# NONLINEAR VARIABILITY IN GEOPHYSICS:

# MULTIFRACTAL SIMULATIONS AND ANALYSIS

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## ABSTRACT

Many geophysical fields show extreme variability over wide ranges of scale. We review and develop theoretical insights and empirical evidence concerning the multiple scaling/multifractal behavior of these fields. We emphasize the very singular behavior of geophysical observables, usually obtained by space-time averaging over scales much greater than that of the homogeneity. On the one hand we render more direct the link between statistical singularities (divergence of high order statistical moments) and singularities per realization (small scale divergence of densities). We recall also that in order to deal with the strong (but scaling) anisotropy of these fields we must generalize the idea of scale invariance beyond the familiar self-similar (or even self-affine) notions. On the other hand we examine the conditions of the existence of two-parameter universality classes of the generic multifractal processes. These have many important theoretical and practical consequences: infinite hierarchies of dimensions depending only on two parameters, the five main subclasses are determined. These facts greatly facilitate both the empirical characterization of multifractals, as well as their numerical simulation.

## INTRODUCTION

A central and common feature of geophysical phenomena and processes is their extreme variability over wide ranges of scale, whose ratios easily reach nine orders of magnitude (earth radius scale/ centimeter scale). Recently it has been increasingly recognized that this feature provides a powerful unifying *problematic* of Geophysics whose advance constitutes key steps both in increasing fundamental knowledge in Geophysics (especially turbulence), as well as in many practical applications (especially remote sensing techniques). More precisely the question of *scaling* behavior -i.e. a common behavior at different scales- became central simply because this (scale) *symmetry assumption is not only the simplest but also the only assumption acceptable in the absence of more information or knowledge*. Since this behavior is the result of nonlinear interactions -leading to nonlinear (i.e. non-proportional) response to a given excitation- between different scales (and/or processes), there arises the general question of *scaling nonlinear variability in Geophysics*. Mushrooming interest in geophysical applications of such nonlinear variability has lead to two workshops on the theme "Scaling, fractals and Nonlinear Variability in Geophysics 1, 2" in August 1986 at McGill University (Lovejoy and Schertzer 1988, Schertzer and Lovejoy 1989), and at the former Ecole Polytechnique in Paris France, in June 1988. There was also a session on "Chaos, Turbulence and Nonlinear variability in geophysics" at the March 1989 European Geophysical Society meeting where many of these questions were discussed.

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small scale or high frequency "ultraviolet" divergences, and unfortunately our observations and measurements are nearly always restricted to resolutions much higher than the scale of the smallest detail (i.e. the inner scale of the process or the scale of homogeneity which is typically of the order of millimeter or less). Note that full knowledge down to this inner scale is usually out of our scope due to large number of degrees of freedom involved. When we are speaking of large numbers, we of course refer to the *physicists' infinity* such as the Avogadro's number ($10^{23}$): indeed the number of $mm^3$ involved in the atmosphere is of order[1] $10^{10}x10^{10}x10^7= 10^{27}$

Let us briefly note that this question of the "details" has a rather long history, as is testified by the introduction of Perrin in his edited thesis (1913) and especially his valuable quotation of E. Borel about abstract vs. "real world" measures. We may also note the permanent question of fine graining vs. coarse graining , or the question of "homogeneization", "renormalization" (how to define smooth macroscopic "effective" fields from irregular microscopic ones), the above quoted question of "ultra violet divergences". Concerning fluid dynamics, the question of the singularities became more precise with the works of Leray (1934), and in Von Neuman's review on turbulence (Von Neuman (1963), but also in the debate between Richarson[2] and Bjkernes: is the characterization of a few large scale singularities (the meteorological fronts) sufficient to forecast the evolution of the weather? The present day debate could be much more precise dealing with characterization of hierarchies of scaling singularities. In the following we hope to give more easier insights into this fundamental question with the help of seemingly (at first glance) simple models (phenomenological models or "mock geophysics"), which nevertheless possess surprising properties which we argue to be quite general.

Let us also emphasize that the (nonlinear!) path historically followed to explore nonlinear variability crossed the geometrical world and was maintained in its restrictive frontiers for too long a period. This period created some unfortunate consequences and attempted to bypass some fundamental problems. Indeed, the development of concrete analytical methods has tended to show that geometrical frameworks can often be misleading and fractal notions have been most fruitful when divorced from geometry. In particular, the abandonment of the dogma of the uniqueness of fractal dimension (Grassberger (1983), Hentschel and Proccacia (1983), Schertzer and Lovejoy (1983,1984) Frisch and Parisi (1985), Halsey et al. (1986), Pietronero and Siebesma (1986), Bialas and Peschanski (1986), Stanley and Meakin (1988), Levich and Shtilman (1989)...) in favour of hierarchies of dimensions and singularities with their non-geometric generators has been one of the most important recent advances.

These new ideas involve both the possibility of very general anisotropic types of scaling (necessary, for example to deal with rotation, stratification or "texture"), as well as "multiple scaling" or "multifractality" associated with highly intermittent processes in which the weak and intense regions have different scaling behavior. In a general manner, a system may be said to be scaling (or scale invariant) over a range if the small and large scale structures are related by a scale changing operation involving only the scale ratio. Hence, scale invariance is not restricted to the familiar self-similar (or even self-affine) notions and we outline the necessary formalism (generalized scale invariance discussed by Schertzer and Lovejoy 1985, 1987a-b).

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[1] Considering the scale of homogeneity of the order of the millimeter, and the (outer) vertical scale of the order of ten kilometers and the horizontal scale of the order of ten thousand kilometers. In a similar manner the Reynolds number of atmospheric turbulence is usually estimated as $\approx 10^{12}$, taking the ratio of injection (1000 km)/dissipation (1mm) (horizontal) scales as $10^9$, since it is the 4/3 power of this ratio.

[2] Recall that Richardson (1926) didn't hesitate to raise the (sacrilegious?) question "does the wind have a velocity?" (i.e. are the time derivatives regular?). Indeed, he pointed out the very irregular Weierstrass function as a counter example.

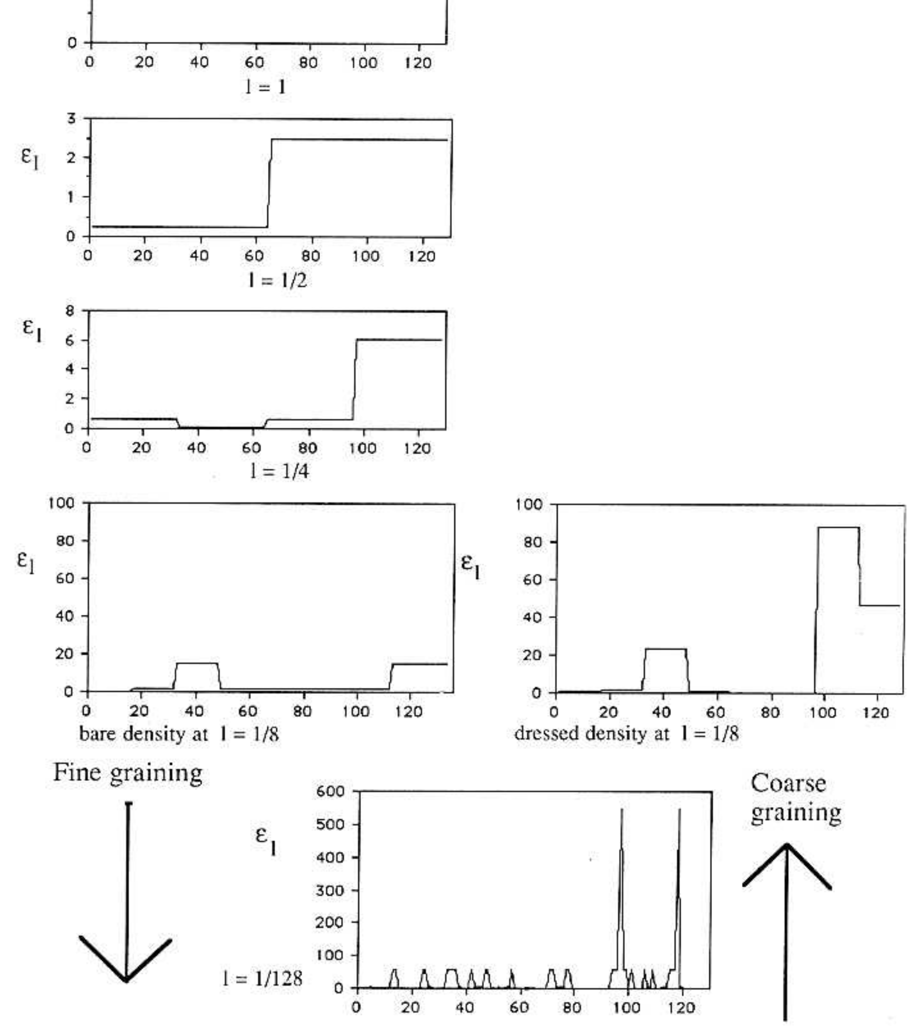


Fig. 1a . Illustration of the "bare"and "dressed" energy flux densities. The left hand side shows the construction step by step of the bare field produced by a multifractal cascade process (the $\alpha$-model, discussed below) starting with an initially uniform unit density. At each step the homogeneity scale is divided by a constant ratio $\lambda$=2 From top to bottom, the number of cascade steps takes the following values n = 0, 1, 2, 3 and 7, with the corresponding length scale values $l$ = 1, 1/2, 1/4, 1/8, 1/128. When the number of steps n increases, some rare regions of high intensities ("singularities") appear, most of the space becomes inactive. At $l$ =1/8, n=3, one may compare the rather more intense dressed density with the bare density. The sharp contrast arise from the smaller scales singularities, as seen on step n=7, which contribute to high fluctuations of the dressed density.

As we will insist that it is now rather obvious that multiple dimensions and singularities are the rule rather than the exception for fields. However, as we will discuss after having left the uniqueness for infinity, the important question of existence of universality classes gives credence to returning to two fundamental parameters!

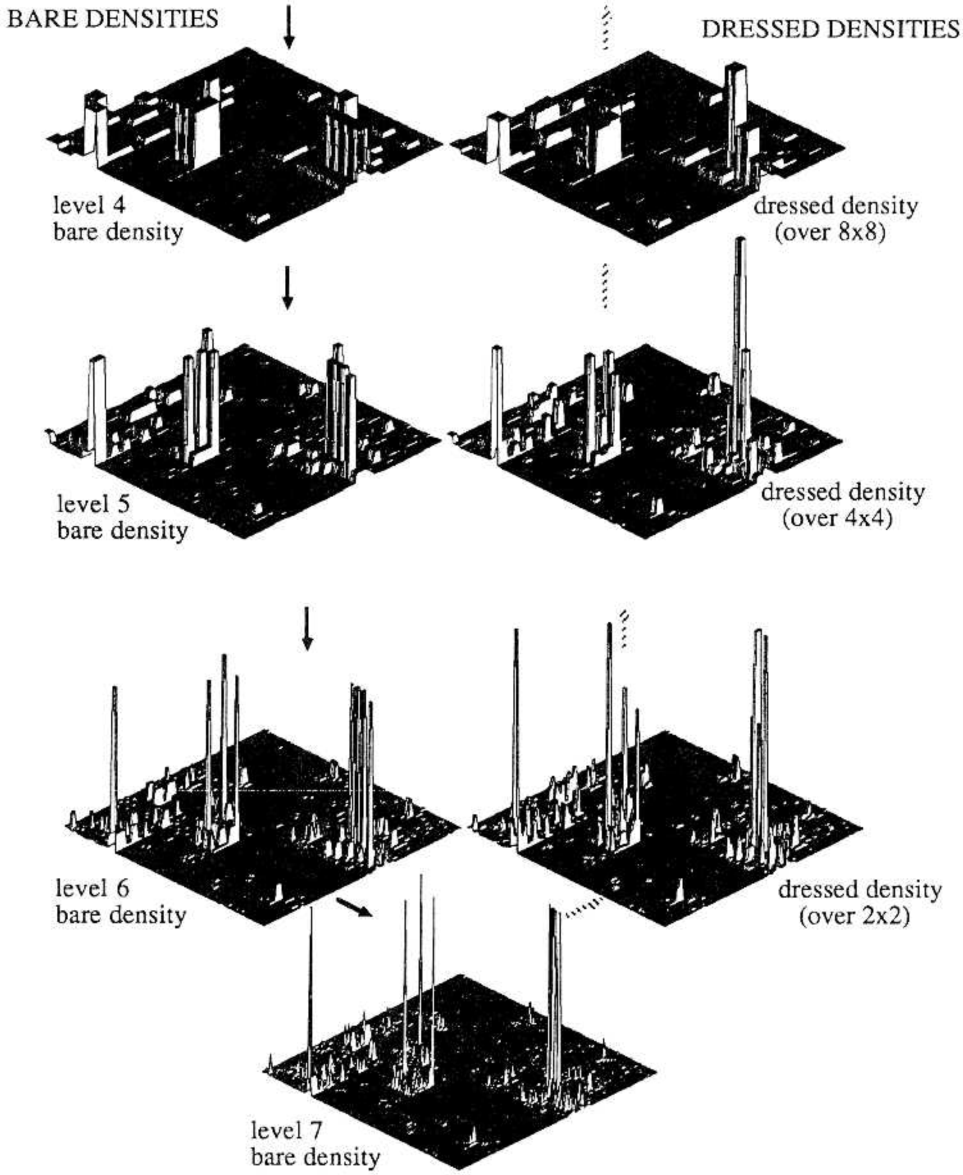


Fig. 1b . as in Figure 1a, illustration of the "bare"and "dressed" energy flux densities, but on a 2 dimensional space. The dressed energy flux densities, obtained by averaging, are presented on the right hand side of the figure. At intermediate scales, level 3 or 4, one may still note the important contributions from smaller scales singularities to high fluctuations of the dressed density.

other words, we must explore the fundamental *symmetry breaking caused by the observation* at a given scale. This is the reason why we will insist on the fundamental difference between *"bare" and "dressed" properties* at a given (non-zero) scale i.e. the important differences between a process with a cut-off of small scale interactions and one with these interactions restored (cf. Fig. 1a-b for illustrations)

The bare properties are related to fine graining (e.g. developing a cascade...) and are the properties of the process with nonlinear interactions at scales smaller than the observation scale being filtered out (i.e. truncation of the process at the scale of observation). The dressed properties are related to coarse graining and are the observed properties at a given scale of resolution (i.e. linear or nonlinear averaging on the observation scale over the smaller details of the same process but with all interactions: the process fully developed down to the smallest scale). In other words, only half the problem has been explored (and even a smaller fraction of the real problem): the "dressed" truth is the one which counts! The terms "bare" and "dressed" are borrowed from renormalization jargon, but here due to the extreme variability, they will become quite different; not only by a renormalizing factor but by different statistical behavior, thus the overwhelmingly important question of singular statistics (divergences of statistical moments, Schertzer and Lovejoy 1987a-b) linked to multiple ultraviolet divergences.

## PIXEL WORLDS AND "MOCK GEOPHYSICS"

On the one hand geophysical phenomena (especially when remotely sensed) are more and more often represented with the help of digitized "images", pixel sets. On the other hand the "theoretical" representations of the same phenomena are still believed to be of a certain continuous type. Such continuous representations are thought to be rather obvious limits of the pixel representation when the resolution (scale of observation) goes to zero. In particular one usually would associate with such an image, a function, a "density", and the digitized field corresponding to averages on a pixel of this density. Hence, from a very rough knowledge of the pixel values, one "naturally" tries to associate a hypothetical function. Such a "natural" hypothesis is far from being physically obvious: it requires ample (mathematical) regularity constraints which are the opposite of the observed strong variability down to smaller scales. Mathematically, it corresponds to very particular measurable properties: one considers only regular measures with respect to the usual line, surface, volume measures, i.e. Lebesgue measures. Indeed, the simplest illustration of scaling and scale invariance is to consider the (apparently "metric" in fact "measure") idea of dimension of a set of points as it often occurs in geophysics. The intuitive (and essentially correct) definition is that the "size" of the set n(L) at scale L is given by:

$$n(L) \propto L^D \qquad (1)$$

where D is the dimension (e.g the length of a line $\propto L$, the area of a plane, $\propto L^2$...or the number of in situ meteorological measuring stations on the earth in a circle radius $L \propto L^{1.75}$ (Lovejoy et al , 1986a,b), the distribution of raindrops on a piece of blotting paper $\propto L^{1.83}$ (Lovejoy and Schertzer , 1989a) and the occurrence of rain during a time period $T \propto T^{0.8}$ (Hubert and Carbonnel (1988), Tessier et al. (1989)) .... The "volume" (actually the measure of the set) is therefore a simple scaling (power law) function, and the dimension is important precisely because it is scale invariant (independent of L). We recall that the Hausdorff dimension D(A) of a (compact) set A may be defined by generalization to non-integer D of the divergence rule "the length of a surface is infinite, its volume is zero..."

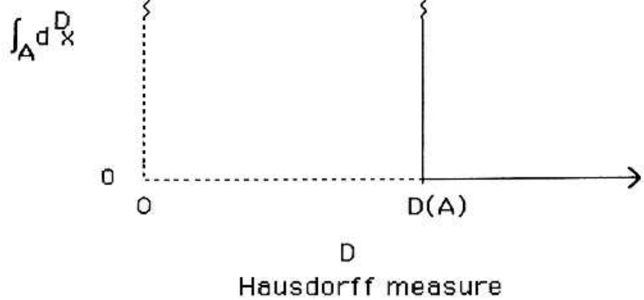


Fig. 2. Illustration of the divergence rule for Hausdorff measures, generalizing the divergence rule "the length of a surface is infinite, its volume is zero...". The transition at D=D(A), from infinity to zero, defines the Hausdorff dimension of the set A.

with the rather straightforward extensions (to non-integer D) of the d-Lebesgue measure (defined for integer d) to the D-dimensional Hausdorff measure. Thus we use the notation $\int_A d^D x$ for the D-dimensional Hausdorff measure of a (compact) set A and the Hausdorff dimension D(A) of A is hence defined by the divergence rule [1] (see Fig. 2):

$$\int_A d^D x = \infty, \text{ for } D<D(A); \int_A d^D x = 0 \text{ for } D>D(A) \qquad (2)$$

One may note that the D(A)-measure of A is not necessarily finite and non-zero: some logarithmic corrections (exponents $\Delta_i$ on the i-th iterate of the logarithm[2], are "sub-dimensions") may be needed to obtain fineness and precise determination of the Hausdorff dimension (they may give rise to the appearance of 'lacunarity', eg. Smith et al. (1986)).

In other words, the "natural" framework for fields is not functional analysis (nor geometry...!), but (mathematical) measures. Indeed, the use of functions rather than the (more general) measures is often purely a mathematical artifact. It is unnecessarily stringent since really what we can empirically measure or describe is not in fact a value at a (geometrical) point, but rather a value on "nearly any" (small) set surrounding this point . Such considerations are at the basis of (mathematical) measure theory which renders quite precise the notion of "nearly any" set [3]. Thus geophysics seems more and more associated with singular measures with respect to Lebesgue measures[4]. Going a step further we will be interested in (random) *linear operators acting on measures*, as fundamental tools to study nonlinear variability. Such apparently abstract questions can be concretely addressed by apparently simple-minded geophysical models, but with rather general non-trivial consequences and properties corresponding to the more abstract tools mentioned above. In fact we will try to give two approaches to the same problem: one which is constructivist (the multiplicative processes) and the other one which is non-constructivist ("flux dynamics").

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[1] It is easy to check that Eq. 1 is consistent with this divergence rule. Indeed, interpreting Eq. 1 as the fact that the number of cubes of size $l = L/\lambda$ needed to cover the fractal set will be of the order $\lambda^D$ and since the D'-volume of an elementary cube is $l^{D'}$, it follows that the sum of their D'-volumes -of the order of the D-Hausdorff measure- will follow the indicated divergence rule.

[2] The volume of an elementary cube ($l^D$) is now 'corrected' by factors of the type $[\mathrm{Log}_i(1/l)]^{\Delta i}$ ; where $\mathrm{Log}_i$ is the i-th iterate of the logarithm.

[3] It needs to be member of the "tribe", usually the borelian tribe...

[4] Regular (respectively singular) measures with respect to Lebesgue measures means that (almost everywhere) they correspond to a product of a density (a function) and a Lebesgue measure (resp. they don't ).

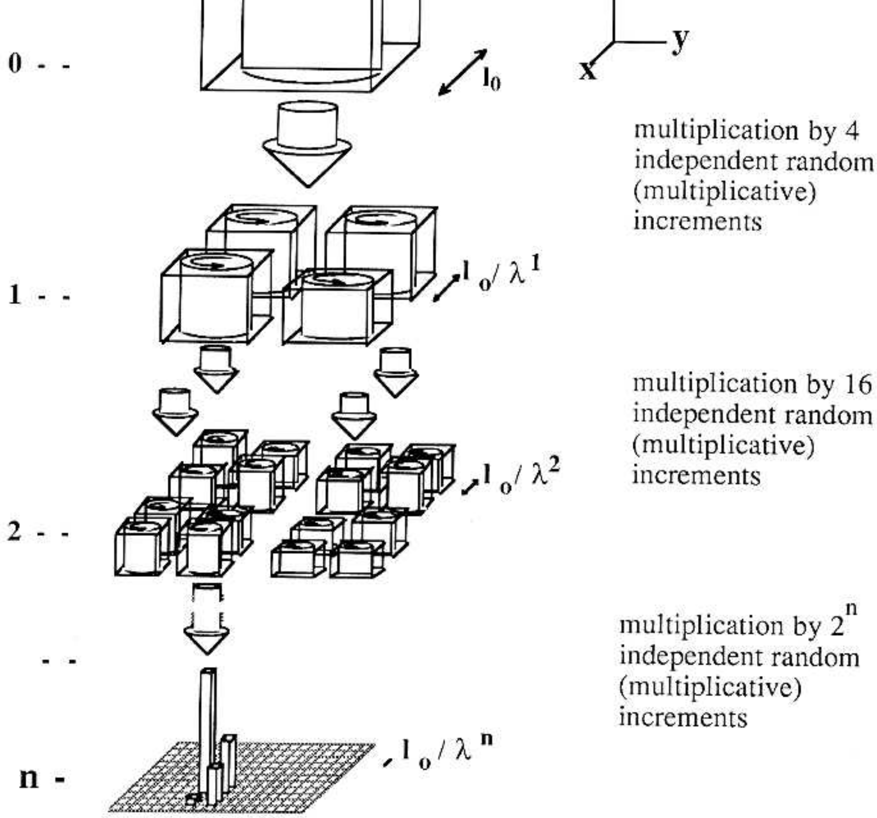


Fig. 3. A schematic diagram showing few steps of a discrete multiplicative cascade.process, here the "$\alpha$-model" with two pure singularity orders $\gamma^-$ ($>-\infty$) and $\gamma^+$ (corresponding to the two values taken by the independent random increments, $\lambda^{\gamma^-}<1$ and $\lambda^{\gamma^+}>1$) leading to the appearance of mixed singularity orders $\gamma$ ($\gamma^- \leq \gamma \leq \gamma^+$).

In both cases, the abstract object studied remains the scale invariant group of geodynamical equations (e. g. Navier-Stokes equations for flows).

The problematic of nonlinear variability, over wide range of scale, has been considered for a long time with respect to the mysterious turbulent behavior of fluid dynamics, especially their asymptotic (and universal) behavior when the dissipation length goes to zero (fully developed turbulence). Conceptual advances occurred using apparently simple models of self-similar cascades, as opposed to the frustratingly tedious developments of renormalization techniques... which still fail to grasp the intermittency problem. From very general considerations (going back to the famous poem of Richardson (1922)), the phenomenological models of turbulence have become more and more explicit (to quote a few: Novikov and Stewart (1964), Yaglom (1966), Mandelbrot (1974), Frisch et al. (1978)...; see for review Monin and Yaglom (1975)), sometimes in an overly restrictive manner. However their common theme -how does the energy flux spreads into smaller scales in successive steps while respecting a scale invariant conservation principle- is far from being restricted to space through the surface of a sphere of radius roughly proportional to this scale. In this sense we can speak of probability flux of points on a strange attractor, e.g. the flux of points flowing to smaller scales on this strange attractor, hence this generality of "flux dynamics" we will discuss, paralleling the classical thermodynamics, but with very strong divergences.... Note the basic fluxes will respect important scale conservation properties (e.g. the ensemble average of the energy flux ...) as some corresponding basic quantities (e. g. ensemble average of the energy ) in the framework of thermodynamics... We will also discuss the related fields which are not constrained to such scale conservation (such as scalar concentration, velocity field...).

## MULTIPLICATIVE PROCESSES AND FLUX DYNAMICS

The key assumption in phenomenological models of turbulence (which has recently became more explicit) is that successive steps define (independently) the fraction of the flux of energy distributed over smaller scales. Note that it is clear that the small scales cannot be perceived as adding some energy but can only (multiplicatively) modulate the energy passed down from larger scales (hence the lack of relevance of additive processes which nevertheless have been sometimes used to simulate such fields (e.g. Voss 1983)). Hence bare densities $\varepsilon_\lambda$, resulting from cascade processes from outer scale $l_0$ (which will be assumed equal to 1, without loss of generality ) to $l$ (the homogeneity scale)$=l_0/\lambda$ are multiplicatively defined (see Fig. 3 for illustration):

$$\varepsilon_{\lambda\lambda'} = T_\lambda(\varepsilon_{\lambda'})\ \varepsilon_\lambda \qquad (3)$$

$T_\lambda$ denotes a spatial contraction of ratio $\lambda$ (>1). In the isotropic case, for any point $\underline{x}$; $T_\lambda \underline{x} = \underline{x}/\lambda$ ; for any set A: $T_\lambda(A)= \{T_\lambda \underline{x} / \underline{x} \in A\}$; for any function f :$T_\lambda [f(\underline{x})] = f(\lambda\underline{x})$; for any measure $\mu$ and any set A $\int_A d[T_\lambda(\mu)] = \int_{T_\lambda(A)} d\mu$ and more generally for any function f (i.e. not only for $1_A$, the indicator function of the set A): $\int f d[T_\lambda(\mu)] = \int T_\lambda(f) d\mu$.

In case of (scaling) anisotropy, more involved contractions of space are required (see Schertzer and Lovejoy 1989b for a review). For instance, in order to avoid the classical but untenable 2D/3D dichotomy between large and small scale atmospheric dynamics, we have proposed an anisotropic scaling model of atmospheric dynamics (Schertzer and Lovejoy (1983, 1984, 1985a,b, 1987a,b), Lovejoy and Schertzer (1985), Levich and Tzvetkov[1] (1985)). In this model, the anisotropy introduced by gravity via the buoyancy force results in a differential stratification and a consequent modification of the effective dimension of space, involving a new "elliptical" dimension ($d_{el}$, see below), with resulting anisotropic shears. In isotropy, $d_{el}=3$, while in completely flat (stratified) flows, $d_{el}=2$. Empirical and theoretical evidence were given indicating $d_{el}$ is rather the intermediate value $d_{el} \approx 23/9=2.5555$... Indeed, the requisite scale changes $T_\lambda$ can be far more general than simple magnifications or reductions. It turns out that practically the only restrictions on $T_\lambda$ are that it has group properties, viz: $T_\lambda = \lambda^G$ where G is a the generator of the group of scale changing operations, and that the balls $E_\lambda = T_\lambda(S_1)$ ($S_1$ being the unit sphere) decreasing with $\lambda$,. In this "Generalized Scale Invariance" ("GSI"), G can be either a matrix -"linear GSI" (Schertzer and Lovejoy 1983, 1984) $E_\lambda$ are self-affine ellipsoids rather than the self-similar spheres of the isotropic case (G=identity)-, or even a non-linear operator (see Schertzer and Lovejoy 1987b,

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[1] They also pointed out the possible breaking of mirror symmetry for atmospheric dynamics, hence the importance of the associated helicity.

$\phi_{el}$. is then defined by the volume of the $E_\lambda$ (hence is a measurable property, rather than a metric property) This anisotropic framework allows rather straightforward extensions of Hausdorff measures and dimensions, still based on the divergence rule (Eq. 2), and the effective dimension of the space, the "elliptical" dimension $d_{el}$ of the space, is simply the trace of $G^2$: $d_{el}=Tr(G)$

Leaving additive (stochastic) processes (which had been used on purely geometrical grounds, e.g. fractional brownian motions -for modelling landscapes etc...- of the fractal geometry) to multiplicative processes, one encounters surprising properties: multiplicity of singularities, scaling and dimensions, rather than uniqueness. Let us discuss these properties briefly: a priori a fairly direct consequence of Eq. 3 is the existence of a generator for the one parameter multiplicative (semi-) group of the bare densities:

$$\varepsilon_\lambda = e^{\Gamma_\lambda} \tag{4}$$

where $\Gamma_\lambda$ is its generator, still with the homogeneity scale $l = l_0/\lambda$. $\Gamma_\lambda$ is a certain operator whose main properties (especially its asymptotic behavior, $l$ going to 0 or $\lambda$ going to $\infty$) we will analyze. $\Gamma_\lambda$ should in *some sense* (see below) become independent of $\lambda$, i.e. approach its limit $\Gamma$ as the homogeneity scale approaches zero. For positive values $\gamma$ of $\Gamma_\lambda$, divergence of $\varepsilon_\lambda$ occurs as $\lambda$ tends to $\infty$, hence such values correspond to (algebraic) orders ($\gamma$) of singularity. Conversely negative values correspond rather to (algebraic) orders of regularity. Nevertheless for brevity, we will keep frequently the expression singularity (instead of regularity) in both cases to shorten the expressions. As soon as this generator does not reduce (Schertzer and Lovejoy 1983 and 1984) to only two values $\gamma^+>0$ and $\gamma^- = -\infty$ (the once celebrated "β-model" (Novikov and Stewart 1964, Mandelbrot 1974, Frisch et al. 1978) corresponding to the alternative of dead ($\lambda^{\gamma^-}=0$) or alive (and $\lambda^{\gamma^+}>1$) sub-eddies, the *pure singularity orders* $\gamma^-$ and $\gamma^+$ lead to the appearance of *mixed singularity orders*. In particular, as soon as $\gamma^- > -\infty$ (the "α-model"), mixed singularities of different orders $\gamma$, are built up step by step (cf. Fig. 3) and bounded by $\gamma^-$ and $\gamma^+$ ($\gamma^- \le \gamma \le \gamma^+$, $\gamma^-$ and $\gamma^+$ corresponding then to the alternative of weak ($1>\lambda^{\gamma^-}>0$) or strong ($\lambda^{\gamma^+}>1$) sub-eddies). In other words, as pointed out by Schertzer and Lovejoy (1983), leaving the far too simple alternative dead or alive ("β-model") to weak or strong ("α-model") leads to the appearance of a full hierarchy of levels of survival, hence the possibility of a hierarchy of dimensions of the set of survivors for these different levels. In this α-model (as in more elaborate ones) the different orders of singularities (or survival levels) define the multiple scaling of the (one-point) probability distribution [3]:

$$\Pr(\varepsilon_\lambda \ge \lambda^\gamma) \approx N_\lambda(\gamma)/N_\lambda \approx \lambda^{-c(\gamma)} \tag{5}$$

where $N_\lambda(\gamma)$ is the number of occurrences of singularity order greater than $\gamma$, $N_\lambda$ is the total number of events examined. We temporarily postpone discussion on the accuracy of the

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[1] i.e. inf Re $\sigma(G)\ge 0$; $\sigma(G)$ being the (generalized) spectrum of G (acting on $\Re^d$): $\sigma(G)=\{\mu \in C \mid G-\mu 1$ non-invertible on $C X \Re^d\}$.

[2] However, G needs to be correctly normalized as discussed by Schertzer and Lovejoy 1987b.

[3] Note we are studying a whole family of measures defined by just one density, this the reason why our notation can't reduce to the very specialized notation ($\alpha$, $f(\alpha)$) introduced by Halsey et al. (1986), since they refer at one dimension (the dimension d of the embedding space) and the corresponding specialized measure. Hence, $\alpha = d-\gamma$ as singularity of the d-dimensional Lebesgue measure, and $f(\alpha))=d-c(\gamma)$.

incomplete statistical descriptors of a field)- and rather convenient empirical analysis technique to test the the probability distribution multiple scaling (PDMS, introduced by Lavallée et al 1989) and to determine $c(\gamma)$. For the moment, we would like to insist on the interest of such a formula in gaining insights in different fundamental aspects of multiple scaling.

Obviously singularities will prevent convergence in the usual sense, i.e. even if the $\varepsilon_\lambda$ are rather smooth functions (for a given $\lambda$), they do not admit a function as their limit. Indeed, their limit will be rather defined by the limit of the fluxes (i.e. integrals of the density) over differents sets. One may note also that $N_\lambda$ will be proportional to $\lambda^d$ -the number of boxes, size $\lambda^{-1}$, required to cover the relevant region of the embedding space (which can be fractal...) of dimension d (integer or not)- multiplied by the number ($N_i$) of realizations (e.g. images) examined. Hence, when $c(\gamma)$ is smaller than d it has a rather immediate meaning of a codimension[1] $= d-d(\gamma)$; $N_\lambda(\gamma) \approx \lambda^{d(\gamma)}$, where $d(\gamma)$ (>0) is the dimension of the fraction of the space occupied by the singularities of order greater than $\gamma$ on "nearly" each realization. Larger values of $c(\gamma)$, which have often been disregarded, correspond to more rare events: singularities of orders which "nearly" never appear on a realization. At first glance they seem to correspond to negative dimensions, sometimes mysteriously called "latent dimensions". However, there is no mystery at all, since $c(\gamma)$ still has a meaning of a codimension: no any longer in an individual realization, but in the subspace of the (infinite dimensional) probability/state space that our finite sample size enables us to explore by a cut of finite dimension. Indeed the dimension of this subspace can be estimated as $d + d_s$, where $d_s$ - termed as the "sampling dimension" (at scale $\lambda^{-1}$)- is estimated by writing the number of images (or realizations) $N_i$ as $\lambda^{d_s}$. Indeed when $c(\gamma)$ is smaller than $d + d_s$, $\gamma$ occupies a fraction of the accessible subspace having dimension $d(\gamma) = d + d_s - c(\gamma)$. Of course, increasing the number of images, hence the sampling dimension, allows us to encounter more easily higher singularities occupying a fraction of the accessible subspace, with well defined dimension ($d(\gamma) = d + d_s - c(\gamma) > 0$). The corresponding mathematical subtlety underneath the important difference between cases $c(\gamma)<d$ and $c(\gamma)>d$, is the "almost surely" or not properties, the latter do correspond to extremely rare events.

However, these extremely rare events are in fact of overwhelming importance since they imply divergence of statistical moments, i.e.these singularities prevent convergence of all statistical orders: by integrating the density over a set A with dimension D (to obtain the flux through A), the resulting smoothing may be sufficient so that convergence is obtained for low order statistics, but not for orders higher than a critical order $h_D$ of divergence. Indeed, let us point out this rather immediate consequence of Eq. 5, by introducing first the trace (paralleling the definition of the trace of the density operator in Quantum Statistical Mechanics, see below) of the $h^{th}$ power of the flux $\Pi_\lambda$ over an (averaging) set A of dimension D (integration performed with resolution $\lambda^{-1}$ on $A_\lambda$, A measured with the same resolution):

$$\mathrm{tr}_{A\lambda}\, \varepsilon_\lambda{}^h = \int_{A_\lambda} \varepsilon_\lambda{}^h\, d^{hD}x$$
$$\approx \sum_{A_\lambda} \varepsilon_\lambda{}^h\, \lambda^{-hD} \qquad (6)$$

[1] In particular, in the case of the $\beta$-model there is a unique codimension c, characterizing the fraction of the space occupied by alive sub-eddies. The parameter $\beta$ is $\lambda^{-c}$.

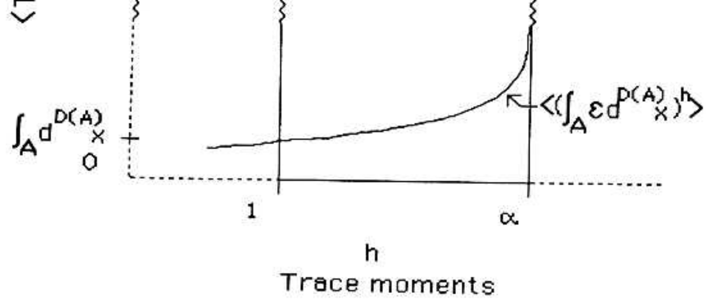


Fig. 4. the twin divergence of trace-moments.

a-priori any singularity of order higher than D, may create divergences of the trace but are extremely rare (since their frequency of occurrence tends to zero as $\lambda^{-c(\gamma)}$ ). One may evaluate the importance of these by considering their statistics (trace-moments introduced by Schertzer and Lovejoy 1987 a-b) for an arbitrary singularity of order $\gamma$:

$$Tr_{A_\lambda}\, \varepsilon_\lambda^{h} = \langle tr_{A_\lambda}\, \varepsilon_\lambda^{h} \rangle \geq N_\lambda(\gamma)\, \lambda^{h\gamma}\, \lambda^{-hD} = \lambda^{[h\gamma - c(\gamma)]-(h-1)D} \qquad (7)$$

thus diverges, for some orders of singularity, as soon as:

$$K(h) \geq (h-1)D \qquad (8)$$

where:

$$K(h) \equiv \sup_\gamma[h\gamma - c(\gamma)] \quad \{\text{hence: } c(\gamma) \equiv \sup_h[h\gamma - K(h)]\}$$
$$\text{or: } h = dc(\gamma)/d\gamma,\ K(h) = h\gamma - c(\gamma) \quad \{\gamma = dK(h)/dh,\ c(\gamma) = h\gamma - K(h)\} \qquad (9)$$

On the one hand, Eq. 9 corresponds to the Legendre transform of $c(\gamma)$ as pointed out by Frisch and Parisi (1985), Halsey et al. (1986) and as the resulting K(h) does correspond -by the method of steepest descent to the exponent of the moment of the density of the flux (at least to first order, i.e. omitting logarithmic corrections):

$$\langle \varepsilon_\lambda^{h} \rangle = \lambda^{K(h)} \langle \varepsilon_1^{h} \rangle = e^{K(h)\mathrm{Log}(\lambda)} \langle \varepsilon_1^{h} \rangle \qquad (10)$$

the Legendre transform establishes a well defined relation between orders of singularities and orders of moments. Note that conservation in ensemble average of the flux requires conservation of densities ($\langle\varepsilon_\lambda\rangle = \langle\varepsilon_1\rangle$) thus $K(1)=0$. On the other hand, as pointed out by Schertzer and Lovejoy 1983, the divergence rule, Eq. 8, introduces a hierarchy of critical codimensions C(h), simply defined as:

$$C(h)\,(h-1) = K(h) \qquad (11)$$

since the former divergence rule (Eq. 8) can be rewritten ($\lambda \to \infty$, $\varepsilon_\lambda \to \varepsilon$):

$$Tr_A\, \varepsilon^{h} = \infty,\ D < C(h),\ \text{i. e. } h > h_D,\ C(h_D) = D \quad \{\gamma_D = dK(h)/dh|_{h_D}\} \qquad (12)$$

$$Tr_{A_\lambda}\,\varepsilon_\lambda{}^h \le \langle\Pi_\lambda{}^h(A)\rangle \text{ any } h\ge 1 \qquad (13)$$

Conversely, as discussed more thoroughly by Schertzer and Lovejoy 1987b, convergence of statistical moment of order h (h>1) is assured by the convergence of the $h^{th}$ trace moment ; for h<1 divergence of the trace moment implies degeneracy of the flux (the set A has a so small dimension ($D<C_1\equiv C(1)$) that almost surely the flux is null). We thus obtain a twin divergence rule for the trace moments (represented in Fig. 4) implying non-degeneracy of the flux ($h\le 1$) and divergence of the flux ($h\ge h_D>1$). Note that non-degeneracy of the flux implies conservation of the ensemble average flux[1]:

$$\langle\varepsilon_\lambda\rangle = \langle\varepsilon_1\rangle = 1 \text{ and } D>C_1\ (\equiv C(1)) \Rightarrow \langle\Pi_\lambda(A)\rangle = \langle\Pi_1(A)\rangle\ (\equiv \int_A d^Dx\,) \qquad (14)$$

Note that $C_1$ ($\equiv C(1) = K'(1)$, due to Eq. 11) is at same time the codimension of singularities contributing to the average (h=1) and the order of these singularities, since by virtue of Legendre transform it is the fixed point of $c(\gamma)$:

$$c(\gamma) = \gamma \Rightarrow \gamma = C_1\ (\equiv C(1) \equiv K'(1)) \qquad (15)$$

Multiple scaling (for the statistical moments) corresponds to the fact K(h) is no longer linear ($= C_1(h-1)$) as in the β-model but depends on a whole hierarchy of codimensions C(h) ($\neq C_1$, for $h\neq 1$). As the first characteristic function (or moment generating function) $Z_\lambda(h)$ and second characteristic function (or cumulant generating function) $K_\lambda(h)$ of the generator $\Gamma_\lambda$, are by definition:

$$Z_\lambda(h) = e^{K_\lambda(h)} = \langle e^{h\Gamma_\lambda}\rangle \quad (\equiv \langle\varepsilon_\lambda{}^h\rangle) \qquad (16)$$

multiple scaling corresponds to algebraic divergence ($\lambda\to\infty$) of $Z_\lambda(h)$ and thus to logarithmic divergence of $K_\lambda(h)$ (see Eq. 10), a fundamental property we will exploit below. Note here, we are dealing with characteristic functions in the Laplace sense, since $Z_\lambda(h)$ is obtained by Laplace transform (instead of Fourier transform) of the probability distribution. In order to make some crude connections with statistical physics, $\Gamma_\lambda$ can be considered as the negative of a pseudo-hamiltonian ($-H_\lambda=\Gamma_\lambda$), with h as the inverse of temperature (h=1/T, the Boltzmann constant being set equal to 1), $Z_\lambda$ is called a partition function and the "free-energy" ($F_\lambda$) would correspond to $K_\lambda(h)/h$. More generally (in statistical Quantum Mechanics), the "density operators" $\rho_\lambda = e^{-H_\lambda/T}$ (corresponding to $\varepsilon_\lambda{}^h = e^{h\Gamma_\lambda}$) are considered along with their trace over different sub-spaces of states, each trace corresponding to a partition function. The densities $\rho_\lambda$ and $\varepsilon_\lambda$ are both defined on a fairly abstract space [eg. in quantum mechanics the space of wave functions]. The trace moments correspond to the trace of the density operator but here on the space of the measures of (compact) supports (the different sets A, used for averaging). Finally, as $c(\gamma)$ characterize the logarithm of the probability distribution of $\Gamma_\lambda$, they correspond to entropies ($S_\lambda$) (of the state $\gamma$), and indeed the Legendre duality between $K_\lambda(h)$ and $c(\gamma)$ does correspond to the same duality between $F_\lambda(T)/T$ and $S_\lambda(E)$ (the conjugate variables being 1/T and the energy E). Let us emphasize that in both cases, this property simply results from the fact that the Laplace transform of the

[1] As it corresponds to a "martingale "property, it assures a "weak measurable' convergence of the process (see Schertzer and Lovejoy 1987b for discussion).

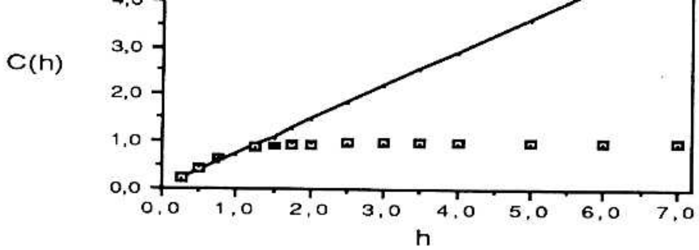


Fig. 5. Spurious scaling obtained with a Gaussian generator, the theoretical line is the solid line $C(h) \propto h$, squares are observed points and lead to the spurious appearance of bounded $C(h) < C_\infty < \infty$!

probability distribution[1] (or conversely of the partition function) reduces to Legendre transform of the exponents. In order to develop a nonconstructive approach, which can be called "fluxdynamics", considering $\varepsilon$ *per se* (the limit $\varepsilon$ of the $\varepsilon_\lambda$, at zero homogeneity scale, $\lambda$ going to infinity) as a linear operator on the measures (converting the D-volume, D being the dimension of A, integer or not, into the flux over the set A) we need to investigate some basic properties of this limit and its generator.

## WILD/SMOOTH MULTIFRACTALITY AND CANONICAL/MICRO-CANONICAL CONSERVATION

In this section we emphasize the consequences of divergence of moments before proceeding to characterizing the generator. Divergence of moments is a wild statistical behavior very far from gaussianity (or quasi-gaussianity), and is due to "hyperbolic" (algebraic) fall-off of the probability distribution:

$$\Pr(|X| \geq s) \approx s^{-\alpha} \ (s >> 1) \Rightarrow \text{any } h \geq \alpha : \langle |X|^h \rangle = \infty \qquad (17)$$

It turns out, that among these "hyperbolic" random variables some are rather well defined, since they are mostly (but surprising!) generalizations of gaussian laws. These are the Lévy stable random variables ($0<\alpha<2$) satisfying "generalized central limit theorems", hence intervening in additive processes as discussed in subsequent sections and especially with the help of Appendix A dealing with a particular type of them. However we (Schertzer and Lovejoy 1985) already used the expression "hyperbolic intermittency" to describe the effect of this strong variability for a wider range of $\alpha$ (i.e. $\alpha \geq 2$), as we pointed out this divergence as a general consequence of multiplicative processes and that the corresponding critical order of divergence $\alpha = h_D$ (theoretically, determined by Eq. 12) has no absolute bound. Waymire and Gupta (1985) have used the expression "fat-tailed" for such (asymptotically algebraic) distributions, and "long-tailed" for the log-normal law, to distinguish these distributions from standard exponential "thin-tailed" distributions. In the preceding section we showed that hyperbolic behavior is expected from averaging a multifractal field on a set of too small dimension D. It has been empirically estimated in a variety of meteorological fields: $h_D \approx 5$, for temperature (Lovejoy and Schertzer 1986a,b, Ladoy et al 1986), $h_D \approx 1.66$ in changes in storm integrated rainrates (Lovejoy 1981), $h_D \approx 1.06$ in radar reflectivity factors of rain (Schertzer and Lovejoy 1987), and respectively

[1] It implies also the convexity of K(h) (or F((T)/T), hence of $c(\gamma)$ (or S(E)).

spurious (or pseudo) scaling exponents. Indeed, these methods are based extensively on the law of large numbers which blows up due to the statistical divergences. Fig. 5 shows how misleading can be the appearance of spurious scaling, on a well known simulated field: classical estimation will lead to bounded codimensions C(h)... even though the true C(h) increases linearly with h! Conversely, clear understanding of spurious scaling can be used to explain most of the behavior of certain data. For instance we argue (Schertzer and Lovejoy 1983,1984, 1985, Lovejoy,and Schertzer, 1986a) that the presumed critical moment order $h_D \approx 5$ for wind speed, may well explain the overall behavior of the observed scalings exponents of the structure functions of the velocity field collected by Anselmet et al. (1984)!

A direct consequence of the hyperbolic behavior of the dressed densities $\varepsilon_{\lambda, D}$ (obtained by D-dimensional averaging, at scale $\lambda^{-1}$) is that their singularity codimensions $c_D(\gamma)$ are quite different from their bare counterparts $c(\gamma)$, since they become linear for orders greater than the critical singularity order $\gamma_D$:

$$c_D(\gamma) \approx h_D(\gamma-\gamma_D),\ \gamma \geq \gamma_D \tag{18}$$

this is an immediate consequence of hyperbolic behavior as described by Eq. 17, as well as from the corresponding divergence of characteristic function ($K_{\lambda, D}(h)=\infty$, $h \geq h_D$) and the fact that Legendre transform breaks down[1] for linear functions. Conversely, for the same reasons, K(h) becomes linear as soon as there is an upper bound ($\gamma_0$) of the singularity order:

$$c(\gamma) \rightarrow \infty, \text{ when } \gamma \rightarrow \gamma_0 \Leftrightarrow K(h) \approx \gamma_0 h\ (h>>1), \text{ hence } C_\infty=\gamma_0. \tag{19}$$

However, the hyperbolic behavior is expected only for singularities of order greater than the dimension of the averaging set A. It obviously can't occur if we are imposing a much more strict conservation than conservation in ensemble average i.e. a strict conservation on A of the flux in each realization, since we have in the latter case:

$$\varepsilon_\lambda\, \lambda^{-D} \leq \Pi_\lambda(A)=\Pi_1(A) \tag{20}$$

Paralleling, once again classical thermodynamics, one can speak respectively of canonical conservation (or cascade) in the former case, and micro-canonical conservation (or cascade) in the latter (see for instance[2] Benzi et al. (1984) Pietronero and Siebesma (1986), Sreenivasan et Meneveau (1988)). Micro-canonical conservation assumption has many defects: not only we are usually dealing with open systems (as in thermodynamics), but this assumption turns out to be quite demanding. Indeed it requires in fact conservation at every scale, so we can even speak of "*pico-canonical*" assumption: strict conservation is implied not only on the largest scale of A, but on the smallest scale due to scaling behavior of the consider process! Hence, we have rather sharp distinctions[3] between wild multifractality associated with canonical conservation on the one hand, and smooth multifractality associated with micro- (in fact pico-) canonical conservation.

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[1] A rather direct consequence of the geometrical interpretation of the Legendre transform as the envelope of the tangencies.

[2] In fact a micro-canonical version of the $\alpha$–model is often called a "random $\beta$–model". The latter expression (trying to designate that the fraction of the space occupied active sub-eddies is randomly chosen) is somewhat misleading, since the "$\beta$–model" is already a random model...

[3] On the other hand, one may note that micro-canonical conservation refers to a given set and dimension: it no longer holds on sets of smaller dimensions. Hence, micro-canonical conservation is at the same time too precise and too vague...

generator $\Gamma$ -the limit of the $\Gamma_\lambda$ - is fundamental. Corresponding to the definition of the $\Gamma_\lambda$ (Eq. 4), we have, at least formally:

$$\varepsilon \equiv \text{limit}_{\lambda\to\infty}(\varepsilon_\lambda) = e^{\Gamma} \qquad (21)$$

One may note also we have (corresponding to Eq. 4) the following "dynamical" (and somewhat formal) equation for the $\varepsilon_\lambda$:

$$\partial\varepsilon_\lambda/\partial\lambda = \gamma_\lambda\, \varepsilon_\lambda \;\; ; \gamma_\lambda = \partial\Gamma_\lambda/\partial\lambda \qquad (22)$$

where $\gamma_\lambda$ is the infinitesimal generator. This equation gives the cascade dynamical sense when we are studying a cascade on a space-time domain[1]. As discused by Schertzer and Lovejoy (1987a-b, 1989a), the generator must satisfy four main properties:

i) $\Gamma$ is a random noise,with infinite band-width $[1, \infty]$. The bare or (finite resolution) generator $\Gamma_\lambda$ are rather to be understood as the corresponding filtered noises restricted to the wave number band $[1, \lambda]$.

ii) the second characteristic function (or cumulant generating function) $K_\lambda(h)$ of the bare generator $\Gamma_\lambda$ has a logarithmic divergence ($\lambda \to 0$) in order to assure multiple scaling, i.e.:

$$K_\lambda(h) \approx \text{Log}(\lambda)\ K(h). \qquad (23)$$

iii) in order to have some finite moments of positive orders ($Z_\lambda(h)$ and $K_\lambda(h) < \infty$ for $h>0$), the probability distribution of positive fluctuations of the bare generators $\Gamma_\lambda$ must fall off more quickly than exponentially.

iv) the generator needs to be normalized ($K_\lambda(1)=0$) in order to assure (canonical) conservation of the flux.

It turns out that properties i) and ii) correspond (Schertzer and Lovejoy 1987b) to the fact that the (generalized) spectrum $E_\Gamma(k)$ of the generator should be then proportional to the inverse of the wave-number:

$$E_\Gamma(k) \propto k^{-1} \qquad (24)$$

since the characteristic function will correspond to its integral. Such noises are often called "1/f noises" or "pink noises". Usually, one considers only gaussian noises or quasi-gaussian noises. We have already indicated that there is no fundamental reason to restrict our attention to quasi-gaussianity, and thus consider hyperbolic noises. Indeed, among the hyperbolic noises, Lévy stable noises ($0<\alpha<2$) are particularly important, since they define a family of universal generators as we will discuss latter. However, the third property indicated, which is due to the fact the moments of $\varepsilon_\lambda$, are Laplace transforms of the probability density of $\Gamma_\lambda$, lead us to restrict our attention to extremely unsymmetric hyperbolic noises, since we can accept a hyperbolic fall-off of the probability distribution only for the negative fluctuation of $\Gamma_\lambda$. Considering Lévy stable noises (or hyperbolic noises $0<\alpha<2$), one has to generalize the notion of spectrum (the usual one diverges, since it corresponds to a second order moment) as discussed by Schertzer and Lovejoy (1987a-b, 1989). The fourth property is easy to satisfy since if $\Gamma'_\lambda$ is not yet normalized, we can deduce a normalized generator $\Gamma_\lambda$ by:

$$e^{\Gamma_\lambda} = e^{\Gamma'_\lambda} / \langle e^{\Gamma'_\lambda} \rangle \qquad (25)$$

---

[1] The scaling is usually then strongly anisotropic on the space time domain.

Note that the properties of the generator stressed above are on the one hand quite different from the usual properties of (pseudo-) Hamiltonians, especially their scale dependency[1]. On the other hand, they give a precise definition of multiple scaling, especially by requisite properties of the second characteristic functional or of the (more or less) equivalent of the free energy (which might be called "free flux"?).

As in additive processes, one may look for universality classes in the sense that for whatever generator is used (here the flux generator, the infinitesimal increment in the additive ones) under repeated iteration -through (renormalized) multiplication or addition- it may converge to a well defined limit which depends on relatively few of its characteristics. Appendix A first recalls the classical (but not well enough known) results for additive processes associated with generalized central limit theorem, here the classes and "basins of attraction" are primarily[2] defined by the Levy index $\alpha$, the critical moment order (i.e. higher order moments diverge) of the increments

One has to be careful about the definitions of convergence and universality, since it has been obscured by some misplaced claims (Mandelbrot, 1989)that such universality cannot exist in multifractal processes. Indeed, it is easy to check that repeated multiplications corresponding to a process with fixed discretization (i.e. a fixed elementary ratio of scale $\lambda_0>1$) fails to create a simplifying convergence to universal generators (eg. the $\alpha$-model remains an $\alpha$-model), and it seems that this is the reason why Kolmogorov (1962) postulated a lognormal behavior, without postulating convergence[3] to it . However, if we are discussing continuous cascade processes, i. e. processes which have an infinite number of cascade steps over any *finite* range of scales (i.e. elementary ratio of scale $\lambda_0 \rightarrow 1_+$), we are facing quite a different problem. Indeed, such processes may be obtain from a discrete model (finite number of discrete steps over the given ratio of scales) by introducing more and more steps up to an infinity of infinitesimal ones and keeping some properties (e.g. the variance of the generator on this given scale ratio). Obviously while such properties are best mathematically studied directly on the generator, we should also establish the physical relevance of doing so. Indeed, -generalizing the test field method introduced in homogeneous turbulence by Kraichnan (1971)- we may introduce new intermediate scales first as rather passive components, advected by the others, and then include them in the whole set of "active" scales. In this respect, the passive scalar example studied by Schertzer and Lovejoy 1987a is illustrative: the density of the flux ($\varphi$) controlling the passive scalar diffusion is a product powers of densities the energy flux ($\varepsilon$) and the scalar scalar variance flux ($\chi$) -mainly from dimensional arguments, we have: $\varphi=\chi^{3/2}\varepsilon^{-1/2}$. In the first step, $\chi$ and $\varepsilon$ can be considered as rather independent, then in the second step considered of the same type, and identify $\varphi$ as a more complete $\varepsilon$. Hence, we are multiplying densities by densities, or simply adding generators to generators...

Now, we have to investigate which classes of generator are stable and attractive under addition and such that for the corresponding density $\varepsilon_\lambda$ will at least converge for some positive order moments (i. e. the probability density of the generator admits a Laplace transform as already discussed). Either we examine those Levy stables -usually studied in a Fourier framework (e.g. Lévy (1924, 1925, 1954), Gnedenko and Kolmogorov (1954), Gnedenko (1969), Feller (1971), Zolotarev (1986)- which also satisfy a Laplace transform or

---

1 However, the related log divergence may be loosely understood as a phase transition at low temperature (i.e. considering the h $\mathrm{Log}(\lambda)$ as the inverse of the temperature $1/T$.

2 There are two subsidiary parameters which are fixed in our case: the 'location parameter' (fixed by the normalization constraint) and the 'skewness' (set to its extremal value -1 by the condition iii, as explained below). The third subsidiary parameter, the 'scale parameter' is defined by $C_1$.

3 Yaglom (1966) seems to be less cautious on that point.

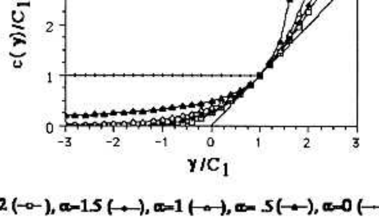


Fig. 6a. universal (bare) singularities codimension $c(\gamma)/C_1$ corresponding to the five classes; here $\alpha$=2, 1.5, 1, .5, 0.

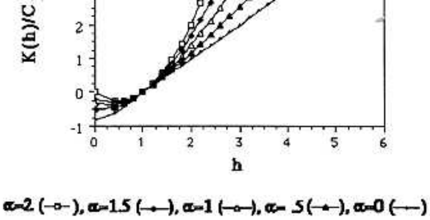


Fig. 6b. universal (bare) second characteristic function $K(h)/C_1$ ($\equiv h.F(h)/C_1$, F(h) being the "free energy"), corresponding to the five classes; here $\alpha$=2, 1.5, 1, .5, 0.

we directly study the generalized central limit theorem in the Laplace framework, as done in Appendix A[1]. In any case, it is immediately clear that the restriction imposed by Laplace transform is that we need (as condition iii) already discussed) a steeper than an algebraic fall-off of the probability distribution for the (positive) orders of singularities, hence with the exception of the Gaussian case ($\alpha$=2), we have to employ strongly asymmetric, "extremal" Levy laws. In our case, we are not considering random variables but noises, however the same characterization are relevant (characteristic functionals intervene instead of characteristic functions).

Let us examine the universal generator classes (from $\alpha$ =2 down to $\alpha$ =0), recalling that the corresponding characteristic function K(h) and codimension functions $c(\gamma)$ estimated by Legendre transform, are (Schertzer and Lovejoy, 1987a-b, 1989a) since $h^{\alpha}/\alpha$ and $\gamma^{\alpha'}/\alpha'$ are Legendre dual ($0\leq\alpha\leq2$, $1/\alpha+1/\alpha'=1$):

$$\alpha\neq1:\ K(h) = \frac{C_1\alpha'}{\alpha}(h^{\alpha}-h) \qquad \text{(only for } h\geq0 \text{ when } \alpha<2;\ =\infty \text{ for } h<0\text{);} \qquad (26)$$

$$\alpha=1:\ K(h)= C_1\, h\, \mathrm{Log}(h) \qquad (27)$$

and (restricted to increasing branches when $\alpha<2$, since $dc/d\gamma\equiv h$):

$$\alpha\neq1: \quad c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'} \qquad (dc/d\gamma>0 \text{ when } \alpha<2) \qquad (28)$$

$$\alpha=1: \quad c(\gamma)= C_1 \exp\left(\frac{\gamma}{C_1}-1\right) \qquad (29)$$

We recall that C1 ($\equiv C(1) = K'(1)$) is the fixed point of $c(\gamma)$, being at same time the codimension of singularities contributing to the average and the order of these singularities (see Eq. 15). We may introduce another convenient characteristic order of singularity :

$$\gamma_0= -\frac{C_1\alpha'}{\alpha} \qquad (30)$$

[1] One may note that only the case $0<\alpha<1$ is classically treated by Laplace transform, Appendix A extends the result for $1\leq\alpha\leq2$.

($\gamma_0$=K'(0); $\alpha$>1) or of the asymptote ($\gamma_0$=K'($\infty$); $\alpha$<1) We may then rewrite Eq. 28 ($\gamma \geq \gamma_0$ when $2 \geq \alpha > 1$; $\gamma < \gamma_0$ when $\alpha < 1$) as:

$$\alpha \neq 1,\ \gamma_0 = -\frac{C_1 \alpha'}{\alpha},\ c_0 = c(\gamma):\ c(\gamma) = c_0\left(1 - \frac{\gamma}{\gamma_0}\right)^{\alpha'} \qquad (31)$$

One may note that the c($\gamma$) introduced here corresponds rather to the probability density, instead of the probability distribution. Both are equal when c($\gamma$) is increasing (eg. for extreme singularities: $\gamma$>>0), but obviously decreasing c($\gamma$) (eg. for extreme regularities: $\gamma$<<0) of the probability is offset for the probability distribution by its minimum value (see below the role of $\gamma_0$ for the gaussian case). On the other hand, the c($\gamma$) don't coincide with the log of the probability density due to (at least!) some logarithmic terms (corresponding to sub-codimensions) which are missed by the Legendre transform, but are of no fundamental importance (as easily seen by considering the exact log of the probability density).

Let us review briefly the main properties of the five classes ($\alpha$ going from 2 to 0, hence $\alpha'$ going from 2 to $\infty$, then from $-\infty$to 0), from the gaussian generator to the $\beta$-model, crossing three Levy cases (see the corresponding Fig. 6a and Fig. 6b):

i) $\alpha=\alpha'=2$: the Gaussian generator is almost everywhere (almost surely) continuous. K(h) and c($\gamma$) are parabolae, c($\gamma$) is tangent on the $\gamma$ axis at $\gamma_0=-C_1$, C(h) is linear (= C1 h). The corresponding c($\gamma$) of the probability distribution, will remain on the $\gamma$ axis for $\gamma \leq \gamma_0$.

ii) $2>\alpha>1$ ($2<\alpha'<\infty$): the Lévy generator is almost everywhere (almost surely) discontinuous and is extremely asymmetric. The lower bound $\gamma_0$ (=-C1 a'/$\alpha$) of fractal singularities is decreasing from $-C_1$ to $-\infty$, as $\alpha$ decreases from 2 to 1. c($\gamma$) will remain on the $\gamma$ axis for $\gamma \leq \gamma_0$ and is thus strongly asymmetric (even for the probability density, since $K_\lambda(h)=\infty$ for h<0), the large orders of singularities order give rise to a steeper algebraic branch than before ($c(\gamma) \propto \gamma^{\alpha'}$, $\alpha'$>2).

iii) $1>\alpha>0$ ($-\infty<\alpha'<0$) the generator is everywhere (almost surely) discontinuous, and is obtained in fact by a one-sided unnormalized generator hence the orders of singularities are bounded by $\gamma_0$ (thus decreasing, with $\alpha$, from $-\infty$ to $C_1$), which defines thus a vertical asymptote, and now the algebraic asymptote intervenes for the large orders of regularity ($\gamma \rightarrow -\infty$, $c(\gamma) \propto \gamma^{-|\alpha'|}$). As the singularities are bounded, the same occurs for the hierarchy of critical codimension C(h) of the different moments, since we can now smooth out the highest singularity on a set A of high enough dimension D. Indeed $\gamma_0$ bounds also C(h) (see Eq. 19), hence to obtain convergence of every (positive order) moment of the flux it suffices that: $D>C_\infty=\gamma_0=\frac{C_1|\alpha'|}{\alpha}$.

iv) $\alpha=1$ ($\alpha'=\pm\infty$!): it is the special in-between case, associated with the ambiguity on $\alpha'$ (note the opposite occurs to $\gamma_0=\pm$-$\infty$), this corresponds in fact to a special case of quasi-stability (or not strict stability) briefly outlined in Appendix A. Note that the curves K(h) and c($\gamma$) are nevertheless the limits ($\alpha \rightarrow 1\pm$, $\alpha' \rightarrow \pm -\infty$) of the two preceding cases, especially

[1] This is the negative of the former $\gamma_0$ introduced by Schertzer and Lovejoy 1987a-b. The change of sign is required to obtain directly the bounds of singularities/regularities as explained.

Fig. 7a. a gaussian white-noise ($\alpha$=2), fluctuations are symmetric.

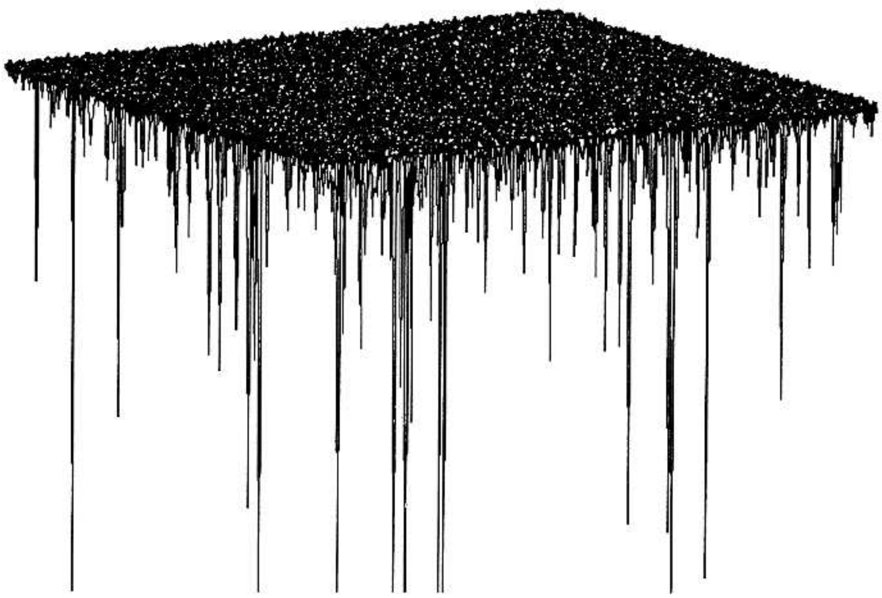

Fig. 7b. an extremal Lévy white-noise ($\alpha$=1.5), fluctuations are extremely asymmetric: only *negative* hyperbolic jumps are allowed for the generator, "digging" wild regularities ("holes")

the former algebraic asymptotes of $c(\gamma)$ tend to exponential behavior since: $(x/\alpha'+1)^{\alpha'} \rightarrow e^x$ when $\alpha' \rightarrow \infty$.

v)$\underline{\alpha=0+}$ $\underline{(\alpha'=0-!)}$ : this limiting case corresponds to divergence of every statistical moment of the generator and seems at first glance very strange, but one of its representations is none other than the once celebrated $\beta$-model ($\gamma^-=-\infty$, $\gamma^+=C_1$ =limit $_{\alpha\rightarrow 0+}(\gamma_0)$)! This fact, in turn, shows clearly the peculiarities of the $\beta$-model, once thought to be a more or less crude approximation of intermittency...

Let us point out briefly some consequences:

- the Levy cases fill the gap between the two more or less classical cases the so-called lognormal ($\alpha$=2) and the $\beta$-model, which now represent just two extremes of the whole spectrum of universal generators.

- the very symmetric gaussian case is the exception which assures the existence of negative order moments, on the contrary the asymmetric "extremal" character of Levy case corresponds to the fact that we are "digging" wild regularities ("holes") with the algebraic extremes of the Levy generator which preventing convergence of any negative order moment. (see the relevant Fig. 7a-b ).

of C(h) ($\equiv\infty$ for $\alpha\geq1$; $\equiv\gamma_0<\infty$ for $\alpha<1$ ) is decreasing with $\alpha$, leading to the finiteness of all moments for set A of dimension D>.$C_\infty$ . However, it is merely due to the fact that the wild behavior of the generator is restricted to regularities (hence the particular problem of negative order moments since they interchange regularities with singularities and conversely. The $\beta$-model yields the extreme regularity since: $C_\infty= C_1=C(h)$, any set A where the process is not degenerate,will have regular flux at all positive orders (but still none at negative orders!).

One may note that exact mathematical results have been obtained on the case $\alpha=2$ (Kahane 1985, 1987) and $\alpha<1$ (Fan, 1989).

## MULTIFRACTAL SIMULATIONS AND ANALYSIS

### Stochastic simulations

Although concentrating their attention on a particular problem, the advection of passive scalar field (eg. concentration of a passive substance) by a turbulent velocity field, Schertzer and Lovejoy (1987), Wilson et al. (1989) show a rather general procedure for simulation of multifractal fields. Indeed, we may first readily produce conserved fluxes. Indeed, due to the existence of universality classes, a Gauss or Lévy generator is rather easily obtained by "coloring", via fractional integration, a corresponding Gauss or Lévy white-noise (represented in Fig. 7a-b in order to obtain a desired Gauss or Lévy pink-noise (most of the details are given by Wilson et al. (1989), especially the Fourier techniques needed). From these fluxes, we may build up others by taking products of them or raising them to different powers. We may even fractionally integrate over them, which is especially desirable when we want to obtain for instance the concentration field itself, rather than the flux of the scalar variance. However, doing so, we will fundamentally add only an extra parameter (the order of fractional integration) to our two basic $C_1$ and $\alpha$. Indeed, a fractional integration (order -b) on a power (a) of a conserved flux.corresponds (as pointed out by Schertzer and Lovejoy (1987a)) to an affine transformation on the orders of singularity (leaving $c(\gamma)$ invariant):

$$\gamma'=a\gamma+b;\ \ h'=h/a;\ \ K'(h')= K(h) +bh';\ \ c'(\gamma')=c(\gamma) \qquad (32)$$

staying in the same type of universality (same $\alpha$). We could mainly restrict our attention to transformation with a=1, since it corresponds, at least formally[1], to the result of a fractional integration of order -b'=-(b+K(a)) on a new conserved flux (obtained by power and fractional integration K(a)).

Fig. 8 shows the two main steps needed to obtain a concentration field over a $2^8$x$2^8$ pixel grid simulated (a=-b=1/3) on a personal computer, Fig. 8a shows the corresponding conserved flux (identified to $\varepsilon$), then the resulting concentration field after fractional integration. Larger simulations are expected to be quite useful in studying radiative transfer in highly inhomogeneous (i.e. multifractal) clouds, which could be notably important to developing remote sensing techniques (cf. Gabriel et al. (1988)). See also Fig. 9a-b

### Probability Distribution Multiple Scaling (PDMS)

In analyzing empirical data of a field f known at a given resolution ($\Lambda=L/l$, L being the larger scale of the sample, eg. the size of a satellite image, $l$ being the smaller scale , eg. the size of a pixel), we seek, as proposed by by Lavallée et al. (1989), to directly apply Eq. 5 to

[1] Indeed there is no equivalence between the different ways of maintaining conservation of fluxes.

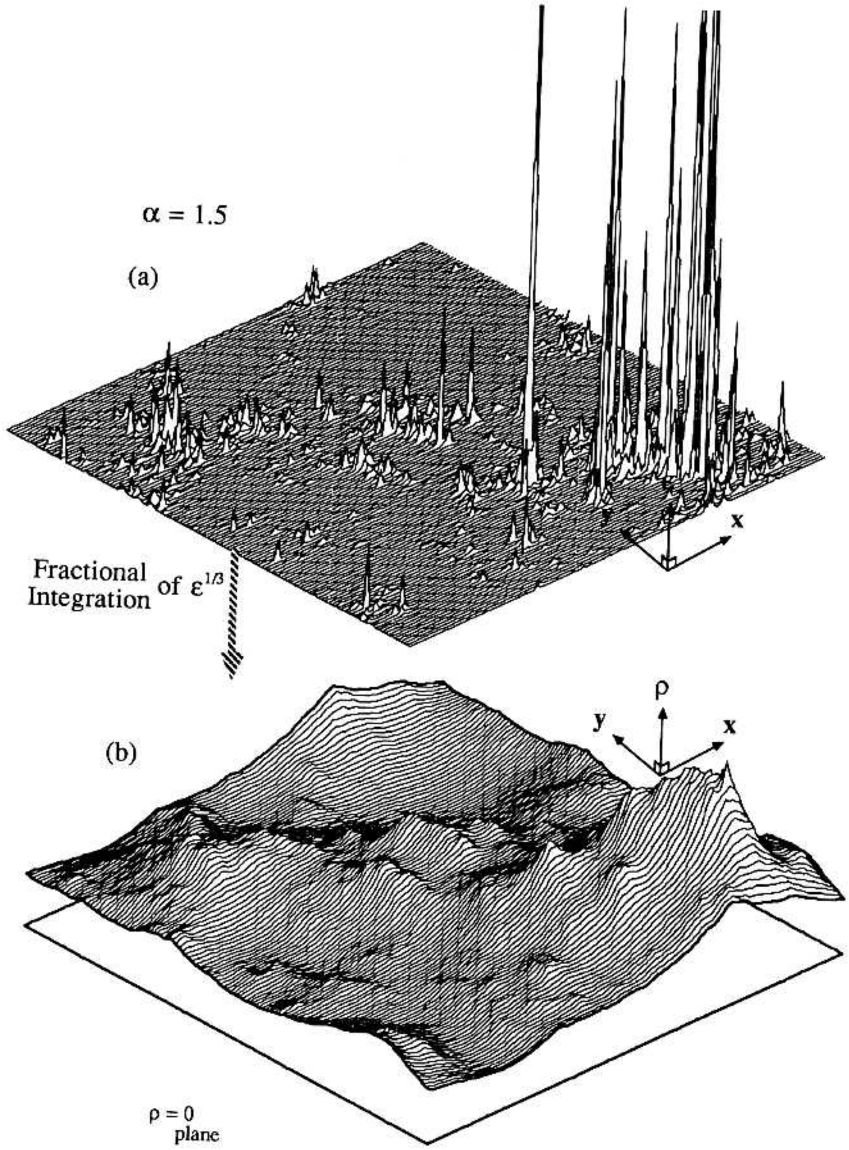


Fig. 8 . It shows from top to bottom, over a 256x256 pixels grid,
(a) the density of a conserved flux, obtained with a Lévy generator $\alpha = 1.5$
(b) the associated concentration field obtained by
fractional integration (of order 1/3).

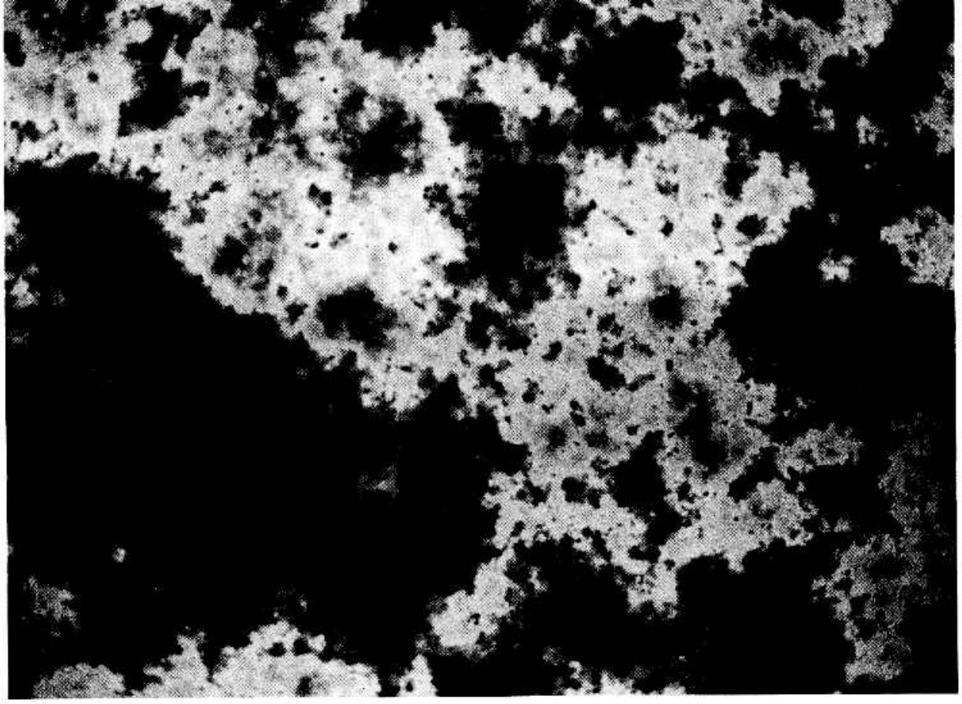

Fig. 9a. a cloud field obtained with a fractional integration (of order 1/3) and a Lévy generator ($\alpha$=1.7). See Wilson et al. (1989) for more discussion.

determine the scale invariant probability distribution (characterized by the codimension function $c(\gamma)$). To do this, we must first estimate f at different intermediate scale ratios $\lambda$ ($f_\lambda$; $\Lambda \geq \lambda \geq 1$) by coarse graining, eg. averaging on larger and larger pixels (successive factors of 2 can be easily implemented recursively until reaching the whole image) and the corresponding probabilities distributions with algebraic thresholds $\propto \lambda^\gamma$, estimating thus the codimensions of the singularities:

$$c(\gamma) \approx -\mathrm{Log}_\lambda \Pr( (\mathrm{Log}_\lambda(f_\lambda) > \gamma ) \qquad (33)$$

In order to avoid contributions given by different correcting terms (such as various logarithmic corrections...), it is often better to estimate $c(\gamma)$ as the slope of the probability distributions in Log-Log plots as discussed by Lavallée et al. (1989). This method can be called Probability Distribution Multiple Scaling (PDMS) and be tested on simulated fields whose codimension function is known. We can check also the validity of our assumptions on the limit of accuracy given by the sampling dimension we have introduced. Fig. 10 presents such a simulation, with a gaussian generator, which rather supports this method (cf. Lavallée et al.(1989) for more discussion).

Fig. 11a-b show the results when this technique is applied to 5 visible and 5 infra red GOES (Geostationary Operational Environment Satellite) pictures respectively over Montréal. The pictures were resampled on an 8X8km grid over a region of 1024X1024km. As can be

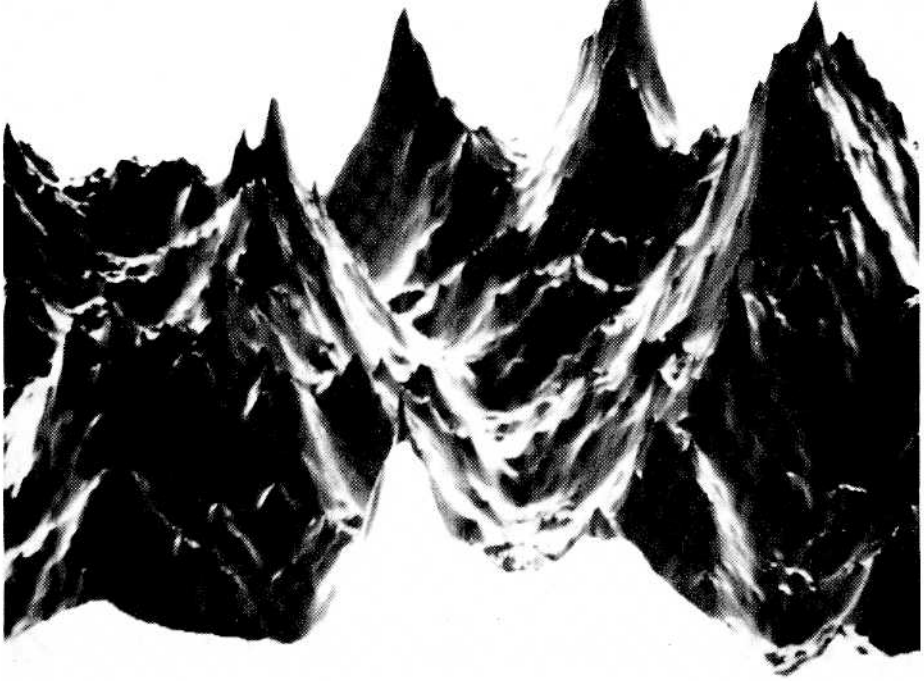

Fig. 9b. a landscape topography obtained with a fractional integration (of order 1/3) and a Gauss generator ($\alpha$=2). See Sarma (1989) for more discussion.

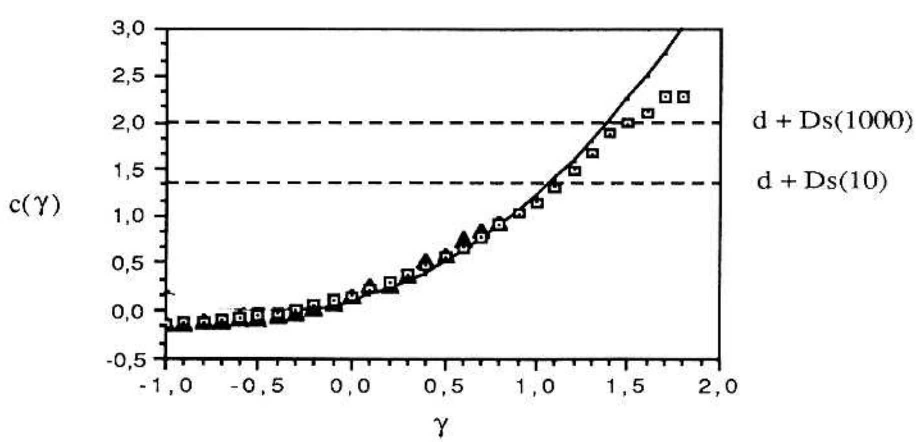


Fig. 10. Probability Distribution Multiple Scaling estimated on a multifractal field, generated by a gaussian generator. Solid line is the theoretical curve, for the probability density, black triangle are estimated codimensions of the probability distribution with a sampling dimension equals to 10 independent samples, open squares correspond to 1000 independent samples . Horizontal dashed lines, indicate the (estimated) upper limit of the validity of the estimations due to their limited sampling dimensions (resp. Ds(10) and Ds(1000). See Lavallée et al. (1989) for more discussion.

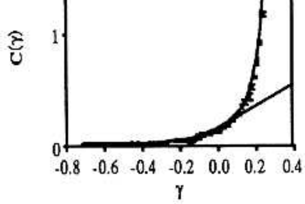


Fig. 11a PDMS estimates of the singularities codimension $c(\gamma)$ from five visible GOES images over 1024x1024 km at 8, 16, 32, 64, 128, 256 km scales. The error bars indicate one standard deviation

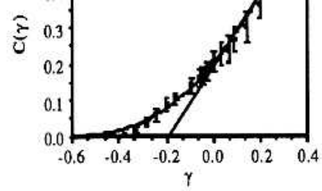


Fig. 11b PDMS estimates of the singularities codimension $c(\gamma)$ from five infra-red GOES images with same range of scales. The error bars indicate one standard deviation. The mean standard error was $\pm.023$, and the best regression to universal function (Eq. 31) yielded $\alpha'=2.52$, $\alpha=1.66$, with a standard error of the fit $\pm.015$

seen, all the distributions are nearly coincident, in accord with the multifractal nature of the fields. To judge the closeness of the fits, we calculated the mean $c(\gamma)$ curves as well as the standard deviations for 8, 16, 32, 64, 128, 256 km, finding that the variation is very small, being typically about $\pm0.02$ in $c(\gamma)$ which is appreciably more accurate than estimates obtained using functional box counting on very similar data (Gabriel et al 1988 found accuracies of $\approx\pm0.05$).

We have already argued that the resolution independent codimension function $c(\gamma)$ is of considerably more interest than particular values of the function and may depend only on very few parameters due the universality classes we discussed: 2 in case of a conserved flux, mainly 3 in case of fractional integration (order -b) on a power (a) of a conserved flux. As discussed in the preceding sub-section, we may restrict our attention to a=1 and rewrite slightly differently this transformation, by introducing the empirically determined fixed point $C_t$ (the subscript "t" indicates "tangent" to a line slope 1) of $c(\gamma)$ and the corresponding translation $\gamma_t$ necessary to reach it, i.e.

$$c(C_t - \gamma_t) = C_t \qquad [c'(C_t - \gamma_t) = 1] \tag{34}$$

The difficulty in testing these ideas empirically is that the key parameter $\alpha'$ (recall $1/\alpha'+1/\alpha=1$) characterizes the concavity of $c(\gamma)$ which is only pronounced when $\gamma$ and $c(\gamma)$ vary over a substantial range. From the point of view of non-linear regression, to fit $\gamma_t$, $C_t$, $\alpha'$ to the data we find that $C_t$ and $\alpha'$ are highly correlated and hence parameter estimates are not very sharp. In Gabriel et al 1988, functional box-counting was used yielding less accurate estimates of $c(\gamma)$ than those obtained here. The issue was side-stepped by assuming $\alpha'=2$ and testing the consistency of the data with that hypothesis.

Here we outline a very simple graphical method which proves quite accurate. The easiest parameter to estimate graphically is $c_0=c(0)$, which yields $c_0=0.16$, 0.20 for visible and IR curves respectively. However, $C_t$, $\gamma_t$ can also be found rather easily: a line slope 1 is tangent to $c(\gamma)$ at the point $c(\gamma)=C_t$ and will intersect the $\gamma$ axis at the point $\gamma=-\gamma_t$. Note that all three parameters estimated this way depend on the values of the curve $c(\gamma)$ in the statistically well

Using Eq. 31 we can obtain the estimates $\alpha \approx 0.65$, 1.7 for visible and IR respectively.

To improve on these results requires nonlinear regression (cf. discussion by Lovejoy and Schertzer (1989)). Here we determined $\alpha'$ by a least squares regression on the mean of the 8 to 256km curves in Fig. 11a-b. Maximum likelihood estimates for the parameter $\alpha$ were found to be: $\alpha$=0.63±0.035 and $\alpha$=1.66±0.37 for the visible and infra red data respectively. The large difference in the maximum likelihood errors cited here is due at least in part to the fact that we directly estimate $\alpha'$ and $\Delta\alpha = (1+\alpha)^2\Delta\alpha'$ hence this effect alone accounts for a factor 2.7 in difference. Fig. 11a-b shows the best fit and mean visible and infra red curves. The standard errors in the fit are ±0.011 and ±0.015 respectively. These results show the accuracy of the graphical method. We can also estimate the critical order of moments divergence $h_D$ (cf. Eq. 11) and the corresponding critical order of singularity $\gamma_D$. We find, (using D=2) $h_D \approx 13.80$, $\gamma_D \approx 3.50$ for infra red images, but for the visible data (with $\alpha<1$, recall that the singularities are bounded) $C_\infty \approx 0.42<2$, hence no divergence.

One may note also that, due to the fundamental discussion (Schertzer and Lovejoy, 1987b) on the method of elliptical dimensional sampling, the scaling anisotropy of the field can be investigated by this method in connection with PDMS, instead of the Functional Box Counting (as done earlier by Lovejoy et al. (1987)).

## CONCLUSIONS

We sharpened the theoretical foundations of the singular statistics of multifractal fields, discussing in a rather general manner the conditions of their appearance, depending on the type of the process, as well as on the observation (scale and dimension). Thus we emphasized the non trivial behavior of geophysical observables. We clarified the fundamental difference between "bare" and "dressed" properties at a given (non-zero) scale i.e. the important differences between a process with a cut-off of small scale interactions and one with all these interactions. We point out also general properties of the generators of multifractal fields and their links with classical statistical physics notions, emphasizing their particularities.

We discussed in some detail the basic two-parameter (average singularities $C_1$, Lévy index $\alpha$ of the generator) family of universal (canonical) multifractal fields having strong attractive properties, with five important sub-classes: gaussian generator ($\alpha$=2), unbounded Lévy generator ($2>\alpha>1$), bounded Lévy generator ($1>\alpha>0$), a very special in-between case Lévy generator ($\alpha$=1)... as well as the once celebrated $\beta$-model ($\alpha$=0)! We also showed that multifractal fields not subject to some flux conservation will nevertheless depend primarily on only three parameters.

In our opinion, these findings may provide keys advances in Geophysics, especially in many practical applications (eg. remote sensing techniques) since they may well lead us to explore a hidden face of multifractality: bare universality under dressed pandemonium. We illustrated these ideas with passive cloud simulations and satellite data analysis.

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## APPENDIX A

## GENERALIZED CENTRAL THEOREM, EXTREMAL LEVY STABLE GENERATORS
(in collaboration with R. Viswanathan[1])

Fixed points for sums of independent and identical distributed (i.i.d.) random variables and central limit theorems

In this sub-section we review briefly the classical features of Lévy stable variables, stressing that these variables emerge as generalizations of gaussian variables, which then are seen to be a very particular case of Lévy stable variable. Indeed, we are interested in the universal stable and attractive fixed points of renormalized sum of i. i. d variables, consider first the stable fixed points of renormalized sum ( $=^d$ means equality in probability[2]):

$X_i =^d X_1 \quad i=1,n$ are stable points under renormalized sum iff (A1)
for any (integer) n (≥2), ,there exists a (positive) $b_n$ and a (real) $a_n$
$$\sum_{i=1,n} X_i =^d b_n X_1 + a_n$$

The well-known gaussian case corresponds to:

$$\langle X_1^2\rangle \leq \infty \Rightarrow b_n = n^{1/2},\ a_n = (n-1)\langle X_i\rangle \qquad (A2)$$

hence the assumption of finite variance which has been considered as so "natural" that it has become a kind of dogma. The usual central limit theorem corresponds simply to the limit $n\to\infty$ in Eq. A1:

$$X = \lim_{n\to\infty} [(\sum_{i=1,n} X_i) - a_n]/b_n \qquad (A3)$$

the $X_i$ on the r.h.s. are not assumed to be gaussian but the X will be, hence the Gaussian law is attractive :

$$\langle X^2\rangle = \langle X_i^2\rangle \leq \infty \Rightarrow b_n = n^{1/2}, a_n = n\langle X_i\rangle - \langle X\rangle \qquad (A4)$$

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[1] Banque Indo-Suez, Paris, France.

[2], Note in order to be consistent, the use of this symbol requires that indicated variables should be mutually independent

variance for the $X_i$ (which implies finiteness of every statistical moment for the limit itself) introducing on the contrary an order of divergence ($\alpha$, $0<\alpha<2$) for the moments of the $X_i$ ($\alpha$ is often called the Lévy index) which satisfies either A1 or A3:

$$h<\alpha \Rightarrow \langle |X_i|^h \rangle < \infty \text{ and } h' \geq \alpha \Rightarrow \langle |X_i|^{h'} \rangle = \infty \quad \text{i. e. : } \Pr(|X_i| \geq s) \sim s^{-\alpha} \qquad \text{(A5)}$$
$$\text{A1 or A3} \Leftrightarrow b_n = n^{1/\alpha} \quad (\text{and, if } \alpha>1;\ a_n = n\langle X_i \rangle - \langle X \rangle)$$

the variables $X_i$ are very often termed "hyperbolic variables" (or even "hyperbolics") due the algebraic fall-off of their probability distribution tails, which are themselves sometimes termed "fat tail" due to their (unusually) important contribution. Hence, the Lévy stable variables are the stable and fixed points of (renormalized) sums of i. i. d. hyperbolic variables. Note that for $\alpha \leq 1$, as the mathematical expectations of the $X_i$ and their sums are divergent, the required recentring is a bit more involved than that indicated for $1<\alpha\leq 2$ (subtracting out the averages) and will be only discussed later. The very special gaussian case appears as the (extreme) regular case $\alpha=2$, after a highly critical transition since for any $\alpha=2-\varepsilon$ ($\varepsilon$ arbitrarily small) we have divergence of all orders greater than $\alpha$ whereas all divergences are suppressed for $\alpha=2$. One may note that the stable variables were introduced in a slightly different form (Lévy 1925, 1954) addressing the stability under any linear combination:

$X_i =^d X_2$ are said stable under linear combination iff (A6)
for any (positive) $b_1$ and $b_2$ , ,there exists (real) a and (positive)b, such that:
$b_1 X_1 + b_2 X_2 =^d b X_1 + a$

It is rather easy to check (by induction) that Eq. A1 and Eq. A6 are equivalent . One may furthermore note that "any n" in A1 can be equivalently reduced to "n=2,3" due essentially to the density of numbers $2^j 3^k$ among positive numbers, j and k being relative integers, (see for instance Zolotarev 1986).

Note that there exists a sub-class of stable variables which do not require recentring (i.e. $a\equiv 0$, -it is rather obvious in the cases $1<\alpha\leq 2$). These special cases (to which, Lévy (1925) restricted his study) are frequently called "strictly stable" (Feller 1971, Zolotarev 1986), more rarely the complementary cases (i.e. $a\neq 0$) are called (Lévy 1954) "quasi-stable".

Characteristic functions of Levy laws

With the few notable exceptions $\alpha=2$, 1, 1/2 (and some further restrictions on the two latter cases, since they must be symmetric) the probability distributions of Levy stable variables are not expressible in a closed form. However, the second (Fourier or Laplace) characteristic function is easily expressible due to the basic properties of stability. K(h) is the logarithm of the first characteristic function Z(h), i.e. the (Fourier or Laplace) transform of the probability distribution dP(x)) and the argument h is purely imaginary in case of Fourier (h=ih'), real in case of Laplace (we will discuss later the restrictive conditions under which such a transform is possible) and a complex number (h+ih') in the case of Fourier-Laplace (or two-sided Laplace) transform:

$$e^{K(h)} = Z(h) = \langle e^{hX} \rangle = \int e^{hx}\, dP(x) \qquad \text{(A7)}$$

the fundamental property of the fixed point (Eq. A1) or the equivalent form (Eq. A6) are easily transposed for the characteristic functions: .

$X_i =^d X_1$ ( i=1,n) of second characteristic function K(h) (A1') and:

$X_i =^d X_2$ of second characteristic function K(h) (A6')
are said stable under linear combination iff. for any (positive) $b_1$ and $b_2$
there exists a (real) a and a (positive)such that: $K(b_1h)+K(b_2h) = K(bh) + ah$

the limit theorems correspond to ($K_i$ second characteristic function of the $X_i$, K second characteristic function of X):

$$K(h) = \lim_{n\to\infty} n[K_i(h/b_n) - ha_n/nb_n] \quad \text{(A3')}$$

We may infer, especially from A6', that these characteristic functions, up to the recentring term, should be of power law form whose exponent $\alpha$ is bounded above by 2 (the extreme regular gaussian case) and must of course be positive (to avoid divergence at h=0). It is obvious that the case $\alpha$=1 is very special since this hyperbolic exponent becomes equal to the (linear) recentring term exponent, we may guess that conflict and compensation between the two terms will introduce logarithm corrections. For other values of $\alpha$ the (linear) recentring term has no importance and we can restrict our attention to strictly stable cases ($a_n$=0). As $h^\alpha$ (or h Log(h) for $\alpha$= 1) is not analytical (except once again in the gaussian case $\alpha$= 2) in the complex plane, we clearly expect on the one hand divergence of moments of order greater or equal to $\alpha$, on the other hand that the inferred "power law form" may be rendered more precise in order to obtain a second characteristic function (e.g. Z(h) must be positive definite in case of Fourier transform (Bochner's theorem) or absolutely monotone in case of Laplace transform). Indeed, considering the symmetric (or symmetrized) probability distributions lead to the following law (partially known... since Cauchy 1853, but essentially obtained by Lévy (1925) of Fourier characteristic functions, since K(h) must be also symmetric:

$$K(ih') = -\lambda_\alpha |h'|^\alpha \quad \text{(A8)}$$

with the obvious gaussian case when $\alpha$=2, and Cauchy case when $\alpha$=1. The $\lambda_\alpha$ characterizes the width of the probability distribution as in the gaussian case ($\lambda_2 = \sigma^2/2$) but doesn't correspond to the evaluation of an $\alpha$-moment, since it diverges, but rather the rate of divergence of this moment.

Particular properties of extremal Levy laws

The symmetric case corresponds to limit sums of symmetric hyperbolics or mixing with equal probability (p=q=1/2) positive (with probability p) and negative (with probability q) one-sided hyperbolic distributions (concretely: just multiplying by a random sign positive one-sided hyperbolics) Asymmetric cases correspond to $p\neq q$. It is time to stress that if we want to have a Laplace transform. we can only consider extremal (asymmetric) hyperbolics, simply because algebraic fall-off could not tame an exponential divergence, hence we restrict here our attention to negative hyperbolics (p=0, q=1). However, note that the corresponding limits, the extremal Lévy stables, are not always one sided -precisely one sided probability distributions only occur for $0<\alpha<1$.

In order to assess different statements, it is interesting to consider the characteristic functions under their "canonical form" i. e:

$$K(h)= \int (e^{hx} -1 +hx)\, dF(x) \approx Z(h) -1+a'h \quad \text{(A9)}$$

development (near h=0) when needed (for $1<\alpha<2$). This form corresponds on the one hand to a first order term development of $\log[1+(Z(h)-1)]$, which is the only term kept in the limit theorem besides recentring and normalization of K(h) (i.e. K(0)=0 corresponding to $\int dP(x)=1$). On the other hand, it corresponds to the limit of (Poisson) random (renormalized) sums of i. i. d. variables, instead of uniform sums as discussed up to now. Indeed considering the characteristic function of the limit ($n\rightarrow\infty$) of the Poisson compound probability distribution generated by renormalized sum of n i. d. d. hyperbolic variables (as given by A1) lead us to a new version of the limit theorem (earlier stated under the forms A3 and A3') keeping in mind that the second characteristic function K(h) of the Poisson compound probability distribution is c(Z(h)-1), where c is the parameter of the Poisson process, Z(h) the first characteristic function of the generating probability distribution):

$$K(h) = \lim_{n\rightarrow\infty} K_n(h)$$
$$K_n(h)=n[Z_n(h) -1]$$
$$Z_n(h)= Z_i\,(h/b_n)\exp(- ha_n/nb_n) \qquad \text{(A3'')}$$

the "canonical form" (Eq. A9) is obtained by slightly recasting this equation to take directly into account arbitrary centering directly on K (no longer on $Z_n$ or Z).

Let us consider the negative hyperbolic generation of extremal Lévy stable by negative hyperbolic, it suffices to put $dF(x) \propto 1_{x<0}x^{-\alpha}\,dx/x$ ($1_{x<0}$ being the indicator function of the negative x) and with repeated uses of the identity:

$$\Gamma(\beta)= z^{-\beta} \int_0^\infty e^{-zt}t^{\beta}dt/t \ ; \ \mathrm{Re}(z) \geq 0 \qquad \text{(A10)}$$

and integrations by parts, we obtain easily for $dF(x) = 1_{x<0}C(2-\alpha)\, x^{-\alpha}\, dx/x$ :

$$\alpha\neq 1:\ K(h) =C\, h^{\alpha}\, \Gamma(3-\alpha)/\alpha(\alpha-1);\ \alpha\neq 1$$
$$\alpha=1:\ K(h) =C\, h\log(h) \qquad \text{(A11)}$$

one may note that the expressions for the corresponding Fourier transforms are a bit more complex (i.e. Fourier transforms, so convenient for symmetric laws, are inconvenient for extremal (and more generally for asymmetric laws), on the contrary Laplace is only fitted for the extremal, useless for the others):

$\alpha\neq 1$:

$$K(h)= |h|^{\alpha}\ C\, [\Gamma(3-\alpha)/\alpha(\alpha-1)]\ [\cos(\pi\alpha/2) + -i(\mathrm{sgn}(h)\ (p-q)\ \sin(\pi\alpha/2)]$$

$\alpha=1$:

$$K(h)= -|h|\ C\,[\pi/2 +i\ (\mathrm{sgn}(h)(p-q)\log|h|] \qquad \text{(A12)}$$

As a last general remark, one may note (from Eq. A11 or Eq. A12) it is only in the case of extremal stable distribution ($p-q=\pm 1$) that an analytic extension on the whole complex plane of K is possible (but with a cut along the ray $\arg(h)=-3\pi/4$), as it is for $\alpha=2$, or that the double-sided Laplace transform applies only to extremal stable variables.