Fractals in the Natural and Applied Sciences (A-41)
M.M. Novak (Editor)
Elsevier Science B.V. (North-Holland)

Multifractal Generation of Self-Organized Criticality

D. Schertzer[a], S. Lovejoy[b]

[a]LMD, Boite 99, Université P.& M. Curie, 4 Pl. Jussieu, Paris 75252 Cedex 05, France

[b]Department of Physics, McGill University, 3600 University St.., Montréal, Québec, Canada, H3A 2T8

Abstract

We clarify the links between scaling and self-organized criticality in demonstrating that multifractal processes -with nonvanishing input reach self-organized criticality via the analog of a first order phase transition. We emphasize that the first order transitions are intrinsic consequences of the scale and dimension of the observations, whereas second order transitions arise from finite sample sizes. We point out implications and applications for nonlinear physical systems.

Keywords codes: G.3; I.6; J.2
Keywords: Probability and Statistics; Simulation and Modeling; Physical Sciences and Engineering.

1. INTRODUCTION

We consider the defining features of self-organized critical (SOC) phenomena [1] to be scaling coupled with algebraic probability distributions. While the classical origin of SOC is both deterministic and with vanishing input (vanishing flux of particles), our alternative is stochastic with a finite input (e.g. a non zero flux of turbulent energy, see also [2] for non vanishing flux of particles). In fact, since we do not rely on any specific model, we consider a rather generic statistical mechanism for open dissipative nonequilibrium systems: the analogue of a *non-zero critical temperature* associated with a first order multifractal phase transition [3-4]. Indeed multifractal behavior is determined by exponent functions which have analogues in thermodynamics [5-6], and therefore discontinuities of the analogues of the free energy and the thermodynamic potential correspond to phase transitions. "Frozen free energy" (second order) transitions [7-8] arise from finite sample sizes [9], whereas much more wild first order transitions are consequences of the scale and dimension of the observations on large samples [4].

These transitions are purely scaling phenomena and are thus totally different from the high temperature transitions found in (low dimensional) deterministic chaos [10-11] which are basically created by breaks in the scaling symmetry of the probability measure in phase space. Such transitions can occur in multifractal fields ranging from strange attractors [12-13], turbulence [14-16], statistical physics [17], high energy physics [8, 18], astrophysics [19], and geophysics [20]. They have direct implications for extreme, catastrophic, events in physical space. They can explain recent results in turbulence [21] and geophysics. Indeed, we survey empirical evidence from various geophysical sources that show that first order multifractal phase transitions are actually quite common.

2. MULTIFRACTAL CODIMENSION FORMALISM AND THERMODYNAMICS ANALOGUES

The multiple scaling behavior of *stochastic* field ε_λ at scale ratio λ (=L/*l* the ratio the largest scale L to the scale *l*), can be either characterized by its probability distribution or by the statistical moments since they are dual for the Mellin transform, see for instance [22] (here and below the sign ≈ means equality within slowly varying or constant factors):

$$\Pr(\varepsilon_\lambda \geq \lambda^\gamma) \approx \lambda^{-c(\gamma)}; \qquad \langle \varepsilon_\lambda{}^q \rangle \approx \lambda^{K(q)} \approx \int \lambda^{q\gamma} \lambda^{-c(\gamma)} dc(\gamma) \tag{1}$$

the exponent $c(\gamma)$ is [23, 24] a statistical codimension since -as discussed below- the probability measures the fraction of the probability space occupied by the singularities exceeding the order γ. Each realization corresponds to a finite D-dimensional cut of a process in an infinite dimensional probability space. Multiplicative processes [14, 23-25] produced by scaling random multiplicative modulations of larger structures by smaller ones which yield highly intermittent space/time fields are generic processes of stochastic multifractality.

At scale ratio λ, the probability can be estimated as the ratio of the number ($N_\lambda(\gamma)$) of structures with singularities $\geq\gamma$ to the total number of structures (N_λ): $\Pr(\varepsilon_\lambda \geq \lambda^\gamma) \approx N_\lambda(\gamma)/N_\lambda$. Whenever $D \geq c(\gamma)$, $c(\gamma)$ also has a geometrical interpretation over a D-dimensional observing set A. In this case, not only $N_\lambda \approx \lambda^D$, but also, on almost any single realization, $N_\lambda(\gamma) = \lambda^{D(\gamma)}$, with a positive $D(\gamma)$ which is then a geometric fractal dimension. This restrictive geometric interpretation corresponds to the starting point of [15]. However, the "hard" singularities (see below) which are the most interesting have $c(\gamma)>D$ and are only present in "canonical" multifractals [3, 23] whose invariants (e.g. turbulent energy flux) are conserved in the "canonical" sense, i.e. only on the ensemble average. One may note that recently the need of a codimension formalism has been implicitly acknowledged by Mandelbrot [26].

Typical examples of the multifractal observables are the D-dimensional integrals ($\Pi_\lambda(A)$) over A:

$$\Pi_\lambda(A) = \int_A \varepsilon_\lambda d^D\underline{x}, \tag{2}$$

For example, in turbulence $\Pi_\lambda(A)$ is the energy flux or in chaotic systems it is the multifractal measure on a strange attractor. Due to the D-dimensional integration (on a ball B_λ of size L/λ), the codimension characterizing ε_λ leads to a dimension characterizing Π_λ; similarly, the singularities of ε_λ are related to the singularities of Π_λ etc.:

$$\Pr(\Pi_\lambda(B_\lambda) \geq \lambda^{-\alpha_D} \approx \lambda^{f_D(\alpha_D)}/\lambda^D; \;\; \alpha_D = D-\gamma\, ; \; f_D(\alpha_D) = D-c(\gamma) \tag{3}$$

where we render explicit by the subscript "D" the extrinsic D dependencies [3] of the usual strange attractor notation [13] for the multifractal measure exponents α, f, τ. A similar dependency intervenes for the exponents of the "trace moments" [24] generalizing the partition function by combining ensemble with spatial averaging ("superaveraging"):

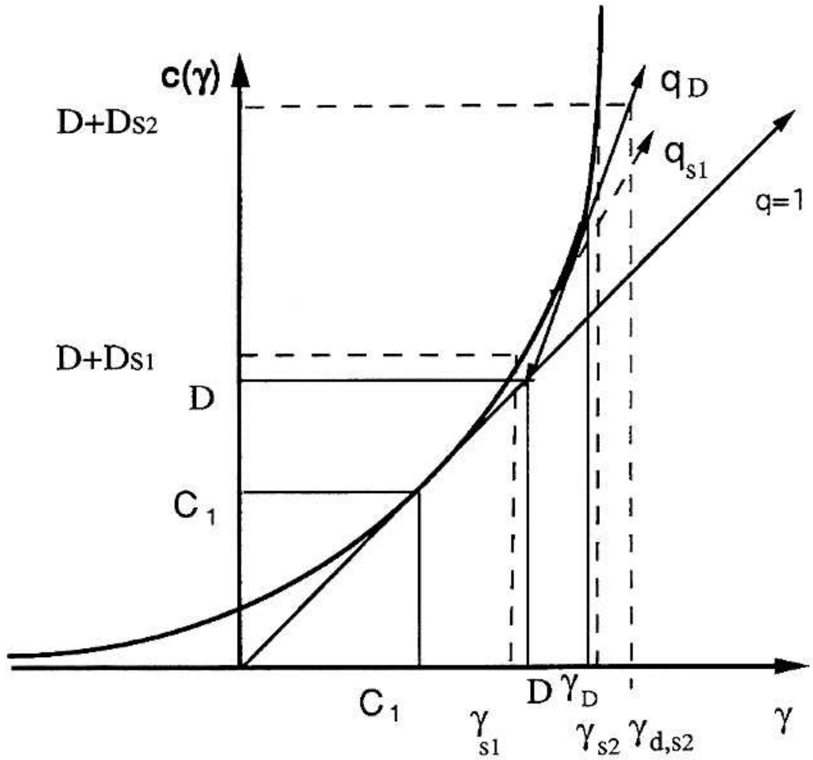

Fig. 1a: Schematic diagram of $c(\gamma)$, $c_d(\gamma)$ indicating two sampling dimensions D_{s1}, D_{s2} and their corresponding $\gamma_{s1}<\gamma_D<\gamma_{s2}<\gamma_{d,s2}$; the critical tangent (slope q_D) contains the point (D,D).

$$Tr A_\lambda(\varepsilon_\lambda{}^q) \approx < \sum_i \Pi_\lambda(B_{\lambda,i})^q > \approx \lambda^{-\tau_D(q)}; \quad \tau_D(q) = (q-1)D - K(q) \tag{4}$$

summing over A at resolution λ, i.e. on a covering of $N_\lambda(A) \approx \lambda^D$ disjoint balls $B_{\lambda,i}$ ($\Pi_\lambda(B_{\lambda,i}) \approx \varepsilon_\lambda \lambda^{-D}$). It is worthwhile to note that α_D, $f_D(\alpha_D)$, $\tau_D(q)$ diverge unfortunately as we increasingly explore the infinite dimensional probability space ($D \rightarrow \infty$; contrary to the finite phase space dimension D for strange attractors).

If we follow [6,27] (rather than [5]), the probability description (γ, $c(\gamma)$) is the multifractal analogue of the (energy, entropy) description of standard thermodynamics, whereas the moment description (q, K(q)) is the analogue of the (inverse temperature, Massieu potential) description. Since the free energy analogue is $C(q)=K(q)/(q-1)$discontinuities (phase transition analogues) will be apparent in either the free energy or Massieu potential description. (Entropy, Massieu potential) and (c ,K) are Legendre transform [15] pairs:

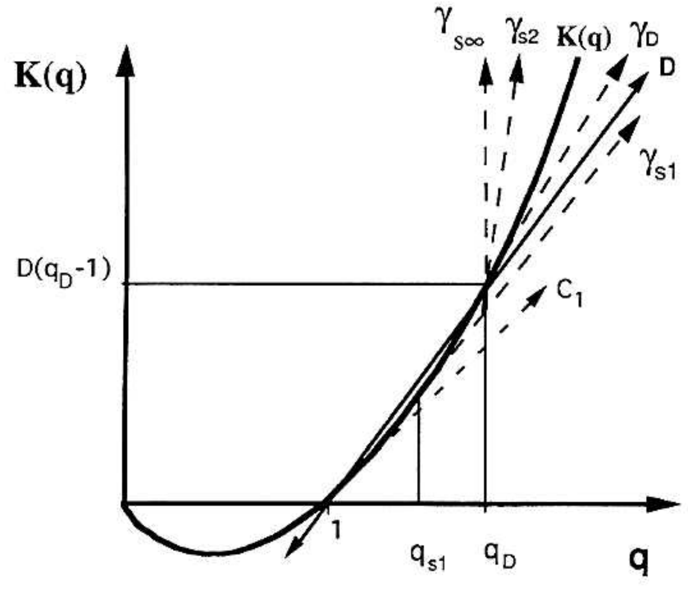

Fig. 1b: Schematic diagram of K(q), with straight lines of slopes $\gamma_{s1}<\gamma_D<\gamma_{d,s2}<\gamma_{d,s\infty}$ $(=\infty)$ indicating the behavior for increasing sample size N_s ($N_{s\infty}=\infty$). The line of slope D defining q_D is also shown.

$$K(q) = \max_{\gamma} (q\gamma - c(\gamma)) \qquad\qquad c(\gamma) = \max_{q} (q\gamma - K(q)) \qquad (5)$$

these relations establish the one to one correspondence $q = c'(\gamma)$, $\gamma = K'(q)$ (Figs. 1a,b).

3. MULTIFRACTAL PHASE TRANSITION ANALOGUES

3.1 Second order multifractal phase transitions

As we increase the number of independent realizations (N_s), each of dimension D and covering a range of scales λ, we gradually explore more and more the probability space; encountering more and more extreme events that would be almost surely missed on any smaller sample (Fig. 1a). This corresponds to the fact that we are increasing the dimension of observation D to an (overall) effective dimension Δ_s, which may be quantified with the help of the sampling dimension [28] D_s. This determines the highest order singularity (γ_s) we are likely to observe on N_s independent realizations and is estimated by:

$$c(\gamma_s)=D+D_s=\Delta_s ; \qquad\qquad D_s \approx \log N_s/\log \lambda \qquad (6)$$

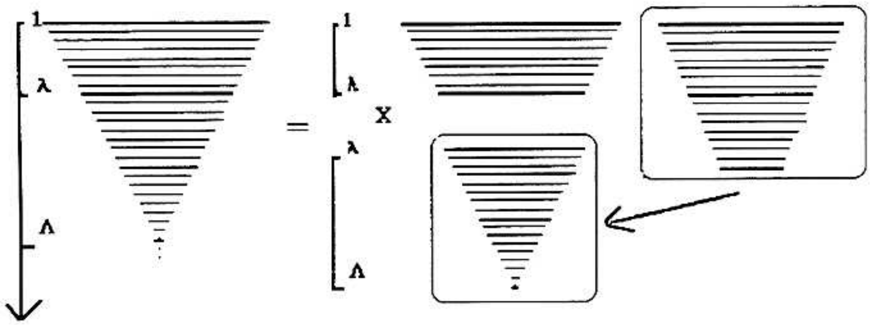

Fig. 2: A schematic diagram showing a cascade constructed down to scale ratio Λ, dressed (averaged) up to ratio λ. This is equivalent to a bare cascade constructed over ratio λ, multiplied by a hidden factor obtained by reducing by factor λ a cascade constructed from 1 to Λ/λ.

This follows from the Eq. 1 and the fact that there are a total of $N \cdot N_s = \lambda^{D+D_s}$ structures in the sample. More extreme singularities having codimensions greater than this effective dimension ($c \geq \Delta_s$) remain almost surely not present in our sample.

The upper bound γ_s [28] for observable singularities leads to a second order phase transition. Indeed, the Legendre transform of $c(\gamma)$ with $\gamma \leq \gamma_s$ leads to a spurious linear estimate K_s instead of the nonlinear K for $q > q_s$; $q_s = c'(\gamma_s)$ being the maximum moment that can accurately be estimated:

$$K_s(q) = \gamma_s(q-q_s) + K(q_s)\,,\quad q \geq q_s;\quad K_s(q) = K(q) \quad q \leq q_s \qquad (7)$$

hence there is a second order transition associated with a jump in the second derivative of the free energy/Massieu potential:

$$\Delta K''_s = -K''(q_s) \qquad (8)$$

and the corresponding (spurious) free energy $C_s(q)$ becomes quickly frozen at the value γ_s.

3.2. First order multifractal phase transitions

As discussed in [3-4], more violent first order transitions may occur for a multifractal process typically observed by spatial and/or temporal averaging on scales $l \gg \eta$ (the inner size of the process), i.e. with corresponding ratios $\lambda = L/l$, $\Lambda = L/\eta$ with $\Lambda \gg \lambda$. Indeed, the

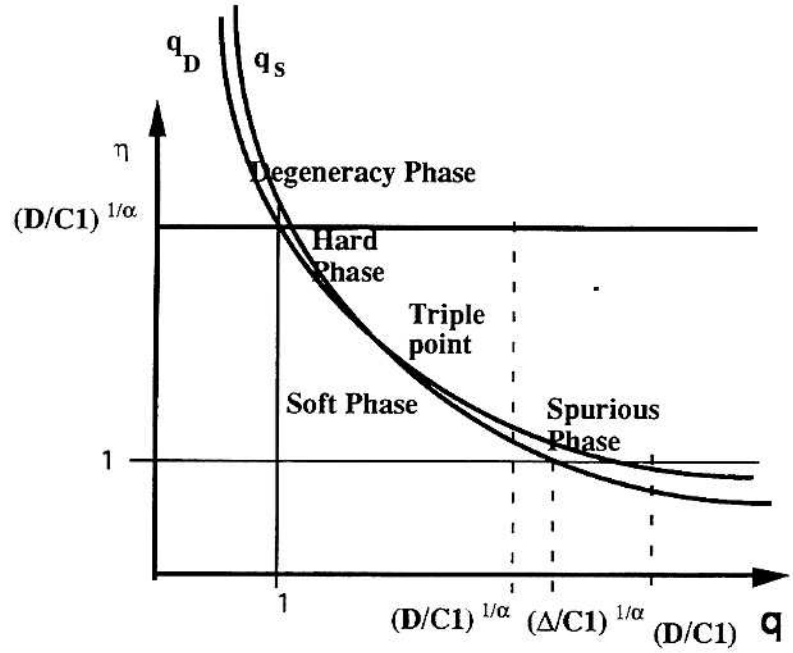

Fig. 3.—General outline of phase diagram in (q,η) plane.

"dressed" [24] variability observed at scale ratio λ will generally be wilder than the corresponding "bare" field obtained by stopping the cascade process at the same scale ratio and therefore stripped of its small scale activity (see [4] for a striking numerical example). The much more violent dressed variability results from the "hidden" interactions and fluctuations of the field on scale ratios between λ and Λ.

The hidden small scale component may indeed contribute as a highly variable prefactor having occasional avalanche-like effects on the large scale as soon as a singularity of order greater than D occurs. In the case of multiplicative processes, which are generic multifractal processes, the dressed field (ε_d) then factors (see Fig. 2) into a bare ε (large scale) and a hidden ε_h (small scale) component ($\varepsilon_d=\varepsilon\varepsilon_h$).). As soon as the D-dimensional integration cannot smooth it down to the scale of observation, the observation scale (l) is no longer effective and the scale of homogeneity η prevails. Since $\Lambda/\lambda>>1$, the dressed singularity (γ_d) computed using the observation scale will be much larger. These events will remain statistically negligible until a critical singularity order γ_D ($\geq$D) which we will estimate below. For $\gamma<\gamma_D$ the dressed codimension (c_d) coincides with the bare codimension (c), but for $\gamma>\gamma_D$ c_d will be determined [4 by simply maximizing the probability, i.e. minimizing c, *with the only constraint* being the convexity, c_d thus follows the tangent (Fig. 1a):

$$c_d(\gamma_d)= q_D(\gamma_d-\gamma_D) +c(\gamma_D) \qquad \gamma_d\geq\gamma_D; \quad c_d(\gamma_d)= c(\gamma) \quad \gamma_d\leq\gamma_D. \tag{9}$$

The inverse of the critical temperature $q_D=c'(\gamma_D)$ is the slope of the algebraic fall-off of the dressed probability distribution. It is also the critical order of divergence of statistical moments ($q\geq q_D$, $<\varepsilon_h^q>=\infty \Rightarrow <\varepsilon_d^q>=\infty$) a fact which can be seen by noting that a Legendre transform of a linear function diverges ($K_d(q)=\infty$, $q\geq q_D$). As pointed out in [4], the critical exponents γ_D and q_D can be rather easily determined. Indeed, for a given γ (and its corresponding $q=c'(\gamma)$), the (statistical) scaling exponent of the trace moment [14, 24] density, (i.e. before performing a D dimensional integration), is given by:

$$q(\gamma-D) -c(\gamma)\equiv K(q)-qD \tag{10}$$

the D dimensional integration does not prevent divergence (as $\Lambda\rightarrow\infty$) as soon as $q(\gamma-D)-c(\gamma)\geq D$. As $q(\gamma-D)-c(\gamma)$ in eq. 10 corresponds to the equation of the tangency to $c(\gamma)$, the critical tangent contains the point (D,D) (Fig. 1a):

$$D=q_D(D-\gamma_D)+c(\gamma_D)\equiv q_D D-K(q_D) \tag{11}$$

this yields a simple and direct determination of $c_d(\gamma_d)$:

$$c_d(\gamma_d)= q_D(\gamma_d-D) +D \qquad \gamma_d\geq\gamma_D \tag{12}$$

and of the corresponding hard behavior [3] of the dressed field:

$$\Pr(\varepsilon_{\lambda,d}'\geq \varepsilon_{\lambda,d}) \approx (\varepsilon_{\lambda,d})^{-q_D} \qquad \varepsilon_{\lambda,d}>>1 \tag{13}$$

which is a fundamental consequence of the D-dimensional integration/dressing. Indeed, as discussed elsewhere [29] this integration has an action analogous to that of an external field raising the critical temperature T_c from zero to a positive value. However, it is important to remember that the sample size must be large enough ($\Delta_s=D+D_s\geq c(\gamma_D)$) in order to observe this first -order phase transition. Indeed, following the argument for Eqs. 6-7 (see Fig. 1a), the maximum observable dressed singularity ($\gamma_{d,s}$) is given by the solution of $c_d(\gamma_{d,s})=\Delta_s$. By taking the Legendre transform of c_d with the restriction $\gamma_d<\gamma_{d,s}$ we then obtain the finite sample dressed $K_{d,s}(q)$:

$$K_{d,s}(q)= \gamma_{d,s}(q-q_D) + K(q_D) \qquad q>q_D; \quad K_{d,s}(q)=K(q) \quad q<q_D \tag{14}$$

In the limit $N_S\rightarrow\infty$, $\gamma_{d,s}\rightarrow\infty$, and for $q>q_D$, $K_{d,s}(q)\rightarrow K_d(q)= \infty$ as expected. For N_s large but finite, there will be a high q (low temperature) first order phase transition, whereas the scale breaking mechanism proposed for phase transitions in strange attractors [10-11] is fundamentally limited to high and negative temperatures (small or negative q). This transition corresponds to a jump in the first derivative of the K(q):

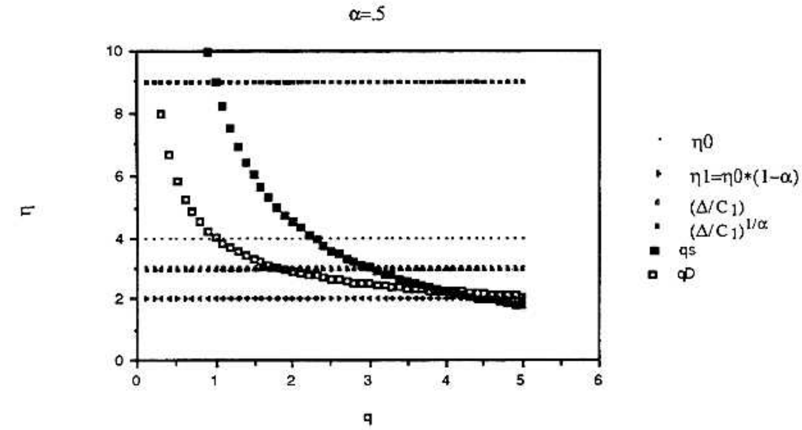

Fig. 4a.—Phase diagram in (q,η) plane for conditionally soft universal multifractal fields ($\alpha=1/2$).

$$\Delta K'(q_D) \equiv K'_{d,s}(q_D) - K'(q_D) = \gamma_{d,s} - \gamma_D = \frac{\Delta_s - c(\gamma_D)}{q_D} \quad (15)$$

On small samples ($\Delta_s \approx c(\gamma_D)$), this transition will be missed, the free energy simply becoming frozen and we obtain: $K_{d,s}(q)=(q-1)D$, which was already discussed with help of some precipitation experiments [9], whereas eq. 14 corresponds to an improvement of earlier works on "pseudo scaling" [14, 24]. Note that the above relations, especially eq.14, were tested numerically with the help of lognormal universal multifractals [4].

3.3. Multifractal phase transitions of (normalized) powers of a multifractal process:

We may now consider the normalized η-power of a multifractal process, defined as the following process:

$$\varepsilon_\lambda^{(\eta)} = \frac{\varepsilon_\lambda^{\eta}}{\langle \varepsilon_\lambda^{\eta} \rangle} \quad (16)$$

These various powers are introduced because they are fundamental to the study of *η-powers of a multiplicative process ε measured at ratio of scale Λ.*, which bare counterpart is:

$$\varepsilon_{\lambda,\Lambda}^{(\eta)} = \varepsilon_\lambda^{(\eta)} \langle \varepsilon_\Lambda^{\eta} \rangle \quad (17)$$

and which trace moment corresponds to the double trace moment [9,21] of ε_λ and gives therefore more directly access to properties of the double trace moment.

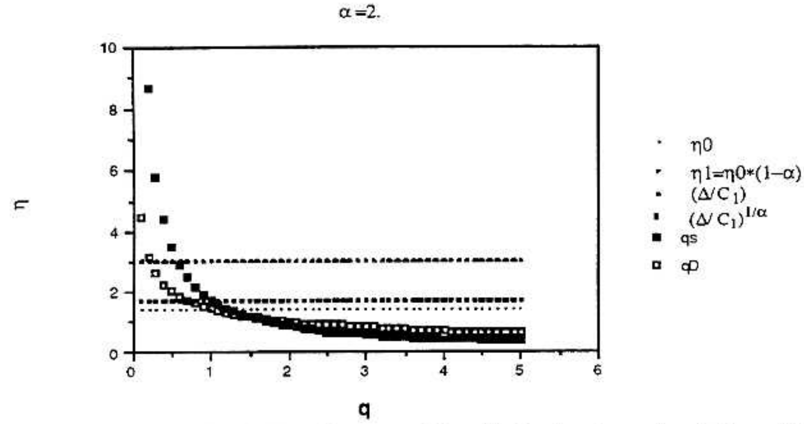

Fig. 4b—Phase diagram in (q,η) plane for unconditionally hard universal multifractal field $(\alpha=2)$.

Multifractal phase transitions can be best studied on either the (q,η) or (γ,η) planes, in which we have merely to determine the phase transition *lines* $q_s^{(\eta)}$, $q_D^{(\eta)}$ (or equivalently $\eta_s(q)$, $\eta_D(q)$). Fig. 3 gives a general outline. For very large η ($>\eta_0$) we have only the degeneracy phase, i.e. almost surely the process converges to zero, after having huge fluctuations. Indeed, for conserved processes with codimension of the mean singularity C_1 ($=c(C_1)$), the process is "degenerate" for $D<C_1$, therefore the transition line to degeneracy corresponds to $C_1^{(\eta_0)}=D$.

The behaviour is particularly easy to investigate in the case of *universal multifractals*, [23-24, 30-31] which are characterized by only two basic parameters, the Levy index $0\leq\alpha\leq2$ and the codimension of the mean singularity C_1 (for nonconservative processes there is a third basic parameter H). Universal multifractals have the following bare $K(q)$, $c(\gamma)$:

$$K(q)=\frac{C_1}{\alpha-1}(q^{\alpha}-q);\qquad c(\gamma)=\left(\frac{\gamma}{C_1\alpha'}+\frac{1}{\alpha}\right)^{\alpha'}\qquad \text{where } \frac{1}{\alpha'}+\frac{1}{\alpha}=1 \quad (17)$$

It is rather trivial to check that the only transformation required to obtain $K(q,\eta)$ and $c(\gamma,\eta)$ from the corresponding $K(q)$ and $c(\gamma)$ is the following:

$$C_1\rightarrow C_1^{(\eta)}=C_1\eta^{\alpha} \quad (18)$$

This transform allows us to derive all the statistical properties of the η-power of ε, e.g. $\eta_0=(D/C_1)^{1/\alpha}$, whereas it was unfortunately perceived as an indetermination problem in [9]. We next consider the maximum observable singularity $\gamma_s^{(\eta)}$ (for second order phase

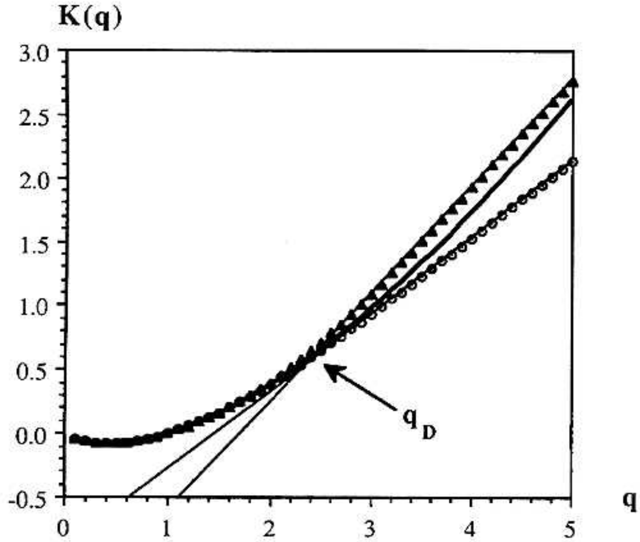

Fig. 5: Empirical observation of first order multifractal phase transition in turbulence [39]: the slope of the observed asymptotic dressed $K_{d,s}(q)$, $q>q_D\approx 2.3$ increases with sample size. Indeed, open circles correspond to the observed $K_{d,s}(q)$ for a number of samples $N_s=4$, whereas closed triangles for $N_s=700$, solid line corresponds to the theoretical bare K(q) (α=.1.45, C1= .25).

transitions) as well as the critical order for the first order transition $q_D^{(\eta)}$:

$$c(\gamma_s^{(\eta)}/\eta^\alpha)=\Delta_s/\eta^\alpha \quad (19)$$

$$K(q_D^{(\eta)}) = (D/\eta^\alpha)\,(q_D^{(\eta)}-1 \quad (20)$$

whereas the corresponding second order phase transition occurs at a moment of order $q_s^{(\eta)}$:

$$q_s^{(\eta)}= (\Delta_s/C_1^{(\eta)})^{1/\alpha} \quad (21)$$

With the help of the trivial rescaling of C_1 already discussed, (eq. 18), the distinctive features between the cases $\alpha<1$ and $\alpha\geq 1$ are given by:

$$q\downarrow 0; \qquad \eta_D(q)\approx q^{-1}\ (\alpha<1);\ \eta_D(q)\approx q^{-1/\alpha}\ (\alpha>1)$$

$$q\uparrow\infty: \qquad \eta_D(q)\downarrow \eta_1=\eta_0\,(1-\alpha)\ (\alpha<1);\ \eta_D(q)\approx q^{-1}\ (\alpha>1) \quad (22)$$

which is shown in Figs. 4a-b (using the examples α=1/2, 2). One may note the existence of the asymptote η_1 (which should be renamed η_∞) which corresponds to the conditional soft behaviour of $\alpha<1$ processes.

The overall behaviour can be summarized as follows: starting at large $\eta>\eta_0$, we obtain degeneracy. For moderate η ($\eta_T<\eta<\eta_0$) and large enough q, we obtain a (first order) transition from a soft to a hard phase, whereas for low enough η ($<\eta_T$) we obtain a transition to "spurious scaling" (only a second order transition). The triple point (q_T, η_T) is obviously interesting.

4 EMPIRICAL EVIDENCE

In the 1980's many attempts were made to empirically search for evidence of divergence of moments, especially in geophysical fields (see the table below for relevant references). More recently, preliminary results on the mechanism of the first order multifractal transition gave new impetus to studying algebraic or "fat-tailed" probability distributions in various areas. In contrast, monofractal processes generally only involve divergence in either an incidental way[1] , or the divergence is connected with the distribution of the size of sets (themselves only indirectly related to the process, a typical example being the size distribution of island areas which is indirectly related to the topography process-see however [32] for review-, the moments divergence becomes in the multifractal framework the result of a rather precise and general space time critical mechanism.

Examples of processes whose fluctuations or averages over various scales were empirically shown to have a probability distribution with power law tails are relatively recent and are given in the table 1. With the exception of the Gutenberg-Richter (1944) law for earthquakes [33] (more on this below) and the purely ad hoc regression performed by [34] , all the studies were explicitly motivated by efforts to test space-time scaling models of the processes. In addition, they were all motivated by the connection between multifractals and divergence of moments (the only exception was the rain analysis [35] interpreting the $q_D\approx1.7$ for temporal scaling in terms of (additive) Levy processes, and [36]). The amplitude-frequency relation for earthquakes (the Gutenberg-Richter law) is - along with the distribution of galactic luminosity - in a rather special position since in both cases the phenomena are considered to be essentially point-like, and although the "events" have well defined locations, they have been invariably analysed either via their marginal probability distributions (i.e. irrespective of the location of the events), or using the density of events (i.e. irrespective of their intensity). For the case of earthquakes, (long before it was justified on the basis of scaling models) the hyperbolic nature of the distribution has been very widely accepted whereas for galactic luminosity, it was considered exponential, but it is argued in [37] that this was mainly because of insufficient sample sizes and hyperbolic distributions do indeed give excellent fits to the large sample sizes now available. However, they have been more recently studied as seismic and luminosity fields taking both position and intensity into account. In particular, in [37-38], the normalized η-powers of these fields (see section 3.3) are analyzed and especially the dependence on η of $q_D^{(\eta)}$ (eq. 20), whereas the value in the table below is $q_D = q_D^{(1)}$.

Perhaps the two most important features of the results in the table is first that most are for turbulent atmospheric fields -which have long been recognized to be highly intermittent- and second, that the exponents are surprisingly low. This is especially true when one realizes that

[1] e.g. divergence of D-dimension Hausdorff measures on fractal set of dimension D'>D

Table 1:
Empirical evidence of first order multifractal phase transitions in geophysics

Field	Type	Data	q_D	References
Turbulence, wind field, atmosphere	fluctuations in horizontal wind in the vertical	Radiosonde≈50m resolution	5	[14]
			5.2*	[41
	fluctuations in horizontal wind, time	Sonic Anemometer 16Hz	6.9*	[40]
	Energy flux	Aircraft, horizontal ≈100m resolution	1.7	[14]
		≈12.5 m resolution	2.2*	[42]
	Energy flux	Hot wire anemometry	2.2*	[40]
	Energy flux	Sonic Anemometer 16Hz	2.3*	[40]
Temperature field, atmosphere	Potential Temperature, fluctuations in vertical	Radiosonde≈50m resolution	3.3	[14]
	Fluctuations in daily mean temperature	France single station (Macon)	5,	[43], [54]
		Regional average (53 stations)	5	[44]
	Fluctuations in hemispheric annual mean temperature	1880-1980 from Jones et al 1982 series	5	[45]
	Fluctuations in paleo-temperatures	ice cores, 350-22,400 years	5	[45]
Passive scalar field, atmosphere	UF_6	100m resolution, horizontal (in air)	3	[46]
	CO_2 at 50m above crops	≈3m in the horizontal	5.2	[53]
Precipitation, radar	horizontal reflectivity	1km resolution	2	[35]
	reflectivity isolated storm in time	5 minute resolution	1.7	[35]
	Reflectivity (absolute)	1km resolution	1.1	[24]
Rain, gauges	Tipping buckets, time	(0.01mm, 50 stations in Canada, 10 years)	2.5±0.5	[34]
	Daily accumulations	rain gauges (Nimes, 30 years)	3-3.5*	[48]
	Total accumulation of a single storm	spatial variability over a gage network	4.4	[55]
Rivers	Daily streamflow	4096 days, 50 rivers, France	3-4*	[36], [49]
Ocean surfaces	Far Red radiances	1m resolution	3*	[50]
Networks	Density of meteo-rological network	8000 stations	3.6*	[51]
Astrophysics	Apparent total galactic luminosity	8000 galaxies	1.25	[37] , [4]
Seismicity	Number- amplitude relation	California earthquakes	0.5-1.5	[33], [52] for a review;
	Gridded total amplitude	2X2km resolution, 300,000 events, 1980-1990, Parkfield, Ca.	0.9*	[38]

in many of the cases (indicated with asterisks), the universal multifractal parameters (eq. 17) have been estimated for the same data sets as q_D, which allows us to simultaneously obtain theoretical estimates of q_D as functions of D (using eq. 11 and corresponding to Fig1b), especially the dependency of $\gamma_{d,s}$ on the sample dimension can be tested, whereas q_D remains fixed. Fig. 5 gives an example of such study [39] where the number of samples is increased from $N_s=4$,to $N_s=700$.

Without exception, agreement with the empirical values is obtained only when relatively low values of D are used; often D<1. If the divergence really is a result of a multifractal dressing mechanism, it will not generally be a trivial matter to determine the effective D: the dynamically significant D may be smaller (and hence dominate) the one introduced by the measuring device. One possible mechanism yielding a dressing with D<1 which may be relevant in turbulence can be sketched. Recall the Kolmogorov relation between the observed velocity fluctuations (Δv_λ), and the energy flux (ε): $\Delta v_\lambda \approx \varepsilon^{1/3}\lambda^{-1/3}$. A simple interpretation of the linear scaling exponent -1/3 is that it represents a fractional integration of order D=1/3 of $\varepsilon^{1/3}$ and as elaborated elsewhere the estimates [39-40]- of universal multifractal exponents support a low $q_{\varepsilon D} \approx 2.3$, consistent with empirical estimates going back ten years[1] [14] as indicated in the table 1..

5. CONCLUSION

Many nonlinear dynamical systems found in nature are scale invariant over wide ranges of scale and exhibit both weak and avalanche-like violent/"hard" events. In this paper we have considered the possibility of a direct and surprising connection between the mean field and these extremes events. If we consider a self-organized critical system to be defined by the combination of scaling with algebraic probabilities (divergence of moments), then we have argued that SOC is in fact a general feature of such systems and results from a generic first order multifractal phase transition. With the help of this thermodynamic phase transition analogue, we obtain a detailed and general understanding of the appearance of SOC, including for the different (normalized) η powers of a multifractal process. We emphasize that multifractal processes reach self-organized criticality with nonvanishing input and within a stochastic framework, whereas classical SOC is both deterministic and with vanishing input. This alternative route to SOC considerably enlarges its relevance, since in high dimensional systems scale invariance can be regarded as a basic dynamical symmetry principle, which will be respected in the absence of symmetry breaking mechanisms. Finally we summarized various data which substantiate the relationship between multifractality and SOC.

6. ACKNOWLEDGMENTS

We thank P. Bak, A. Csordas, P. Hubert, C. Hooge, D. Lavallée, F. Schmitt, Y. Tang, T. Tel and Y. Tessier for helpful discussions and comments.

REFERENCES

1. P. Bak, C. Tang and K. Weiessenfeld, *Phys. Rev. Lett.*, **59** (1987) 381. P. Bak, C. Tang and K. Weiessenfeld, *Phys. Rev.* , **A59** (1988) 364.

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[1] The small discrepancy between the present estimates (2.3) and those made ten years ago (1.7) may be due to various causes including a fundametal difference in qD between the horizontal and verticl directions in the atmopshere and the smaller sample sizes used ten years ago which lead to lower accuracies.

2. T. Hwa and M. Kardar, *Phys. Rev. Lett.*, **62** (1989) 1813. T. Hwa and M. Kardar, , *Phys. Rev.* , **A45** (1992) 7002. P.M. Lam and F. Family, *Phys. Rev.* , **E47** (1993) 1570. .F. Family, in this volume, 1993.
3. D. Schertzer, S. Lovejoy, *Physica* , **A185** (1992) 187.
4. D. Schertzer, S. Lovejoy, D. Lavallée, *Cellular Automata: prospects in astronomy and astrophysics*, Eds. J.M. Perdang, A. Lejeune, World Scientific, Singapore, 1993.
5. T. Tel, Z. *Naturforsch*, **43A** (1988) 1154. .
6. H.G. Schuster, *Deterministic Chaos: an introduction.* 2d. rev. ed. , New York, VCH, New York, 1988.
7. B. Derrida and E. Gardner, *J. Phys.* **C19** (1986) 2253. B. Derrida and E. Gardner, *J. Phys.* **C19** (1986) 5783. M. Mesard, G. Parisi, M. A. Virascoro, *Spin Glass Theory and Beyond*, World Scientific, Singapore, 1987.
8. P. Brax, P. , R. Peschanski, *Phys. Lett.*, **B253** (1991) 225.
9. D. Lavallée., *PhD Thesis*, Physics, McGill U. 142pp, 1991.
10. P. Szépfalusy, T. Tél, A. Csordas, Z. Kovas, *Phys. Rev. A*, **36** (1987) 3525. A. Csordas, P. Szépfalusy, *Phys. Rev.*, **A39** (1989) 4767.
11. D. Barkley, A. Cumming, *Phys. Rev. Lett.*, **64** (1990) 327.
12. H.G.E., Hentschel, I.. Proccacia., *Physica* , **8D** (1983) 435. P., Grassberger, *Phys. Lett.*. **A97** (1983) 227. P. Grassberger, I. Proccacia,. *Phys. Rev. Lett.*, **50** (1983) 346.
13. T.C. Halsey et al., *Phys. Rev.* , **A33** (1986) 1141.
14. D. Schertzer, S. Lovejoy in *Turbulence and Chaotic Phenomena in Fluids*, T. Tatsumi ed., Elsevier, Amsterdam, 1984. Schertzer D., S. Lovejoy in *Turbulent Shear Flows*, **4**, B. Launder ed., Springer, 1985.
15. G. Parisi, U. Frisch in *Turbulence and predictability in geophysical fluid dynamics and climate dynamics*, Eds. M. Ghil, R. Benzi, G. Parisi, North-Holland, 1985.
16. R.Benzi et al. J. Phys., **A17** (1984) 3521. G. Paladin, A. Vulpiani, *Phys. Rev. Lett.*, **156** (1987) 147. K.R. Sreenivasan, C. Meneveau, *Phys. Rev.*, **A38** (1988) 6287.
17. H.E.P. Stanley, P. Meakin, *Nature*, **6** (1988) 116. P. Meakin, A. Coniglio, H.E.P Stanley, *Phys. Rev.*, **A34** (1986) 3325.
18 Ratti, S.P., G.Salvadori, G.Gianini, S.Lovejoy, and D.Schertzer, 1993: A Universal Multifractal Approach to Intermittency in High Energy Physics *Z. Physics*, in press, (1993).
19. H. Atmanspacher, H. Scheingraber, G. Wiedenmann, *Phys. Rev.*, **A40** (1989) 3954.
20. D. Schertzer, S. Lovejoy (eds.), *Non-Linear Variability in Geophysics, Scaling and Fractals*, Kluwer, Dordrecht-Boston, 1991.
21. F. Schmitt, D. Lavallée, D. Schertzer and S. Lovejoy, Phys. Rev. Lett. **68** (1992) 305, D. Lavallée, S. Lovejoy, D. Schertzer and F. Schmitt in *Topological aspects of the dynamics of fluids and plasmas*, Eds. K. Moffat, M. Tabor, G. Zaslavsky, Kluwer, 1993.
22. D. Schertzer, S. Lovejoy, *Nonlinear Variability in Geophysics 3 Lectures Notes*, Centre d'Etudes Scientifiques de Cargèse, 292 pp., 1993.
23. D. Schertzer, S. Lovejoy, D. Lavallée, F. Schmitt, in *Nonlinear Dynamics of Structures*. R. Z. Sagdeev et al, eds., World Scientific, Singapore, 1991.
24. D. Schertzer, S. Lovejoy, *J. Geophys. Res.*, **92** (1987) 9693.
25. L. Pietronero, A.P. Siebesma, *Phys. Rev. Lett.*, **57** (1986) 1098.
26. B. Mandelbrot in *Turbulence and Stochastic processes*, Eds. J.C.R. Hunt, O.M. Phillips and D. Williams, The Royal Society, 1991.
27. D. Schertzer, S. Lovejoy in ref.20, 1991
28. D. Schertzer, S. Lovejoy in *Fractals: Physical Origin and Properties* Ed. L. Pietronero, Plenum, New York, 1989. D. Lavallée, D. Schertzer, S. Lovejoy in ref. 20.
29. D. Schertzer, S. Lovejoy, submitted to *Phys Rev E* (1993).
30. D. Schertzer et al. *Fractal Aspects of Materials: Disordered Systems*, Eds. D.A. Weitz, L.M. Sander, B.B. Mandelbrot, pp. 267-270, Materials Research Society, Pittsburg,

1988. A.H. Fan, *C. R. Acad. Sci. Paris I*, **308** (1989) 151.
31. A. Bialas, R., Peschanski, *Nucl. Phys.*, **B308** (1988) 857.
32. G. Korvin: *Fractal Models in the earth sciences*. Elsevier, Amsterdam, 1992.
33. B. Gutenberg, C. F. Richter, *Bull. Seism. Soc. Am.* **34** (1944) 185.
34. B. Segal, *Comm. Res. Centre, Dept. of communications*, Canada, report CRC 1329-E, Ottawa (1979).
35. S. Lovejoy, *20th conf. on radar meteorology*, AMS Boston, (1981) 476.
36. D. Turcotte,.*J. of Hydrol*, (1993).
37. P. Garrido, *M.Sc. Thesis*, McGill University, Montréal, Québec, 1993.
38. C. Hooge, S. Lovejoy, D. Schertzer, F. Schmitt, in *Fractals and Dynamic Systems in Geosciences* , Ed. J. H. Kruhl, Springer Verlag, Berlin, 1993.
39 F. Schmitt, D. Schertzer, G. Brethenoux, S. Lovejoy, Y. Brunet, *C. R. Acad. Sci. Paris* , (1993, submitted)
40. F. Schmitt et al., *C. R. Acad. Sci. Paris, II*, **308**, 151, (1992), F. Schmitt, D. Schertzer, S. Lovejoy, Y. Brunet, *Fractals*, **1**, 3, (1993).
41. A. Lazarev, D. Schertzer and S. Lovejoy. *Annales Geophysicae* , **11** Supp.II (1993) C308
42. Y. Chigirinskaia, A. Ordanovich, D. Schertzer, S. Lovejoy. *Annales Geophysicae*, **11** Supp.II (1993) C307.
43. S. Lovejoy D. Schertzer, *Bulletin of the AMS*, **67** (1986) 21.
44. P. Ladoy, P., D. Schertzer, S. Lovejoy, *La Météorologie*, **7** (1986) 23.
45. S. Lovejoy D. Schertzer, *Annales Geophys.*, **4B** (1986) 401.
46. R. Visvanathan, *M.Sc. Thesis*, Ecole Polytechnique, Palaiseau, France, 1985.
47. M. Duncan phD. thesis, McGill University, Montréal, Québec, 1993.
48. Ph. Ladoy, F. Schimtt, D. Schertzer, S. Lovejoy, *C. R. Acad. Sci. Paris*, **II 317** (1993) 775.
49. Y. Tessier,, P. Hubert, J.P. Carbonel, S. Lovejoy and D. Schertzer, in *Proceedings Hydrofractal93*, Ischia, Oct. 12-15, (1993).
50. B. Kerman and C. Bernier, *J.Geophys. Res.* in press (1993).
51. Y. Tessier, S. Lovejoy and D. Schertzer, *J. Appl. Meteor.*, (submitted, 1993).
52. C.H. Scholz, *The mechanics of earthquakes and faulting*, Cambridge U. Press, Cambridge, 1990.
53. L. B. Austin, G. L. Austin, P. H. Schuepp, A. Saucier, in ref. 20, 1991.
54. Ph. Ladoy, S. Lovejoy, D. Schertzer,in ref. 20, 1991.
55. P. Hubert, J. P. Carbonnel, in ref. 20, 1991.