_{h,E_\pm} &= \max\left(\alpha - \frac{1}{2}, \alpha + h - \frac{3}{2}\right) = \alpha - \frac{1}{2}; \quad h < 1 \\ H_{l,E_\pm} &= \max\left(\alpha - \frac{1}{2}, \alpha - \frac{3}{2}\right) = \alpha - \frac{1}{2} \end{aligned} \qquad (43)$$

We see that for the physically relevant parameters, $H = \alpha\text{-}1/2 = -0.25$ for both *E, O*, over the whole range (close to the data, see SL and fig. 3).

To get a concrete idea of the implications of model, let's use the rough empirical estimates from SL of $\alpha = 0.25$, $h = 0.5$. Plugging these values into eqs. 41, 42, we obtain:

$$H_h = \begin{pmatrix} 0.25 & 0 & 0.25 & -0.25 \\ 0 & -0.25 & 0 & -0.5 \\ 0.25 & 0 & 0.25 & -0.25 \\ -0.25 & -0.5 & -0.25 & -0.75 \end{pmatrix} \qquad H_l = \begin{pmatrix} 0.25 & 0 & 0 & -0.5 \\ 0 & -0.25 & -0.25 & -0.75 \\ 0 & -0.25 & -0.25 & -0.75 \\ -0.5 & -0.75 & -0.75 & -1.25 \end{pmatrix} \qquad (44)$$

(again, for T, E_+, D, E_- left to right, top to bottom). We can see that the Haar fluctuations will be useful for all the series over the whole range of frequencies/scales, the only exception being $\Delta D(\Delta t)$ at long lags (H_l<-1, lower right corner of the H_l matrix with H_l<-1). In this case, the Haar fluctuations "saturate" and the spurious (limiting) value $H_l = -1$ is obtained.

3.1.2 Normalized Correlations

The cross spectra and cross covariances (eq. 38) can be used to determine the normalized correlations that were estimated in SL:

$$\rho_{jk}(\Delta t) = \frac{\left\langle \Delta S_j(\Delta t)\Delta S_k(\Delta t)\right\rangle}{\left\langle \Delta S_j(\Delta t)^2\right\rangle^{1/2}\left\langle \Delta S_k(\Delta t)^2\right\rangle^{1/2}} \tag{45}$$

(Haar fluctuations). However, from eqs. 41, 42, we find that their exponents (whether at high or low frequencies) are $2H_{jk}-\left(H_{jj}+H_{kk}\right)=0$ i.e. they are *not* power laws and only vary at sub power law rates, they are therefore nontrivial (i.e. they are significant) over the whole range of Δt. Since there are six series (T, E, D, O, E_+, E_-) there are 15 pairs whose fluctuation correlations may be determined over the observed range of 3 $\approx < \Delta t \approx <$400 Myrs, see fig. 4. The key correlations are those that correspond to the model parameters: $\rho_E = \rho_{TE}$, $\rho_D = \rho_{TD}$, see below. We can already see that the correlations are quite noisy, a consequence of the low resolution and variable sampling of the series. In order to make a proper model - data comparison, we therefore turn to numerical simulations.

4. Numerical simulations:

4.1 The statistics of the simulated series

The model has two fundamental exponents (α, h), two basic correlations ($\rho_E = \rho_{TE+}$, $\rho_D = \rho_{TD}$) and a cross over time scale τ. The third correlation ρ_{DE} is a derived parameter (eq. 32). In addition, there are two amplitude factors σ_T, σ_E but these will depend on the nondimensionalization/normalization of the series; on log-log plots they correspond to an up-down shift and on (normalized) correlation plots, the normalization eliminates them, they will not be considered further.

We used the results of SL to fix the values α = -0.25, h = 0.5, τ = 32Myrs (this is the nearest power of 2 to the slightly larger – but only roughly estimated - value τ = 40Myrs in SL). This leaves the only unknown parameters as the TE and TD correlations ($\rho_E = \rho_{TE+}$, $\rho_D = \rho_{TD}$), fig. 4.

Before comparing the model directly to the (noisy) data we first check that we are able to numerically reproduce the theoretically expected behaviour. The basic modelling technique is to use convolutions with various (impulse response) Green's functions, this is detailed in appendix A, but follows the methods described in (Lovejoy 2022). The main numerical problems are the small scales that have singular power law filters that are not trivial to discretize, and there are some (easier to handle) long time (low frequency) issues.

Rather than attempting to rigorously determine optimum parameters, as indicated above, we fixed the exponents α =0.25, h=0.5 and the crossover scale τ =32Myrs. With guidance of the fig. 4 correlations for ρ_{TE+} , ρ_{TD} and some numerical experimentation, we took ρ_E = 0.5, ρ_D = -0.1 (hence ρ_{TE+} = 0.5, ρ_{TD} = -0.1, ρ_{DE+} = -0.9 i.e. the sign of r was taken as negative, eq. 32). We then performed simulations at a resolution of 250kyrs resolution, with simulation length of 4 Gyrs (2^{14}= 16384 points), shown in fig. 5. We postpose a discussion of the significance of the correlations to section 4.2.

According to the model (see the diagonal elements in eq. 44), the only series with positive low frequency scaling exponent (H_l >0) is the temperature (H_l = 0.25), it indeed shows "wandering" behaviour (second from the bottom in fig. 5); from the figure, one can see its long range correlations as low frequency undulations. This is also true for D, but only up to the cross-over scale (≈32Myrs) after which consecutive 32 Myr intervals tend to cancel (H_l <0, eq. 44). The other series are on the contrary "cancelling" (H_l <0, H_h <0) especially E_- (eq. 44). We can also visually make out some of the correlations, but this is clearer at lower resolution discussed later.

On these simulations, we can check that the theoretical scaling is obeyed, this was done using Haar fluctuations, see fig. 6 where the theory slopes (from eq. 43, 44) are shown as reference lines. Note that since the Haar analysis "saturates" at H =-1, the low frequency H_l = -1.25 value for E_- (eq. 44, lower right hand diagonal element) yields a slope -1 (not -1.25), the other slopes are however accurately estimated. Note that the theory / simulation agreement is not perfect, mostly because the theory is for the average statistics over an infinite ensemble, whereas fig. 6 is from a single - albeit large - simulation.

We can also work out the 15 correlations as functions of lag, fig. 7. The figure shows the model parameters $\rho_{TE+} = 0.5$ ($=\rho_E = 0.5$), $\rho_{TD} = -0.1$ ($= \rho_D = -0.1$) as solid black reference lines and the derived correlation $\rho_{DE+} = -0.9$ (eq. 32) as a dashed reference lines. Also shown are dashed theory lines for the *TE*, *TO* correlations (predicted to be equal to equal to TE_+ at long lags – eq. 39) and the *DE, DO* correlations (predicted to be equal to DE_+, at long lags, see eq. 39). We can see that the correlations approach the theoretical correlations at large lags, although the results are somewhat noisy.

4.2 The statistics of the simulated series resampled at the data sampling times

Before making more effort at parameter fitting and comparing the model to data, it is important to take into account the small number of empirical data points and their irregular sampling. Fig. 8 shows the result for a simulation with the same parameters, but with a 1 Myr temporal resolution (right hand side), resampled at the same times as the data (left hand side). Since the model and data are only expected to have similar statistics, the detailed "bumps" and "wiggles" are unimportant, but one can nevertheless make out realistic looking variability including correlations between the series. Note that the model respects causality so that when there is a large extinction event, that is asymmetric with a rapid upturn being followed by a slower downturn (however, we have followed convention so that the present is at the left and the past at the right).

We can now consider the fluctuation scaling and correlation statistics on the resampled series and compare them to the both the data and to the results from the same simulations but at a regular 1 Myr resolution (fig. 9). The figure shows a log-log plot of the RMS fluctuations as a function of the lag. In order to make the comparison, they were normalized by their standard deviations, but this is somewhat arbitrary so that the up-down displacement (corresponding to a different nondimensionalization/normalization) is unimportant. To judge the realism of the model, the appropriate comparison is between the shapes of the resampled model output (red) and the data (black). We can see that the two are fairly close although both model and data are noisy due to the small number of points and the irregular sampling. The agreement must be assessed not only by allowing for (relative) vertical shifts, but also noting that the scales on the top D, T comparisons are such that the fluctuations vary only over a small factors (for the data, factors of ≈ 1.7 for D and ≈ 2 for T) for lags varying over range of about a factor 100. In comparison, the E_+, E_-, E, O ranges are closer to factors of 10. Aside from this, these basic fluctuation statistics are fairly close to the data.

The figure also gives important information about the effect of the sampling: compare the resampled (red) and uniformly sampled analyses (brown). The resampling is particularly important for E_+, E_-, E, O although the effects are mostly at small lags for E_+,E, O but for large lags for E_-. This information should prove useful in interpreting a variety of real world extinction and origination data.

Finally, we can compare the 15 pairwise correlations (fig. 10). Again, to judge the realism of the model, compare the red and black correlations. Although – as expected – these are fairly noisy, we see that the agreement is quite good, significantly, it is generally much better than the agreement between the uniformly sampled correlations (brown curves) and data (black). By comparing the red (resampled) and brown (uniformly sampled) correlations, we see that the resampling is especially important for the DE_+, DO, DE, E_+E_-, E_-O, OE, correlations and to a lesser extent the OE, TE_+ comparisons, for the others it is about the same. We could note the successful prediction that the E_+E, E_+O, OE correlations that should be ≈1 and the E_-E correlations that should be ≈ -1. Interestingly, the prediction that the E_-O correlations should be ≈ -1 (eq. 39) is verified with the uniform sampling (i.e. it is indeed a property of the model), yet, the resampling (red in the lower left graph in fig. 10) makes it >0 and aligns it closely with the observations. In other words, when the pure model predictions are poor (brown versus black), there are many instances where the effects of nonuniform sampling are particularly strong so that overall the model explains the data fairly well: overall 6 fluctuation plots (fig. 9) and 15 correlations (fig. 10) with 5 adjustable parameters (α, h, τ, ρ_E, ρ_T).

4.3 Discussion of the model and physical significance of the correlations

The model was motivated by an attempt to model the diversity process as a scaling cross-over phenomenon with wandering climate (paleo temperature) and stabilizing life (turnover) scaling drivers. In the course of the model development, it became clear both theoretically (due to the definition of the diversity, eq. 5) and empirically, that rather than E, O being fundamental, it rather the turnover E_+ that is fundamental (indeed, the E_+ and E_- statistics are quite different (figs. 3, 9) and the E_+E_- correlations are nearly zero (figs. 4, 10). In any event, the model predicted that E, O would follow the E_+ statistics (eq. 39, fig. 3, 9 and the E_+E and E_+O correlations in fig. 3, 10).

A more counterintuitive finding concerns the correlations. To start with, the model specifies that the diversity is primarily driven by the temperature up until the cross-over scale, yet the temperature and diversity are negatively correlated over the entire range! Although at any given time lag, the DT correlation is small (-0.1), it means that there is a (weak) tendency for the diversity fluctuations to decrease when temperature fluctuations increase and visa versa, but this is not enough to offset the overall temperature control of the diversity that implies that consecutive temperature fluctuations tend to add up ($H_T = 0.25 > 0$) and this is a stronger overall effect.

There is an additional more subtle effect. Consider that at each scale, the imposed TE_+ correlation is moderate and positive ($\rho_{TE+} = 0.5$) and together, ρ_{TD} and ρ_{TE+} (with r<0, eq. 32) they imply that at each lag, DE_+ is negatively correlated (reaching the theory value $\rho_{DE+} \approx -0.9$, at long lags, see the DE_+ correlation, the brown curve in fig. 10). Since the turnover E_+ also drives the diversity, (eq. 1), at each scale, we thus have a tendency for T and E_+ fluctuations to increase (or decrease) together but D and E_+ (and hence T and D to have opposite tendencies). The overall result is that the weak anticorrelation of D with T and D with E_+ at any fixed scale is still dominated by the stronger effect of T fluctuations growing with scale and dominating the E_+ driver at lags < τ.

We could remark that $\rho_{TE+} = 0.5 > 0$ indicates a tendency for temperature changes to "stimulate" the turnover: periods of increasing temperatures tending to be associated with increasing turnovers and decreasing temperatures with decreasing turnovers. Also there is a strong anticorrelation between D and E_+ ($\rho_{DE+} \approx -0.9$, although it seems to nearly disappear after the nonuniform sampling, see fig. 10, second in the top row) that indicates that increased turnover decreases with diversity. However over the range of scales that E_+ dominates dynamics of D (i.e. $\Delta t > \tau$), since $H_{E+} \approx -0.25 < 0$), successive E_+ fluctuations tend to cancel and on long time scales, the latter effect is dominant so that $H_D = H_{E+} \approx -0.25$ – this is a scaling region of biotic self-regulation.

5. Conclusions:

The driver of macroevolutionary biodiversity has famously been reduced to a dichotomy between "life" and the "environment": the "Red Queen" versus "Court Jester" metaphor (Van Valen 1973); (Barnosky, 2001). Using genus level time series from the Paleobiology Database (Spiridonov and Lovejoy 2022) (SL) systematically analysed fluctuations in extinction (*E*), origination (*O*) rates, biodiversity (*D*) and paleo temperatures (*T*) over the Phanerozoic. They did this as a function of time scale from the shortest (≈3Myrs) to longest lags available (≈400Myrs) and their analysis included the correlations of the fluctuations at each scale. They concluded that *T, E, O* – the basic climate and life parameters - showed evidence of wide range scaling, supporting the hypothesis that over this range, there is a single biogeological "megaclimate" (Lovejoy 2015) regime with no fundamental time scale. However, they found that *D* followed the *T* fluctuations up until a critical time $\tau \approx 40$Mys, whereas at longer time scales, it followed life (*E, O*): *D* was a scaling cross-over phenomenon. At the shorter time scales $\Delta t < \tau$, - like the temperature – the *D* scaling exponent $H_D \approx +0.25$ (i.e. >0) indicating that fluctuations tended to grow with scale, leading to "wandering" behaviour. In contrast for time lags $\Delta t > \tau$, - like *E, O*, its scaling exponent was $H_D \approx -0.25$ i.e. <0), hence successive fluctuations tended to cancel, resulting in long time stabilization of diversity by life.

In order to clarify our ideas, to better understand the geobiodynamics and to better understand and quantify the limitations, biases and other data issues, we proposed the simple model Fractional Macro Evolution Model (FMEM) to reproduce the observations. It is a model of macroevolutionary biodiversity driven by paleotemperature (the climate proxy) and the turnover rate ($E_+ = O+E$), the "life" proxy. In order to fit with basic empirical scaling statistics and theoretical ideas about the macroclimate regime (form time scales of roughly 1Myr to at least 500Myrs), these drivers were taken to be scaling with climate dominating at short time scales and life at long time scales. Therefore, FMEM suggests a possible way to combine into single stochastic framework both: i) the destabilizing geophysical processes (and possibly astrophysical ((Raup, 1991; Raup, 1992b; Melott & Bambach, 2014 ; Fields et al., 2020)) with ii) the stabilizing, density dependent and self-regulating, biotic processes. The model is specified by a simple parametrization based on two scaling exponents and two pairwise correlations (between T and E_+ and between T and D).

The model had two unusual characteristics: first, it was stochastic so that the crossover from climate to life dominance was thus a scaling (power law) not standard exponential (i.e. Markov process type) transition. Stochastic models involve infinite dimensional probability spaces, they are therefore natural model types in systems with huge numbers of degrees of freedom. We believe that they are intrinsically more realistic than strongly nonlinear but deterministic chaos type models (including those that are deterministic but are perturbed by noises). When the intermittency is strong scaling stochastic models must be nonlinear (e.g. multifractal cascade processes), and this can easily be included in further model improvements – the Gaussian forcing (γ_1, γ_2, eq. 14) need only be replaced by a multifractal one. Here, intermittency was neglected and linear stochastic equations with Gaussian white noise forcings were used (linear stochastic models can often be used even when the underlying dynamics are strongly nonlinear).

The other unusual FMEM characteristic was that it a system of fractional differential equations. Unlike the familiar integer ordered differential equations that typically have exponential impulse response functions (Green's functions), fractional equations typically have power law response functions and are natural ways to model scaling processes. These impulse response functions are physical models of bolide impacts and similar nearly instantaneous processes, and we discussed some implications.

The model was also highly parsimonious with two scaling exponents and a cross-over time τ determined by the Paleobiology Database data as analyzed SL. These determined the basic scaling characteristics of the 6 series: T, E_+, D, E_- (= O - E), O, E. In addition, the model had two correlations that were specified: those between T and E_+ and between T and D. From these, the other 13 pairwise correlations (out the 15 possible pairs of the six series), were implicitly determined and were compared to the data.

The fractional derivatives were of the Weyl type so that their Fourier transforms were simple power laws. Since the system was ultimately forced by two Gaussian white noises, only the second order statistics (i.e. the spectra and correlation functions) were needed and these were easily obtained: the basic solutions were fractionally integrated fractional relaxation noises (ffRn) that were recently introduced (Lovejoy 2022). In future, more realistic intermittent (multifractal) forcings could be used instead of the Gaussian white noise. Beyond exhibiting the full solution to the equations with a full statistical characterization, we then implemented the model numerically first verifying the model against the theoretically predicted behaviour. By producing simulations at 1Myr resolution, were able to resample the output at the same irregular sampling times as the biodata base. The statistical characteristics of the results (the 6 scaling curves showing the fluctuations as functions of time scale), plus the 15 pairwise correlations as functions of time scale, were all quite close to the data and in several cases, the agreement could be clearly attributed to the limitations, biases, etc. of the data. In particular, this was the case of the DE_+, DO, DE, E_+E_- E_-O, OE correlations that were much closer to the data following the irregular sampling than with the original model outputs uniformly sampled at 1Myr resolution.

Given the model's simplicity, it thus was remarkably realistic. This was fortunate since until higher resolution (global scale) time series become available (e.g. (Fan et al. 2020)), more complex models may not be warranted. In any case, the model was able to help explain some subtle points about the interaction of different correlated series that were also strongly selfcorrelated over wide ranges of time scales and this with quantitatively and qualitatively different scaling behaviours ("wandering" versus "cancelling"/self-stabilizing).

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Competing Interests

The authors declare none.

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Appendix A: Numerical simulations

Since the model is linear, the obvious simulation method is to use Fourier techniques. The main problem is that the small scales have singular power law filters that are not trivial to discretize, there may also be some long time (low frequency) issues. A convenient way is to use techniques developed for simulating ffRn processes discussed in (Lovejoy 2022). ffRn processes can be simulated by convolving Gaussian white noises with the ffRn Green's function $G_{\alpha,h}$ (eqs. 9,10). A somewhat better numerical technique is to use the step response Green's function ($=G_{\alpha+1,h}$ it is the smoother – and hence easier to handle integral of $G_{\alpha,h}$), followed by a numerical differentiation.

Figure Captions

Fig. 1: The impulse (delta function) response $G_{\alpha,0}(t)=t^{\alpha-1}/\Gamma(\alpha)$ for fractional integrals of order α normalized for the same response after 1 Myr. The bottom corresponds to the turnover (E_+) response α = 1/4 and the top corresponds to the temperature (T) response with α = 3/4. Notice the long term effects.

Fig. 2: The impulse response $G_{\alpha,h}(t/\tau)$, with α =1/4, h = 1/2 corresponding to the diversity(D) response, for critical transition times τ = 1, 4, 16, 64, 256 Myrs (bottom to top). The empirical value is τ ≈ 40Myrs (SL).

Fig. 3: This shows the Phanerozoic marine animal macroevolutionary analysis of the 6 series discussed in this paper; D, T, O, E are replotted from SL. The dashed lines show the theory slopes (eq. 44) with transition at $\Delta t \approx$ 40My i.e. $\log_{10}\Delta t \approx 1.6$.

Fig. 4: The (normalized) pairwise correlations of the 15 pairs of the 6 series as functions of lag. Several of these are reproduced from SL.

Fig. 5: The previous 2^{14} simulation degraded from ¼ Myr resolution to 1 Myr. Curves normalized by their standard deviations and then offset by 5 units in the vertical for clarity.

Simulation 2^{14} =16384 points with theoretical slopes indicated. The transition scale τ is 2^7 = 128 units, indicated by dashed vertical lines. If the model was at 250kyr resolution, the cross over is at 32Myrs, the length of the simulation is: 4 Gyrs. Parameters α = 0.25, h = 0.5, $\rho_E = \rho_{TE} = 0.5$, $\rho_D = \rho_{TD} = -0.1$ (with derived DE correlation $\rho_{DE} = -0.9$).

Fig 6: Simulation 2^{14} =16384 points with theoretical slopes indicated. The transition scale τ is 2^7=128 units, indicated by dashed vertical lines. If the model was at 250kyr resolution, the cross over is at 32 Myrs, the length of the simulation is: 4 Gyrs. Parameters α = 0.25, h = 0.5, $\rho_E = \rho_{TE} = 0.5$, $\rho_D = \rho_{TD} = -0.1$ (with derived DE correlation $\rho_{DE} = -0.9$).

Fig. 7: The 15 pairwise correlations from the 2^{14} realization above. Only two of the correlations were prescribed and this, only at a single resolution, the rest are consequences of the model, the two exponents a, *h* and the cross-over time $\tau = 2^7$ (shown as short dashed vertical lines). The two prescribed correlations (DT, TE_+) are shown as solid horizontal lines, and the derived correlations (DE_+ from DT, TE+, eq. 32) and then TE, TO (predicted to be equal to equal to TE_+ at long lags – eq. 39) and DE, DO (predicted to be equal to DE_+, at long lags see eq. 39). Note that these are from a single realization of the process not the ensemble average. In addition, the statistics of some are fairly sensitive to irregularly sampled (and small size) of the empirical data, compare with fig. 10 below.

Fig. 8: Model - simulation comparison with all series normalized by their standard deviations. The simulation was at 1Myr resolution and the sampled at the same (irregular) times as the data (84 points over the last 500Myrs). Each curve was displaced by 5 units in the vertical for clarity. Due to causality, the series are asymmetric with time running from right to left. The simulation is on the right.

Fig. 9: From the 1Myr resolution simulations discussed above (Brown) and in fig. 8 and resampled at the data times (red), black is data. The relative vertical offsets of the curves are not significant, they correspond to specific normalizations / nondimensionalizations. We see that in general, the resampling at the data times (red) yields a closer fit to the data (black) than the analysis of the simulation at uniform (1Myr) intervals, this is especially true for E_-, O, E, E_+.

Fig. 10: Same simulation as above, compared with data (black). Brown is a uniform 1Myr resolution, red is the simulation resampled at the data times. The resampling notably improves the correlations for DE_+, DO, DE, E_+E_-, E_-O, OE, and to a lesser extent the OE, TE_+ comparisons for the others it is about the same.