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Multifractal temperature and flux of temperature variance in fully developed turbulence

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Abstract. – We analyse the multifractal properties of the temperature field θ and temperature variance flux χ in atmospheric turbulence, using simultaneous velocity V and temperature measurements in the atmosphere. This permits us first to characterize the multifractal scaling moment function of the flux of temperature variance estimated as $\chi \approx (\Delta\theta)^2 \Delta V/\ell$. We then study the structure function scaling exponent of the temperature field directly, and compare it to the consequences of the assumption of the independence of the fluctuations of θ and V. Up until moment orders of at least order 6, the data analysis is consistent with independence.

The temperature variance flux. – In fully developed turbulence passive scalar scaling laws [1]-[4] were originally derived using the dissipation, but they can also be expressed, as was done in [5], in terms of local fluxes (*i.e.* at scale ℓ), which are conserved by the equations of motion and cascade from large to small scales. The energy and temperature variance fluxes can be estimated as, respectively,

$$\varepsilon_\ell \approx \frac{(\Delta V_\ell)^3}{\ell}\,; \qquad \chi_\ell \approx \frac{(\Delta\theta_\ell)^2 \Delta V}{\ell}\,, \tag{1}$$

where $\Delta\theta_\ell = |\theta(x+\ell) - \theta(x)|$ and $\Delta V_\ell = |V(x+\ell) - V(x)|$ are the velocity and temperature shears at scale ℓ, $\Delta V_\ell/\ell$ is the local eddy turnover time, and "$\approx$" means proportionality. These cascades are usually assumed to be multiplicative. In this case, this leads to multifractal fields, whose statistics can be defined by their moments [6]:

$$\begin{cases} \langle(\varepsilon_\ell)^q\rangle \approx \lambda^{K_\varepsilon(q)}\,; \quad \langle(\chi_\ell)^q\rangle \approx \lambda^{K_\chi(q)}\,, \\ \langle(\Delta V_\ell)^q\rangle \approx \lambda^{-\zeta_V(q)}\,; \quad \langle\left[(\Delta\theta_\ell)^2 \Delta V_\ell\right]^q\rangle \approx \lambda^{-\zeta_{V,\theta}(3q)}\,, \\ K_\varepsilon(q) = q - \zeta_V(3q)\,; \quad K_\chi(q) = q - \zeta_{V,\theta}(3q)\,, \end{cases} \tag{2}$$

where L is a fixed outer scale and $\lambda = L/\ell$ is the corresponding scale ratio, $K_\varepsilon(q)$ and $K_\chi(q)$ are the scaling moment functions for the fluxes, $\zeta_V(q)$ is the usual velocity structure function

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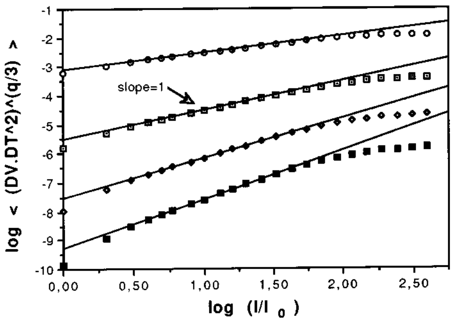

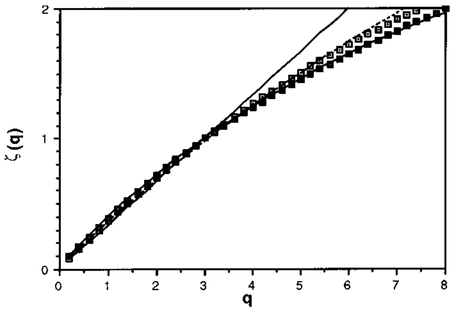

Fig. 1. – Empirical estimates of $\log\langle[\Delta V_\ell(\Delta\theta_\ell)^2]^{q/3}\rangle$ *vs.* $\log(\ell/\ell_0)$, where $\ell_0 \approx 27$ cm is the smallest length scale (obtained using Taylor's hypothesis), for $q = 1.5, 3, 4.5$, and 6 (from top to bottom). The straight lines show the scaling range.

Fig. 2. – Empirical estimates (from $q = 0$ to 9, 0.05 step) of $\zeta_V(q)$ (open squares) and $\zeta_{V,\theta}(q)$ (black squares). The universal multifractal fits obtained with eqs. (2) and (3) and the values given in the text are also shown for comparison (dotted and continuous lines).

scaling exponent and $\zeta_{V,\theta}(q)$ is the joint structure function scaling exponent of the product $(\Delta\theta_\ell)^2\Delta V_\ell$. Because the fluxes are conserved by the non-linear terms of the equations of motion they are conserved multifractals, *i.e.* their mean is scale invariant: $K_\varepsilon(1) = 0$ and $K_\chi(1) = 0$. We may note that this implies $\zeta_V(3) = 1$ and $\zeta_{V,\theta}(3) = 1$, which corresponds to the exact relations for the small-scale dissipation fields given by Kolmogorov [7] and Yaglom [8].

Equations (2) are used here to estimate the universal multifractal parameters for the fields ε and χ. Universal multifractals are the stable and attractive classes which are obtained with continuous multiplicative scaling processes [6], [9], [10]. The scaling moment function $K(q)$ depends in this framework on only two parameters: the mean fractality parameter C_1 (it is the codimension of the mean singularity and satisfies $0 \le C_1 \le 1$) and the Lévy index α ($0 \le \alpha \le 2$):

$$K(q) = \frac{C_1}{\alpha - 1}(q^\alpha - q)\,. \tag{3}$$

These indices have already been estimated for the energy flux using different techniques and datasets [10]-[13]. The results are, for atmospheric data, $\alpha_\varepsilon = 1.5 \pm 0.05$, and $C_{1\varepsilon} = 0.25 \pm 0.05$ when ε is estimated from a fractional differentiation of the wind field, or $C_{1\varepsilon} = 0.15 \pm 0.03$ when it is determined from the velocity structure functions directly ([1]). The atmospheric turbulent temperature has also already been studied in the universal multifractal framework [12], [13], but with an analysis technique which did not deal directly with the variance temperature flux χ (because the velocity field was not recorded simultaneously). Our study is also a development and continuation of previous work [5], [14], which studied the scaling exponents of the second moment of χ, as well as attempted to fit the temperature structure functions with log-normal and β-models. Our structure function method for estimating χ (see below) is also more direct than that used in [15]: the latter used the square of the temperature difference at highest resolution (*i.e.* the convective range) $\chi' \approx (\Delta\theta)^2$, and then studied its multifractal properties. As a study of the intermittency of temperature and its associated flux, our multifractal approach can also be linked to the strong discontinuities of the temperature signals ("ramps") and its associated "anomalous properties" [16].

The data we analyse here consist of the longitudinal component of the wind velocity vector

([1]) This discrepancy is still to be studied with caution, but presumably derives from the absolute value which is taken in the former case.

and temperature, recorded simultaneously with a sonic anemometer in the atmosphere 25 m above ground, over a pine forest in south-west France. We analysed 22 profiles of duration 55 minutes each, all recorded in near-neutral stability conditions (in all cases $-0.10 < z/L_{\mathrm{m}} < 0.15$, where z is the measurement height and L_{m} the Monin-Obukhov length. For a mean wind velocity of $2.7\ \mathrm{m\,s^{-1}}$, the longitudinal velocity spectrum peaks at about 0.03 Hz. The sampling frequency of 10 Hz therefore ensured that a wide enough inertial subrange could be seen (see chapters 6 and 7 of [17]). However, a slight spectral distortion is visible at the high-frequency end of the spectra. It is probably due to line averaging along the probe path (15 cm), since the onset of distortion due to such effects can be evaluated at about 2 to 5 Hz [17]. For more details on the experimental conditions, see [18]. Figure 1 shows that for almost two decades eq. (3) is valid, and also that $\zeta_{V,\theta}(3) = 1$ is quite well respected (Taylor's hypothesis was used to transform temporal into spatial increments, assuming that the small-scale turbulent fluctuations are advected by the large-scale velocity). We computed the scaling exponents of eq. (2) for several values of q between 0 and 9 for the range of scales where $\zeta_{V,\theta}(3) = 1$ is valid; this range of scales is taken here as the inertial range (the small-scale deviation from scaling is probably due to the above-mentioned line-averaging effects). The two scaling moment exponents $\zeta_V(q)$ and $\zeta_{V,\theta}(q)$ are shown in fig. 2: the clearly non-linear behaviour of $\zeta_{V,\theta}(q)$ is direct evidence of the multifractal nature of the flux of scalar variance χ. Here we study and quantify this multifractality using the universal multifractal model. We estimate the first parameter C_1 as $C_1 = K'(1) = 1 - 3\zeta'(3)$. This gives for both curves: $C_{1\varepsilon} \approx 0.16 \pm 0.02$ and $C_{1\chi} \approx 0.22 \pm 0.02$. Then the simplest way to estimate α is to take it as the "best" non-linear fit (for $0 \le q \le 2.5$ and using a simple least-square method) of the data using eqs. (2) and (3). This gives: $\alpha_\varepsilon \approx 1.5 \pm 0.1$ and $\alpha_\chi \approx 1.4 \pm 0.1$. The fits using eqs. (2) and (3) are also shown in fig. 2. For each flux, the universal multifractal fit is very good until a critical moment order where the empirical curve becomes linear. These empirical straight lines for large-order moments are multifractal phase transitions [9], [19] which arise because of sampling limitations possibly in association with the divergence of large-order moments [6], [9], [20]. The two multifractal fields have sligthly different values for the codimension of the mean C_1. The values of α are very close to each other, and—within statistical error—are not distinct. For quantitative comparison with previous studies, we can consider the second moment, which is $K_\varepsilon(2) \approx 0.26 \pm 0.05$ for the energy flux. This value is very close to the usual value 0.25 ± 0.05 [21]. For the field χ, we obtain $K_\chi(2) \approx 0.35 \pm 0.05$, which can be compared to the values $\approx 0.38 \pm 0.08$ [22] and $\approx 0.34 \pm 0.05$ [15]. It could be noted that this is nevertheless larger than the older estimate: $\approx 0.25 \pm 0.05$ [14].

The turbulent temperature field. – The two multifractal fields ε and χ being described by multiplicative cascade processes, we now would like to characterize with the scaling moment function $\zeta_\theta(q)$ the scaling properties of the temperature field itself:

$$\langle (\Delta\theta_\ell)^q \rangle \approx \lambda^{-\zeta_\theta(q)} . \tag{4}$$

To this end there are different possible scaling approaches which we discuss before performing a direct comparison with an analysis of the temperature field.

Using dimensional analysis, Obukhov and Corrsin [3] obtained a well-known relation giving the temperature fluctuations assuming constant fluxes; this can be written

$$\Delta\theta_\ell \approx (\chi_\ell)^{1/2} (\varepsilon_\ell)^{-1/6} \ell^{1/3} . \tag{5}$$

In the multifractal framework, denoting the scaling moment function of the "mixed" flux $\varphi = \chi^{3/2}\varepsilon^{-1/2}$ by $K_\varphi(q)$, we obtain

$$\zeta_\theta(3q) = q - K_\varphi(q) ; \quad \langle (\varphi_\ell)^q \rangle \approx \lambda^{K_\varphi(q)} . \tag{6}$$

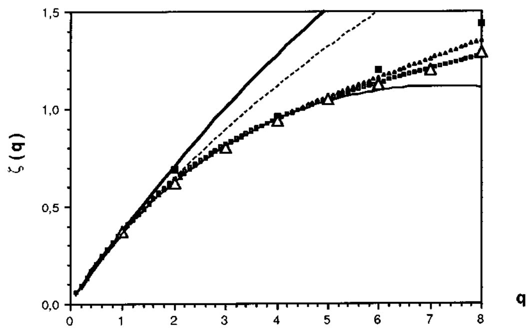

Fig. 3. – Empirical values of $\zeta_\theta(q)$ obtained here (small open squares), from ref. [5] (large squares) and [24] (large open triangles), compared to $\zeta_{V,\theta}(3q/2) - \zeta_V(q/2)$ (small triangles) where $\zeta_V(q)$ and $\zeta_{V,\theta}(q)$ were obtained directly (see fig. 2). Also shown in this figure are empirical estimates of $\zeta_V(q)$ (thick continuous line), theoretical predictions [25] (dotted line), and the universal multifractal fit (continuous line) obtained from eqs. (3), (4) and (9), with the numerical values estimated previously.

The simplest relation between the temperature and wind fields is the independence of ε and χ. This was investigated by Benzi *et al.* [23], who considered the second moment of the temperature, and assumed (using a simple cascade model) the following independence:

$$\langle(\Delta\theta_\ell)^2\rangle \approx \langle\chi_\ell(\varepsilon_\ell)^{-1/3}\rangle\ell^{2/3} \approx \langle\chi_\ell\rangle\langle(\varepsilon_\ell)^{-1/3}\rangle\ell^{2/3} \approx \ell^{1+\zeta_V(-1)} \tag{7}$$

which gives $\zeta_\theta(2) = 1 + \zeta_V(-1)$. The same approach for other moments gives

$$\begin{cases} K_\varphi(q) = K_\chi(3q/2) + K_\varepsilon(-q/2)\,, \\ \zeta_\theta(q) = q/2 - K_\chi(q/2) + \zeta_V(-q/2)\,. \end{cases} \tag{8}$$

Equations (8) are highly questionable due to the independence assumption between ΔV_ℓ and χ (or ε and χ). Indeed, since the velocity field advects the scalar field, it imposes a dependency, especially through its eddy turnover time. In particular, this assumption is not at all acceptable as soon as there are very frequent low values of wind shears (as is the universal multifractal model when $\alpha < 2$), which will render the negative moments divergent ($\zeta_V(-q/2) = \infty$, $q > 0$). These assumptions and problems explain why we now propose a different approach to describe $\zeta_\theta(q)$.

If one considers that ΔV_ℓ and $\Delta\theta_\ell$ are independent (rather than ΔV_ℓ and χ), one obtains from eqs. (1) and (2) the following relation ([2]) for $\zeta_\theta(q)$:

$$\begin{cases} \langle(\chi_\ell)^q\rangle \approx \langle\Delta V_\ell)^q\rangle\langle(\Delta\theta_\ell)^{2q}\rangle\ell^{-q}, \\ \zeta_\theta(q) = q/3 + K_\varepsilon(q/6) - K_\chi(q/2) = \zeta_{V,\theta}(3q/2) - \zeta_V(q/2)\,. \end{cases} \tag{9}$$

Equation (9) for $\zeta_\theta(q)$ is of course different from eq. (8) for multifractal fields (in particular, since $\zeta_V(-q/2) \neq -\zeta_V(q/2)$), and has the advantage of avoiding negative moments; we test it below.

We computed temperature structure functions moments of various orders, for the same range of scales as given by the conditions $\zeta_V(3) = 1$ and $\zeta_{V,\theta}(3) = 1$ (we also used extended self-similarity as in [24], but this did not change the estimated scaling exponents). The results are given in fig. 3, which shows the above empirical values (we also plotted the values reported

([2]) This avoids dividing by ΔV_ℓ, and hence the problem of negative moments.

TABLE I. – *Comparison of empirical estimates of* $\zeta_\theta(q)$ *from ref.* [24] *as well as our estimate of* $\zeta_{V,\theta}(3q/2) - \zeta_V(q/2)$. *Because we used a* 0.05 *increment in* q *for our figures, we only give here some of our values.*

q	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7
$\zeta_\theta(q)$ [24]		0.37		0.62		0.80		0.94		1.05		1.12		1.20
$\zeta_\theta(q)$, present study	0.21	0.38	0.52	0.64	0.74	0.82	0.89	0.95	1.00	1.05	1.09	1.13	1.17	1.21
$\zeta_{V,\theta}(3q/2) - \zeta_V(q/2)$	0.21	0.37	0.52	0.63	0.73	0.81	0.89	0.95	1.01	1.06	1.11	1.16	1.21	1.26

in [5] and in [24]). It also compares the implications independence (eq. (9)) with empirical estimates for $K_\varepsilon(q)$ and $K_\chi(q)$, as well as the universal multifractal fit, and also the theoretical model proposed in [25]. In order to illustrate the discrepancy between velocity and temperature structure functions, empirical estimates for the velocity structure functions are also shown. This figure calls for few comments: up to moments of about order 5, our values are in good agreement with those of Antonia *et al.* [5] and in very good agreement up to moments of about order 8 with the more recent estimates of Ruiz Chavarria *et al.* [24] (for a better quantitative comparison see table I);

i) $\zeta_V(q)$ is clearly much larger than $\zeta_\theta(q)$ as soon as $q > 2$, as already noticed in [5]. This implies that the temperature field is much more intermittent than the velocity field; this high intermittency can be linked to the "ramps" [6].

ii) $\zeta_\theta(q)$ is very close to $\zeta_{V,\theta}(3q/2) - \zeta_V(q/2)$ until $q \approx 6$. This tends to support the hypothesis of independence between the velocity and temperature fields for weak events. However, to completely demonstrate it, a closer inspection of the tensorial nature of the cross moments between the velocity vector and the passive scalar is needed ([3]). This hypothesis of non-correlation between small-scales inertial range temperature and velocity increments is also used in [27]. In this work, Yaglom's equation [8] is generalized, and tested on simultaneous temperature and velocity data. Another recent work [28] considers also simultaneous temperature and velocity data, to study the correlations between temperature fluctuations and local temperature dissipation rates. This study is closely connected with the present one, differing mainly by the analysis of dissipation quantities instead of fluxes, and of second and third moments instead of the general moment scaling function.

iii) Equation (9) also provides a theoretical expression for the temperature structure functions, as soon as the two fluxes of energy and scalar variance are known. Let us recall that up to now there are no satisfying theoretical predictions for $\zeta_\theta(q)$. For example, the prediction of the model proposed in [25] is clearly too large for moments $q > 3$.

iv) This figure also shows that the universal multifractal fit obtained from eqs. (3), (4) and (9), with the numerical values estimated previously, is very good until moments of about order 5; for larger-order moments the difference could be due to the existence of a critical order of divergence of moments.

Conclusion. – Using joint velocity and temperature structure functions (with a different approach than in [15]), we showed the multifractal nature of the temperature variance flux. With this joint structure function, we estimated universal multifractal parameters describing all the statistics of the temperature variance flux. Then we argued that the assumption of independence of the two fluxes is theoretically untenable, whereas the independence of the fluctuations of the fields V and θ is both theoretically and—we find—empirically acceptable

([3]) For instance, translation invariance, plane reflection invariance and incompressibility imply a factorization of $\langle \mathbf{V}(x)\theta(x+r)\rangle$ (see, *e.g.*, [26]).

since the corresponding relation: $\zeta_\theta(q) = (q/3) + K_\varepsilon(q/6) - K_\chi(q/2)$ holds reasonably well, at least for moments up to order 6.

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