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MULTIFRACTAL ANALYSIS OF FOREIGN EXCHANGE DATA

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SUMMARY

In this paper we perform multifractal analyses of five daily Foreign Exchange (FX) rates. These techniques are currently used in turbulence to characterize scaling and intermittency. We show the multifractal nature of FX returns, and estimate the three parameters in the universal multifactal framework, which characterize all small and medium intensity fluctuations, at all scales. For large fluctuations, we address the question of hyperbolic (fat) tails of the distributions which are characterized by a fourth parameter, the tail index. We studied both the prices fluctuations and the returns, finding no systematic difference in the scaling exponents in the two cases.

We discuss and compare our results with several recent studies, and show how the additive models are not compatible with data: Brownian, fractional Brownian, Lévy, Truncated Lévy and fractional Lévy models. We analyse in this framework the ARCH(1), GARCH(1, 1) and HARCH (7) models, and show that their structure functions scaling exponents are undistinguishable from that of Brownian motion, which means that these models do not adequately describe the scaling properties of the statistics of the data.

Our results indicate that there might exist a multiplicative 'flux of financial information', which conditions small-scale statistics to large-scale values, as an analogy with the energy flux in turbulence.

KEY WORDS: finance; scaling; foreign exchange; multifractals; Lévy distributions

1. INTRODUCTION

Until the 1960s, the only stochastic and scaling model in finance was Brownian motion, originally proposed by Bachelier in 1900,[1] and developed several decades later.[2] Some generalisations were then made by Mandelbrot and his followers involving either fractional Brownian motions,[3,4] or Lévy motion[5–9] (see also Reference 10 for a recent review). Closely related additive scaling models have also been developed [11–19]: Brownian, fractional Brownian, Lévy, truncated Lévy and fractional Lévy processes. Recently several publications have proposed new scaling models or empirical analyses, partially inspired by turbulence and statistical physics.[11–22] Indeed, as

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underlined by Stanley *et al.*,[16] in statistical physics when a large number of microscopic elements interact without characteristic scale, universal macroscopic scaling laws may be obtained independently of the microscopic details. The justification for this approach is that economic and financial systems consist of a large number of non-linearly interacting elements; it is also observed that scaling and volatility clustering is a general characteristic of financial systems.

The above approaches have generally involved additive monofractal processes and analyses; in contrast scaling systems are now known to be generally multifractal. In this paper we therefore adopt another approach, considering scaling multiplicative processes, which are the generic multifractal processes. We perform a multifractal data analysis on five daily foreign exchange (FX) data. Such an analysis characterizes in a scale invariant way (hence at all scales) the clustering of the volatility of FX returns, for all intensities. We perform this analysis on the statistical moments of the volatility. In the case of additive processes, the estimate of a few (i.e. 1st and 2nd) moments (q) is generally sufficient to characterize the entire distribution. But *a priori* financial processes are not restricted to this very special monofractal case, and the use of a wider range of moments (e.g. q from 0 to 4 with a 0·1 increment) is a way to explore all the intensities of the fluctuations and allows us to test of the monofractal hypothesis. In the case of multifractality the resulting scaling exponents $\zeta(q)$ are non-linear, showing that the estimate of two moments is by no means adequate for describing the entire distribution. This underlines the necessity of using multifractal analyses and models on financial data.

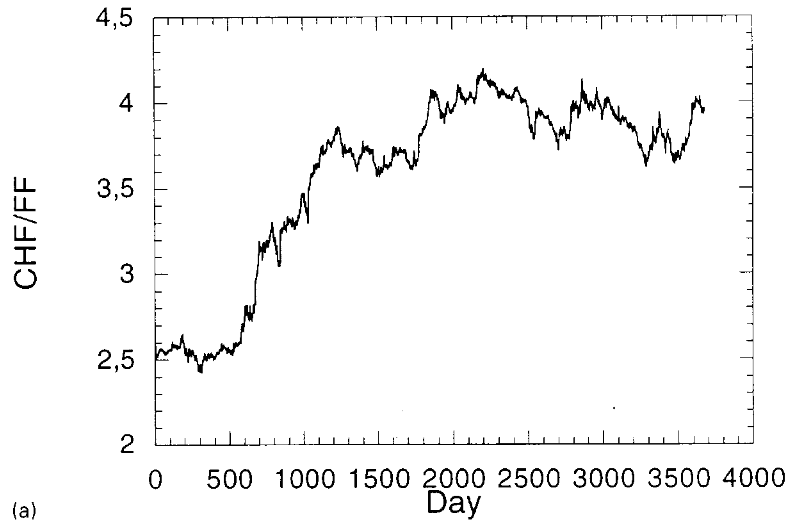
In Section 2, we precisely analyse our data series in the universal multifractal framework, which are stable, attractive multifractal processes, and which yield a full characterization of all small and medium fluctuation intensities with only three parameters: we demonstrate the multifractal nature of the returns, and estimate all the relevant parameters in our theoretical framework. The non-linear $\zeta(q)$ which is obtained shows that the additive scaling models are not compatible with the data: we argue that they were accepted following insufficient and overly restrictive tests. We finally consider another important empirical property of most financial time series: hyperbolic tails of the distributions, 'Pareto' tails.[6] Such extremes are in fact predicted as the result of a multifractal phase transition; we estimate the tail exponents: the fourth parameter in our approach. Unlike additive Lévy processes, where the value of the latter is restricted to be < 2, in multifractal processes, it can be any positive value > 1.

2. MULTIFRACTAL ANALYSIS OF THE DATA

2.1. The data series and the power spectra

The data we analyse are the daily Foreign Exchange rates in French Francs: Swiss Francs (CHF), German Mark (DEM), US Dollar (USD), Great Britain Pound (GBP), Japanese Yen (JPY). Each data series goes from 1 January 1979 to 30 November 1993: taking into account only the active days, we have 3680 data points for each series. In Figures 1(a) and 1(b) we represent two data series, CHF and USD series, which show quite large flucctuations at all scales. We characterize below the statistics of these fluctuations.

We performed two types of analysis: we studied directly the fluctuations of the price data; we also tested the transformation $Z \equiv \log X$, which is commonly used in order to study the proportional returns $r = \Delta X/X$, which are non-dimensional and financially more relevant. Indeed let us recall that $\Delta Z = Z(t+\tau) - Z(t) = \log(X(t+\tau)/X(t)) = \log(1 + \Delta X/X)$. Because



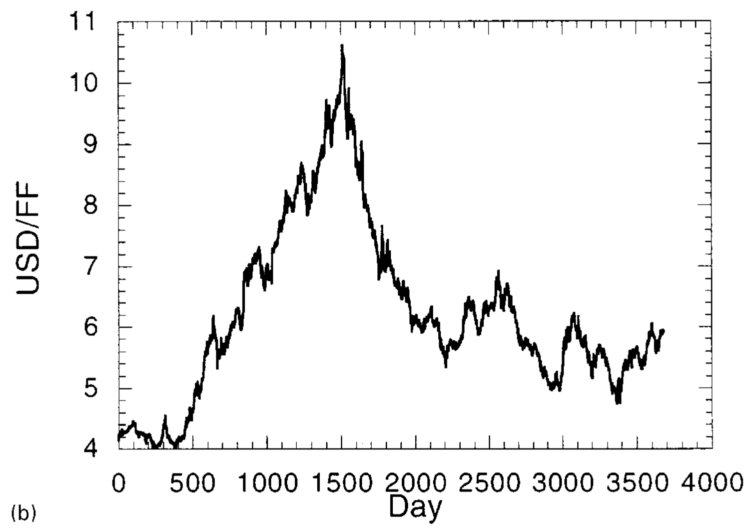


Figure 1. Two of the FX time series which are analysed here: CHF (a) and USD (b)

the fluctuations are usually small compared to the amplitude of the price itself ($\Delta X \ll X$) this gives

$$\Delta Z \simeq \frac{\Delta X}{X} = r \tag{1}$$

where ' $\simeq$ ' expresses equality, up to first order. When the time increment τ is not too large, the factor $1/X$ can be considered as a constant (compared to the highly fluctuating quantity ΔX).

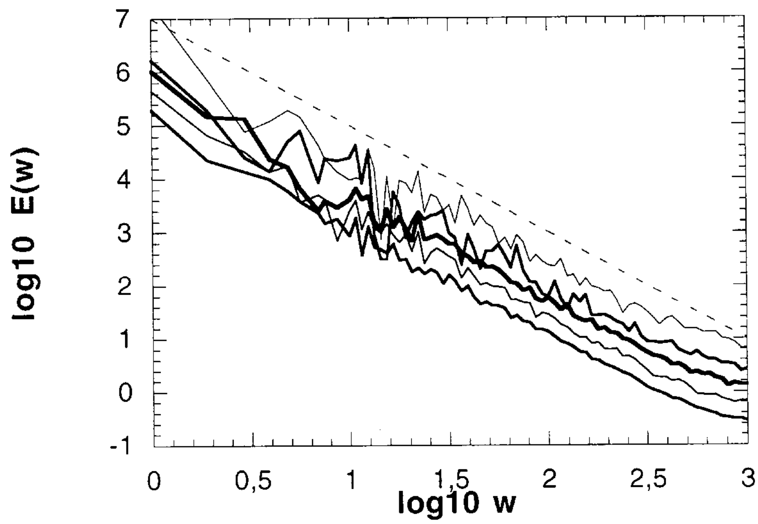


Figure 2. Power spectra of the time series, in log–log plot; from bottom to top: DEM, CHF, JPY, GBP, USD. For comparison, a dotted line of slope of -2 is also shown

Therefore, up to first order, one can predict that the fluctuations of the price and the returns have the same statistical scaling exponents. These exponents are certainly very close for small fluctuations, and less similar for larger fluctuations. Nevertheless, we did all the analysis for the returns and the price fluctuations and compared the results.

First we performed a Fourier (spectral) analysis: this is a sensitive way to estimate the limits of the scaling regime of the returns; the corresponding exponent quantifies the scaling of the variance. As is usually obtained, we find the following scaling spectrum:

$$E(f) \sim f^{-\beta} \tag{2}$$

where f is the frequency, '$\sim$' means proportionality and β, the scaling exponent of the power spectrum, is not far from 2. This is shown in Figure 2 for all the spectra, with a straight line of slope -2 for comparison. We also computed the power spectra of the fluctuations of the prices directly: there is only a shift in the log–log representation (corresponding to a multiplicative constant), but no change in the scaling trend.

We may note that an exact value of 2 is not always accepted: while Brownian motion gives exactly 2, fractional Brownian motion would give different values, and the Lévy models would give a slope depending on the number of realizations studied (see below and Appendix). For example, in a recent study of other data (a New York Stock Exchange Index), Mantegna and Stanley[12] obtain spectra with slopes in the range 1·8–2·1. Concerning studies of Foreign Exchange data, we note that Ghashghaie *et al.*[20] did a scaling analysis of Foreign Exchange USD-DEM data, but without providing the slope of their spectrum.

In any case, the power spectrum is only a second-order statistic and its slope is not enough to validate a particular scaling model: it gives only partial information about the statistics of the process. One would need the knowledge of the probability distribution of the process or, equivalently, of all its statistical moments other than second order. An important application of multifractal analysis is precisely to characterize all order moments for the validation of a scaling model. This is done in next section using structure function scaling exponents. In the following analysis, we will take as the guiding approach the analysis we performed in turbulence.[23–26]

2.2. Scaling of the structure functions, and multifractality

We denote here the value of exchange rate at time t as $X(t)$. The structure function analysis consists in studying the scaling behaviour of the non-overlapping fluctuations $\Delta Z_\tau(t) = |Z(t+\tau) - Z(t)|$ for different time increments τ. We estimate the statistical moments of these fluctuations, which (assuming both scaling and statistical translational invariance in time) depend only on the time increment in a scaling way:[27]

$$\langle (\Delta Z_\tau(t))^q \rangle \sim \langle (\Delta Z_T)^q \rangle \left(\frac{\tau}{T} \right)^{\zeta(q)} \tag{3}$$

where T is the fixed largest time scale of the system (and hence $\langle (\Delta Z_T)^q \rangle$ is a constant), $\langle \, . \, \rangle$ denotes statistical average, q is the order of the moment (we take here $q > 0$), and $\zeta(q)$ is the scale invariant structure function exponent. Equation (3) has already been tested for financial data for $q = 1$ and $q = 2$ by Muller *et al.*[28] The structure function analysis corresponds in fact to studying 'generalized' average volatilities at scale τ, since moments of order 1 or 2 are usually used to define the volatility. Furthermore, the present analysis consists in analysing this generalized volatility for all time scales.

The average of the fluctuations corresponds to $q = 1$, and $H = \zeta(1)$ is the parameter characterizing the non-conservation of the mean. For Brownian motion, $H = 1/2$, and for other models in finance, one can easily obtain $H \neq 1/2$, see below. The second moment is linked to the slope β of the power spectrum: $\beta = 1 + \zeta(2)$. In Figures 3(a) and 3(b), we show this scaling for different orders of moments for the series CHF and USD, in log–log plots. The straight lines in these figures indicate that the scaling of (3) is very well respected; we did this for all moments between 0·1 and 4·0, with a 0·1 increment: for all these values, the scaling of the 5 series is very good, and only for moments larger than about 4·0, it begins to be broken because of the insufficient amount of data analyzed (the statistical become dominated by the extreme gradients present in the sample).

We then estimated $\zeta(q)$ from the slopes of these straight lines, with a least squares regression. The resulting curves $\zeta(q)$ for the five data series are presented in Figure 4. For comparison, we also plotted the curve $\zeta(q) = q/2$, which corresponds to Brownian motion (see below). The corresponding curves obtained directly for the price fluctuations are very close to those corresponding to the prices themselves (this is also shown below in Figures 8(a)–8(e). A first remark which can be made concerning these different curves is that the FRF/DEM and FRF/JPY scaling exponents seem peculiar compared to the others: the relatively liquid exchange rates (FRF/CHF, FRF/GBP, FRF/USD) yield scaling exponents quite close to each other, whereas the rarely traded FRF/JPY rate has a more linear $\zeta(q)$, and FRF/DEM seem to possess a qualitatively different $\zeta(q)$, a behaviour which is likely due to the cap on fluctuations imposed by the European Change Mechanism between these two currencies. Recall that the nonlinear shape of the empirical curves is a signature of multifractality and is incompatible with the additive processes

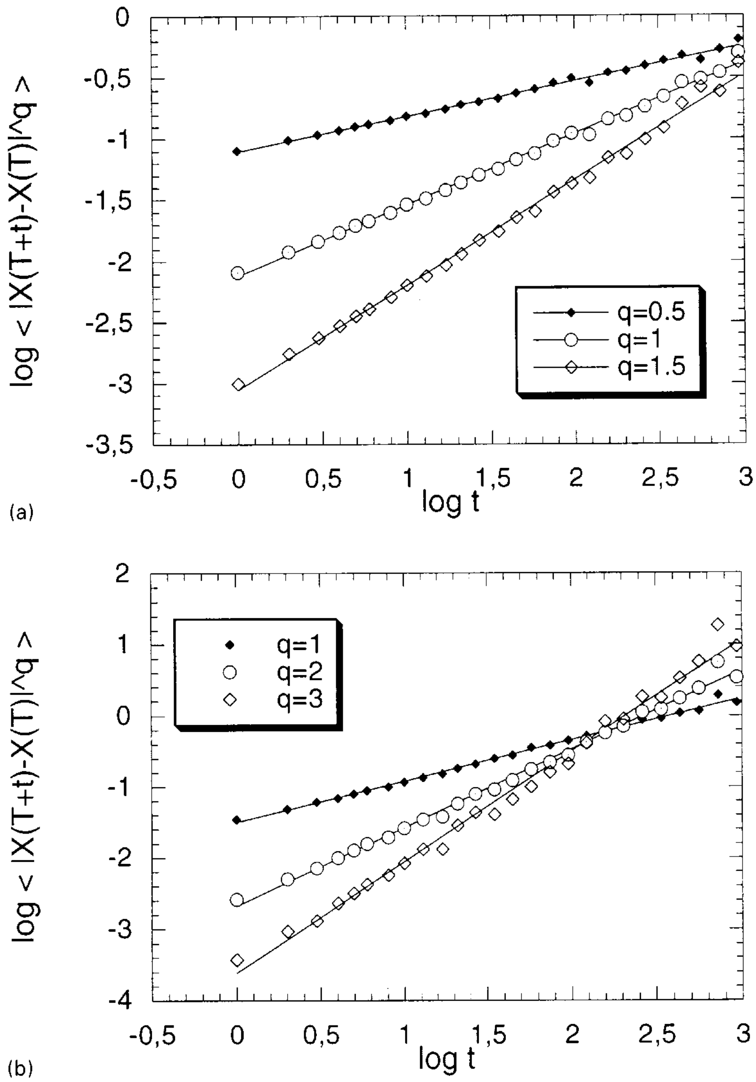


Figure 3. The scaling of the structure functions for CHF (a) for $q = 0{\cdot}5$, 1 and $1{\cdot}5$ and USD (b) for $q = 1, 2$ and 3, in log–log plot. A straight line indicates a perfect scaling. The slopes of these straight lines are estimate of $\zeta(q)$

which are linear (or at most bilinear for finite sample size). Indeed, in the multifractal framework, every order of moment is associated with an order of singularity which depends on the scales in the following way (see Reference 29):

$$\Delta Z_\tau \sim \langle \Delta Z_T \rangle \left(\frac{T}{\tau} \right)^{\gamma} \tag{4}$$

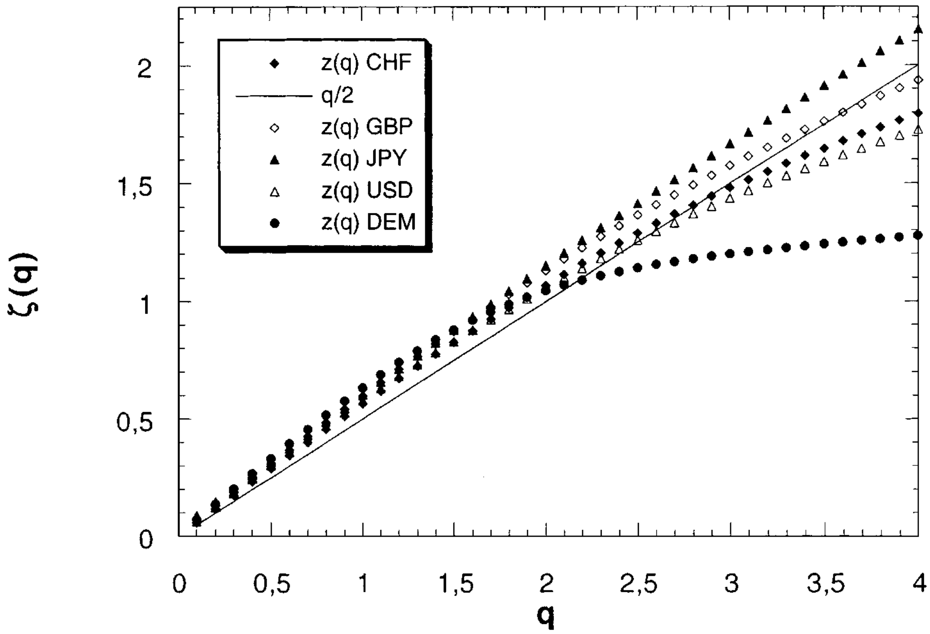


Figure 4. The functions $\zeta(a)$ obtained for the five times series. Their nonlinearity indicates multifractality. The continuous straight line corresponds to $\zeta(q) = q/2$, obtained for Brownian motion

where T is the (fixed) larger time scale of the system, and γ are the orders of singularity, which are usually negative. For monofractal models there is only one singularity, whereas for multifractal models, there is a whole range of singularities, associated to the statistical moments via a Legendre transform:[30]

$$c(\gamma) = \zeta(q) + q\gamma \tag{5}$$

$$\gamma = -\zeta'(q) \tag{6}$$

where $c(\gamma)$ is the codimension of the singularity γ, defined using the singularity probability distribution[29]

$$Pr(\gamma' > \gamma) \sim \left(\frac{T}{\tau}\right)^{-c(\gamma)} \tag{7}$$

Note that the equality sign '$\sim$' is within slowly varying functions. The Legendre transform equations (5) and (6) expresses a one-to-one relation between statistical moments q and singularity exponents γ; it also shows that for multifractal processes, $\zeta(q)$ is non-linear: there is a whole range of different singularities, to that the slope of $\zeta(q)$ is changing. Furthermore, it shows that, when a maximum singularity γ_s is reached for a finite sample size, the function $\zeta(q)$ becomes linear for moments of order $q > q_s = c'(\gamma_s)$:

$$\zeta(q) = c(\gamma_s) - q\gamma_s, \quad q > q_s \tag{8}$$

For additive models, the $\zeta(q)$ function has a different shape. First, there is a very simple linear form for the Brownian motion:

$$\zeta_{\mathrm{Brow}}(q) = \frac{q}{2} \tag{9}$$

and for a fractional Brownian motion (the fractional integration of order h of a Gaussian noise, see Reference 4):

$$\zeta_{\mathrm{Frac}}(q) = q(h - \tfrac{1}{2}) \tag{10}$$

The Brownian motion corresponds to $h = 1$ (i.e. an ordinary integral of Gaussian white noise, which gives, as we saw above, $H = 1/2$).

When Z is a Lévy motion (the integral of a Lévy noise), the behaviour of $\zeta(q)$ is a bit more involved. In this case, there is a Lévy index α ($0 \leqslant \alpha \leqslant 2$), which characterizes the divergence of the moments of the Lévy noise. For an infinite sample, $\zeta(q)$ diverges for $q > \alpha$, but for a finite samples (e.g. one realization) we obtain the following $\zeta(q)$ function for a Lévy motion of index α (see the appendix for the derivation of the corresponding expression for N examples):

$$\zeta(q) = \frac{q}{\alpha}, \quad q < \alpha \tag{11}$$

$$\zeta(q) = 1, \quad q \geqslant \alpha \tag{12}$$

To check this expression, we simulated a Lévy walk: Figure 5(a) shows a portion of it (for $\alpha = 1{\cdot}7$), Figure 5(b) shows the scaling obtained for several orders of moments, and Fig. 5(c) shows the resulting $\zeta(q)$ function which is obtained (for 32 000 data points): the result fits perfectly the theoretical formula.

One may note that the same $\zeta(q)$ was obtained by Bouchaud *et al.*[13] for a truncated Lévy model:[31] the resulting $\zeta(q)$ function is the same as that obtained for a single sample, but the underlying model is different: in the case of a truncated Lévy model, the behaviour $\zeta(q) = 1$ for $q \geqslant \alpha$ is due to the ad hoc introduction of an exponential tail of a Lévy law; for the Lévy model we presented here, this is purely due to finite sampling.

There is another additive fractal model which can be introduced, even if it has not yet (to our knowledge) been proposed in a financial context: a fractional Lévy motion,[29,32] obtained as a fractional integration of order h of a Lévy noise. Similar to the Lévy motion case we presented above, this gives the following $\zeta(q)$ function (see the appendix for the derivation of this result, in the general case of N_r different realizations) for one realization:

$$\zeta(q) = q\left(h - 1 + \frac{1}{\alpha}\right), \quad q < \alpha \tag{13}$$

$$\zeta(q) = q(h-1) + 1, \quad q \geqslant \alpha \tag{14}$$

Usual Lévy motion corresponds to $h = 1$. This gives a bilinear behaviour, with a change of slope of $1/\alpha$, occurring at $q = \alpha < 2$. Even if in order to test this model the empirical nonlinear $\zeta(q)$ in Figure 4 are (poorly) approximated by two lines, the change of slope generally occurs at moments larger than 2, and the empirical changes in slope ($= 1/\alpha \geqslant 1/2$ for fractional Lévy motion) are smaller than 1/2, therefore invalidating this model (see also Figures 8(a)–(e)).

We thus see that the non-linearity of all the empirical $\zeta(q)$ functions is a solid argument against Brownian, fractional Brownian, Lévy, truncated Lévy, and fractional Lévy models, all additive models giving straight lines or two portions of straight lines.

Because of these implications, we performed a direct test on the data: we display in Figures 6(a) and (b) $\langle(\Delta Z_\tau)^q\rangle/\langle\Delta Z_\tau\rangle^q$ vs. τ in a log–log plot: in the case of linear $\zeta(q)$ (a special case of which is the statistical independence of the fluctuations), this plot should be flat for all q's, whereas if the non-linearity is real we should see a trend (because of the concavity of the $\zeta(q)$ curve, the trend

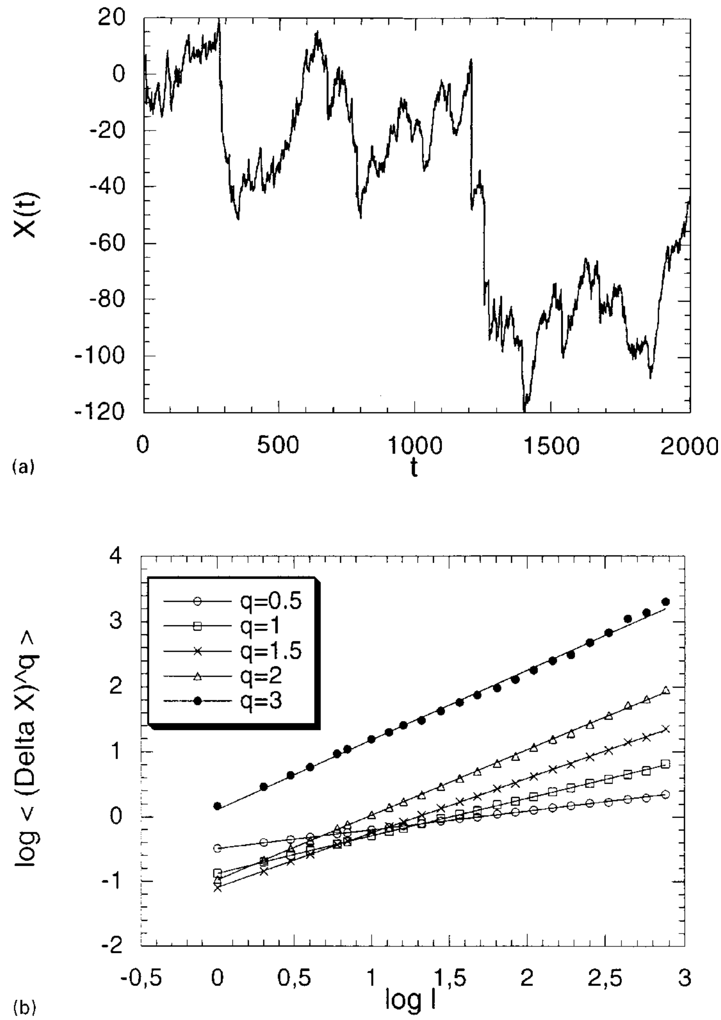


Figure 5. (a) The simulation of a Lévy motion obtained with $\alpha = 1{\cdot}7$; (b) the corresponding scaling of the structure functions for various values of q; (c) the resulting $\zeta(q)$ function: its shape is in excellent agreement with the theoretical expression derived in the text, for one realization. A thick straight line of equation q/α is shown for comparison

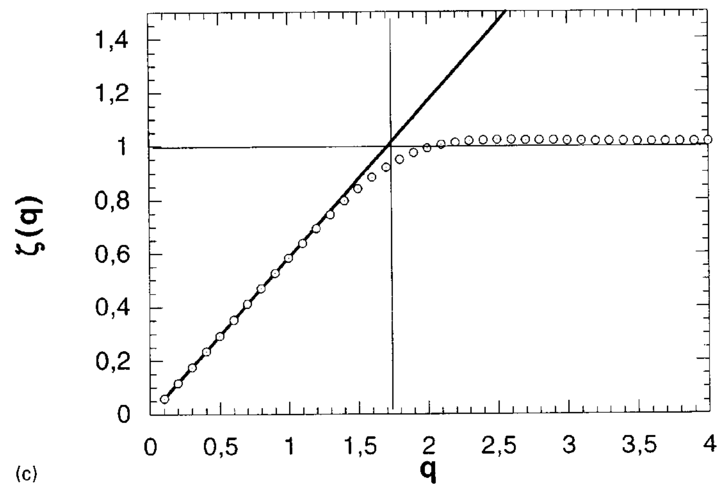


Figure 5. Continued.

should be positive for $q < 1$ and negative otherwise). We show this in Figures 6(a) and (b) for $q = 0{\cdot}5$, $1{\cdot}5$, 2, $2{\cdot}5$, and $q = 2$, 3, 4 respectively, for the series CHF and USD. There is clearly a non-trivial scaling regime over a large range: at large scales we observe noise which is presesumably a finite-size effect. Nevertheless, for more than two decades in time-scale, the trend is clear, and in order to better capture the nonlinearity in computing the slopes for $\zeta(q)$ we took into account only this scale range.

We now turn to a fit of these curves, using the universal multifractal model.

2.3. Universal multifractal parameters

The generic multifractal processes are the (nonlinear, non-additive) multiplicative cascade models. If one densifies (in scales) the cascade, one reaches 'continuous cascade' models[29]; recently the term 'infinitely divisible cascades' (referring to the infinitely divisible probability distributions) has been used.[33] Using the 'canonical representation' (also termed Lévy–Khinchine representation) for infinitely divisible random variables (see Reference 34), the following models were obtained in the turbulence litterature: log-normal,[35–37] log-Lévy,[29,38,39,23] log-Gamma,[40] log-Poisson.[33,41]. Among these, we consider here only the stable (and attractive) models: because of the multiplicative framework, these are the only models which are stable under raising to arbitrary powers, or convolution of different realizations.† This gives the log-Lévy model, and the log-normal model (which is also a special case of the log-Lévy family).

†Multiplicative models have for a long time been invoked via the 'law of proportional effects'. In the case of stable models, this proportional effect conserves the underlying law, the corresponding probability can therefore be received as a (stable) fixed point insensitive to the details.

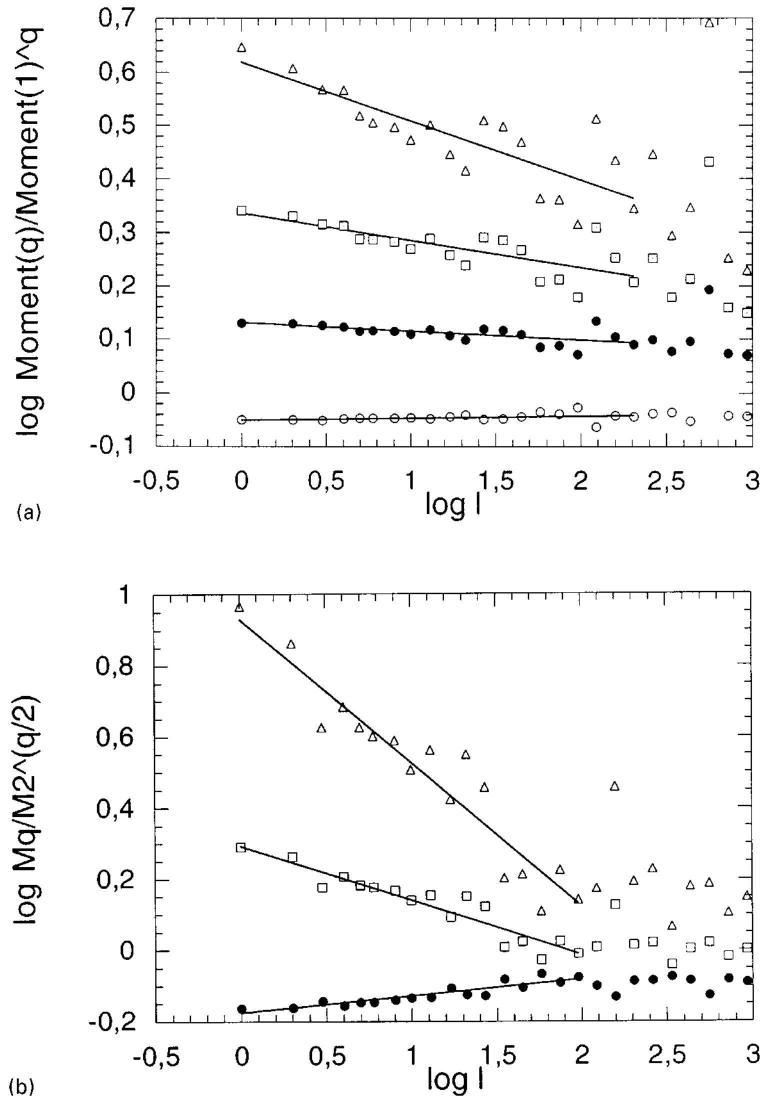


Figure 6. A direct test of the nonlinearity of $\zeta(q)$: $\langle(\Delta X_\tau)^q\rangle/\langle\Delta X_\tau\rangle^q$ vs. τ in a log–log plot; the straight lines which are observed have a slope of $\zeta(q) - q\zeta(1)$, which should be 0 in case of linearity. In (a), this test is performed on the CHF data, from bottom to top for $q = 0{\cdot}5, 1{\cdot}5, 2, 2{\cdot}5$, and in (b), it is applied on USD data, from bottom to top for $q = 2, 3, 4$

The following universal form is obtained for $\zeta(q)$:[23,29,38,39]

$$\zeta(q) = qH - \frac{C_1}{\alpha - 1}(q^\alpha - q) \tag{15}$$

where $H = \zeta(1)$ has the same definition as before; C_1 is the fractal codimension of the mean of the process ($0 \leqslant C_1 \leqslant d = 1$ for a $1 - D$ data set), and α is the Lévy index (of the generator, which is the log of the multiplicative process): $0 \leqslant \alpha \leqslant 2$. The log-normal model corresponds to $\alpha = 2$. We may emphaseze here the difference of this model with addition Lévy models: for universal multifractals, the Lévy distribution is not assumed for the difference of the price, but for its generator, the log of the absolute value of the difference.

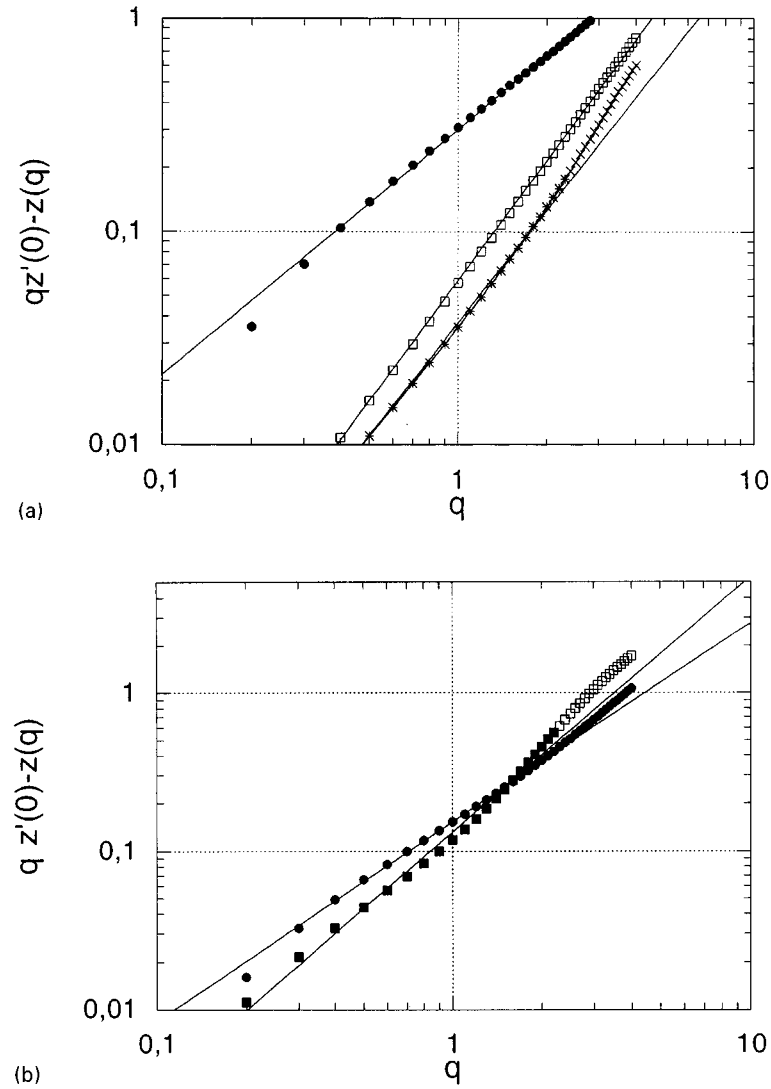


Figure 7. $f(q) = q\zeta'(0) - \zeta(q)$ vs. q in log–log plot. For the universal multifractal model, the straight lines obtained have a slope of α. In (a) this is performed for CHF, USD and JPY data series (from bottom to top), and in (b) for DEM (squares) and GBP (dots). The values of C_1 are given by the intercepts of these straight lines

Table I. The empirical estimates of H, α and C_1 obtained for the five different foreign exchange time series

Series	$H = \zeta(1)$	$\zeta(2)$	α	C_1
CHF	$0{\cdot}56 \pm 0{\cdot}03$	$1{\cdot}07 \pm 0{\cdot}05$	$1{\cdot}8 \pm 0{\cdot}2$	$0{\cdot}03 \pm 0{\cdot}01$
DEM	$0{\cdot}63 \pm 0{\cdot}03$	$1{\cdot}05 \pm 0{\cdot}05$	$1{\cdot}6 \pm 0{\cdot}2$	$0{\cdot}08 \pm 0{\cdot}02$
GBP	$0{\cdot}60 \pm 0{\cdot}03$	$1{\cdot}13 \pm 0{\cdot}05$	$1{\cdot}3 \pm 0{\cdot}2$	$0{\cdot}04 \pm 0{\cdot}01$
JPY	$0{\cdot}60 \pm 0{\cdot}03$	$1{\cdot}15 \pm 0{\cdot}05$	$1{\cdot}2 \pm 0{\cdot}1$	$0{\cdot}05 \pm 0{\cdot}02$
USD	$0{\cdot}58 \pm 0{\cdot}03$	$1{\cdot}06 \pm 0{\cdot}05$	$1{\cdot}9 \pm 0{\cdot}2$	$0{\cdot}05 \pm 0{\cdot}02$

A consequence of universal multifractals is that $\zeta(q)$ and other related quantities such as $f(q) = q\zeta'(0) - \zeta(q)$ (when $\alpha > 1$) are non-analytical at $q = 0$. Using Equation (15), we see that $f(q)$ is precisely a non-integer power law, proportional to q^{α}. Therefore, in a plot of $\log f(q)$ vs. $\log q$ we should obtain a slope of α.[25,42] In Figures 7(a) and 7(b), we show this for all the data series, using for $\zeta'(0)$ an extrapolation of the slope at the origin: this gives very good straight lines, whose slopes are estimates of α; the values of C_1 are given by the intercepts.

We indicative in Table I the different values we obtained, which show that $H = 0{\cdot}60 \pm 0{\cdot}03$ is quite stable for the different series, as well as $\zeta(2) = 1{\cdot}10 \pm 0{\cdot}05$ (hence $\beta = 2{\cdot}10 \pm 0{\cdot}05$). The error bars given here correspond to the range of values shown in Table I for the different times series. Generally speaking, it is not easy to estimate the errors in scaling exponents for a given times series: the method we therefore use is quite direct and consists in estimating several scaling exponents for different portions of the time series, in order to obtain a mean value and a dispersion. The value of $H = \zeta(1)$ we obtained here is quite close to other estimates for Foreign Exchange time series: Muller *et al.*[28] obtain $H = 0{\cdot}59$ for the following currencies against the USD: DEM, JPY, CHF, GBP, for scales between 2 hs and 3 months; Evertsz[14] obtains $H = 0{\cdot}56$ for USD/DEM time series for scales between 3 hs and 2 days.

On the other hand, the values of α and C_1 obtained for the different time series present more variability, in the range 1·2–1·9 and 0·03–0·08 respectively.‡ The procedure used to estimate these latter empirical values being less direct, it may be that much more data are needed in order to have good confidence in the precision of these estimates.

Nevertheless, the fits provided by these values is not so bad: in Figures 8(a)–8(c) we show the empirical functions $\zeta(q)$, compared with the theoretical fit coming from equation (15) with the values of H, α, C_1 given in Table I; the values obtained directly for the fluctuations of the price are also shown in the same plots. They are very close to those obtained for the returns, for weak and medium moments: there is only a slight discrepancy for larger moments. This clearly shows that the processing corresponding to taking the log of the price has no important influence on the statistical scaling exponents.

It is also clear from these figures that the fits we obtain are of different statistical significance. First, the fit for the USD series is excellent until moment of order 4. The empirical curves as well as the theoretical fit are clearly nonlinear, with a clear departure from two possible straight lines:

‡We may note that R. Mellen (private communication, 1992) estimated $\alpha \simeq 1{\cdot}8$ for another financial time series, the Dow Jones Index, using a nonlinear fit of $\zeta(q)$.

$\zeta(q) = q/2$ and $\zeta(q) = qH_{\mathrm{USD}} = 0{\cdot}58H$ (see Figure 8(e)). The JPY series is also well fit (see Figure 8(d)), but the nonlinearity of the curves is not so clear, and a departure from $\zeta(q) = qH = 0{\cdot}60q$ is visible only for moments of order larger than 2. Next, the CHF and GBP curves (Figures 8(a) and 8(c)) are also clearly nonlinear, with good fits until moments of order about 2·5–3·0. Then, there are departures from the fit, which we consider below in the framework of divergence of moments of observables. The last figure concerns DEM data (Figure (8b)), which has a theoretical fit in good agreement with the data until moment of order about 1·8, and then there is a very clear departure, with empirical estimates being linear, with a slope nearly flat. This is the only case where the $\zeta(q)$ function could be said to be not far from that of a (additive) Lévy

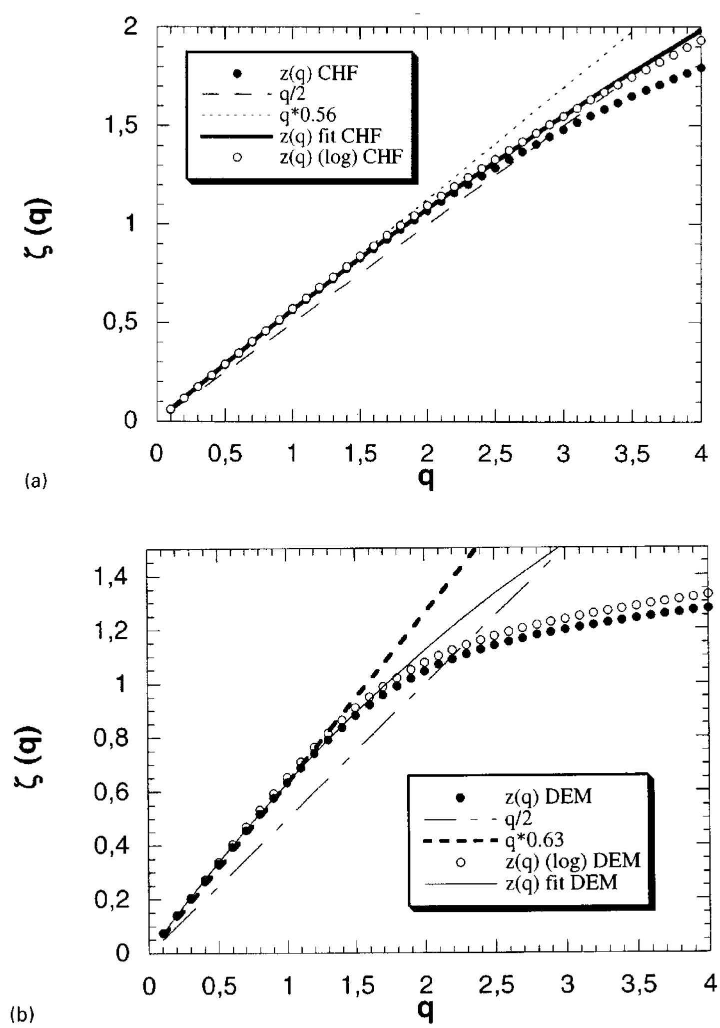
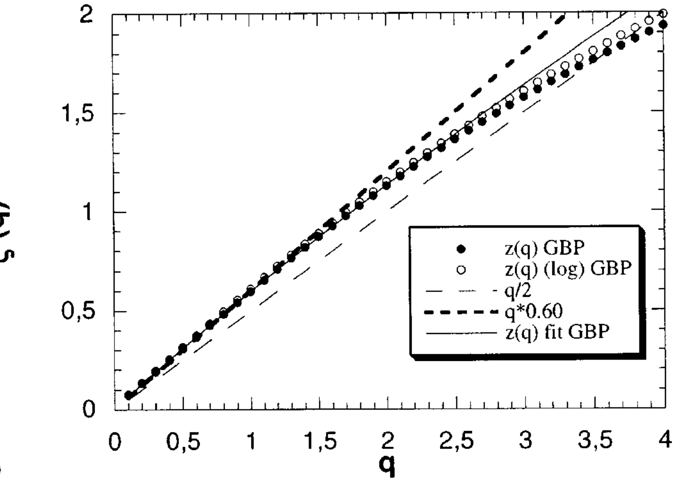
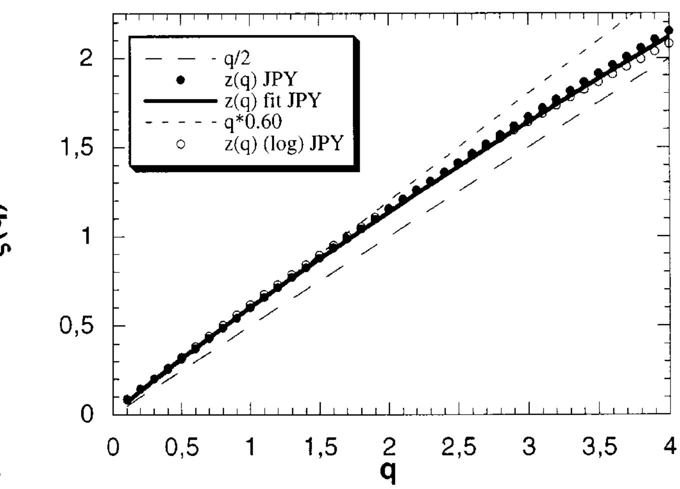


Figure 8. The functions $\zeta(q)$ vs. q: empirical estimates (opened and filled dots) compared to the universal multifractal fits (continuous lines). In (a), CHF; in (b), DEM; in (c), GBP; in (d), JPY; in e), USD. Two dotted straight lines are also shown: the Brownian case $\zeta(q) = q/2$ and the fractional Brownian case $\zeta(q) = q\zeta(1)$



(c)

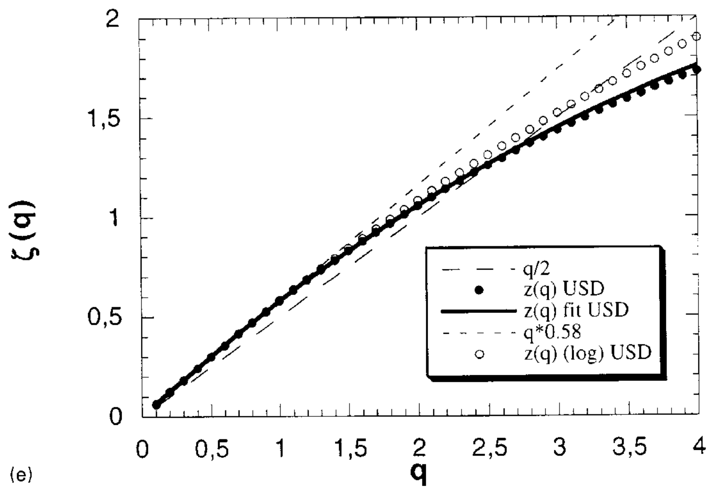


(d)

Fig. 8. Continued

walk (or fractional Lévy model); we may note here that this peculiar behaviour may be a manifestation of the imposed currency area allowing a maximum of 4·5% fluctuations between FRF and DEM inside the European Monetary System since 1979 (there is 15% since August, 1993).§ Nevertheless, we see below that the order of divergence of moments is also smaller for this series ($q_D \simeq 1{\cdot}8$; see next section for details) which gives an explanation of this peculiar behaviour in the multifractal framework.

§A more precise analysis of the GBP between October 1990 and October 1992, when it went inside the European Change Mechanism with fluctuation margins of 12%, was done in Reference 43.



(e)

Figure 8. Continued

2.4. About the fat tails of the distributions

One of the most important properties of economic and financial time series is the so-called "Pareto distributions"[44] of extreme events (see Reference 6), fat tailed (or hyperbolic) probability distributions:

$$\Pr(\Delta X > x) \sim x^{-p}, \quad x \gg 1 \tag{16}$$

This behaviour has been frequently noticed on financial data series,[6–9,45,46] and indicates a divergence of moments of order p. This divergence of moments can be obtained through several scaling models. First, it was one of the main agruments used by Mandelbrot to propose in 1963 Lévy distributions for financial fluctuations, with p being the Lévy index, subject to the limitation $0 \leqslant p \leqslant 2$. But it is not the only scaling model compatible with equation (16). Indeed, in cascade models in turbulence,[47,48,29] the small scale limit is singular; a measurement at larger scale is then only an integration of this singular field, leading to tails following equation (16). In this case, the critical order of divergence of moments is denoted $p = q_D$, and has the only constraint $q_D > 1$, with no upper limit.¶

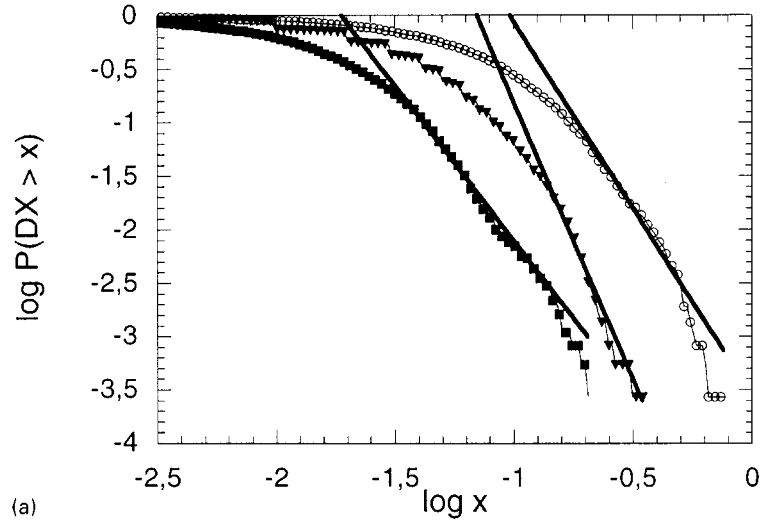
We can therefore obtain, in the multifractal framework[29] nonlinear behaviour of $\zeta(q)$ for $q \leqslant q_D$ and divergence of moments of order q_D. This is a first-order multifractal phase transition;[50,51] because the empirical estimates are always finite, the function $\zeta(q)$, which should be theoretically $-\infty$ for $q > q_D$ (because of the divergence of moments), follows (see Reference 52 for an empirical estimate in turbulence) for a finite sample size:

$$\zeta(q) = c(\gamma_{s,D}) - q_D \gamma_{s,D}, \quad q > q_D \tag{17}$$

¶Note that due to the use of highly restrictive microcanonical[49] or geometric[30] multifractals, which present no divergence of moments, this has not been adequately appreciated in the multifractal turbulence literature.

where $\gamma_{s,D}$ is the largest singularity present in the finite sample size. This singularity increases with the number of samples considered in the statistics, with no bound, but is, of course, always finite for a finite sample.

We performed this analysis on the data. Figures 9(a) and 9(b) shows the probability distribution of the fluctuations, on log–log plot. In both cases, a straight line for the most extreme events indicates the range of values for which equation (16) is valid. Because of the relatively small sample we analysed, straight lines are only formed over a limited range of intensities. The fits we



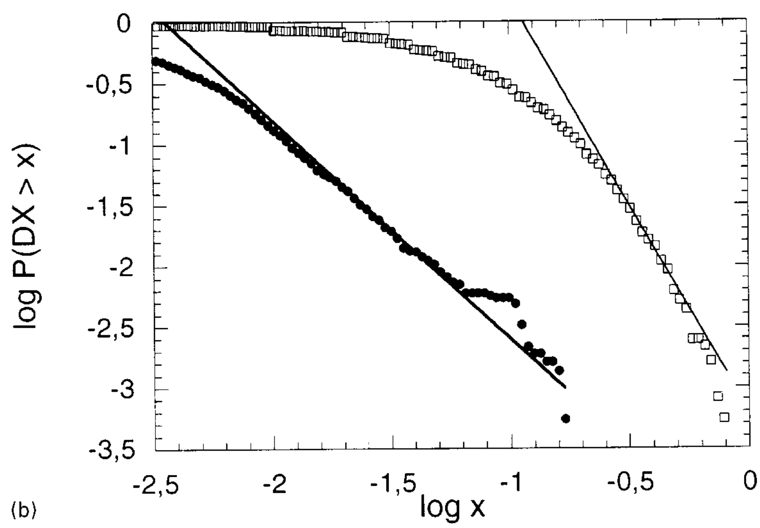


Figure 9. The tails of the probability distributions of the fluctuations $X(t+5) - X(t)$ in log–log plot. In (a): from left to right, for CHF, JPY, USD; in (b) from left to right DEM and GBP. The straight lines indicate hyperbolic distributions, and their slope correspond to q_D

Table II. Empirical estimates of q_D for $\Delta T = 5\tau$ and $\Delta T = 15\tau$ and the value of q_s as given by the estimates of α and C_1 shown in Table I

Series	$q_D(\Delta T = 5\tau)$	$q_D(\Delta T = 15\tau)$	$q_s = \left(\frac{1}{C_1}\right)^{1/\alpha}$
CHF	$3{\cdot}0 \pm 0{\cdot}3$	$3{\cdot}1 \pm 0{\cdot}3$	7·7
DEM	$1{\cdot}7 \pm 0{\cdot}2$	$1{\cdot}8 \pm 0{\cdot}2$	4·8
GBP	$3{\cdot}4 \pm 0{\cdot}4$	$3{\cdot}6 \pm 0{\cdot}4$	12·9
JPY	$4{\cdot}8 \pm 0{\cdot}4$	$5{\cdot}2 \pm 0{\cdot}4$	14·5
USD	$3{\cdot}0 \pm 0{\cdot}3$	$3{\cdot}5 \pm 0{\cdot}3$	4·9

show are simply obtained from a least-squares straight line fit for the tail portion which is roughly linear. More involved and precise methods to estimate the tail index exist, as discussed e.g. in Reference 46, where a subsample bootstrap method is applied, but they necessitate about at least one order of magnitude more data to be implemented. We estimate q_D for time increments of the returns of 5 days and 15 days. As before, we see three different behaviours: CHF, USD and GBP show tail indexes in the range 3–3·6, JPY has a tail index closer to 5 and DEM smaller than 2 (see Table II). We notice that all these values, except the case of DEM (giving $q_D \simeq 1{\cdot}8$), are larger than 2, therefore once again contradicting the additive Lévy walk model. This was also noted by Dacorogna *et al.*,[46] who termed them 'hyperbolic non-stable distributions': they obtained for several foreign exchange series some values of q_D from about 3–4: using a very large dataset, they showed that the second moment was converging, whereas the fourth moment was not. It can also be seen in their Table 7 that currencies belonging both to the European Change Mechanism (as in our case DEM and FRF) have a smaller q_D than other currencies. Muller *et al.* (the same group)[28] argued with reason that 'a valid model must explain both the unstable distribution and the empirical scaling law'. This is what we propose here.

We finally consider the linear behaviour of $\zeta(q)$ for large order moments: we saw that it can have two different origins: in case of finite sampling (see equation (8)) the critical moment is given by $q_s = c'(\gamma_s) = (1/C_1)^{1/\alpha}$ for 1 realization of a universal multifractal;[51] in case of divergence of moments the critical moment is simply q_D, as shown by equation (17). Table II shows the values of q_s obtained from the estimates of α and C_1 for the different times series, as compared to the values of q_D estimated above. This shows that the sampling critical moment q_s is always larger than q_D, confirming that the linear behaviour of $\zeta(q)$ for large order moments is a consequence of divergence of moments, as confirmed visually by the curves in Figures 8(a)–8(e).

The H value, and the two universal multifractal values α and C_1 therefore contain statistical descriptions of the small and medium intensity fluctuations corresponding to the nonlinear part of $\zeta(q)$, whereas the value of q_D corresponds to the most extreme events. These four parameters are argued to provide all the information of the statistics of the amplitude of the fluctuations of the data series, at all scales. They do not contain information of the signs of the fluctuations, which is the subject of further studies, and would need vectorial ('Lie') cascades, see Reference 53.

3. FURTHER COMPARISONS WITH OTHER MODELS

In order to provide a critical comparison to our approach, we will now discuss the scaling models and empirical results contained in several recent proposals. First, we already noticed that the different additive models provide at best bilinear $\zeta(q)$ functions. The truncated Lévy model[11,12,17–19] is a curious proposal, because what is most specific to Lévy distributions is the hyperbolic tail of their distributions; a truncated Lévy distribution is no longer stable, and therefore this model loses its main justification (indeed, except asymptotically where it would be Brownian motion,[31] it would not even be scaling). The good superposition of different scales they observe is nothing more than the scaling hypothesis, with the usual value H being the inverse of their α. We can also add concerning their restriction to a finite variance, that there is nothing special about the variance, and that the 'practical' (empirical) variance is always finite. However, what can be observed is the non-convergence of this variance as the sample size increases, which corresponds to hyperbolic distributions with $q_D < 2$. Furthermore, the empirical fits of the probability distribution of the increments of the data, proposed by the authors in a log-linear plot, is only good for medium events: for large fluctuations, they invoke rather imprecise 'exponential or stretched exponential' behaviour. In fact, it is likely that a log–log plot of their density would show a hyperbolic tail, with a tail exponent larger than their Lévy index.

Bouchaud *et al.*[13] used some empirical results of Muller *et al.*[28] to justify a truncated Lévy model: they used the two values $\zeta(1) = 0{\cdot}59$ and $\zeta(2) = 1{\cdot}02$ to argue the validity of a truncated Lévy model with $\alpha = 1/\zeta(1) \simeq 1{\cdot}7$. These two values are obviously not sufficient to justify one particular model, because several curves could fit into these two empirical values. As we argue in this paper, a whole range of $\zeta(q)$ values should also be considered.

Considering the results of Ghashghaie *et al.*,[20] one may note that their empirical $\zeta(q)$ is nonlinear, but it is not concave: indeed for concave curves the tangency at any point is 'outside' the curve, which is clearly not the case for $n = 4$, 5 and 9 in Figure 2(b) of their paper. This is impossible in the multifractal framework (because $\zeta(q)$ is a second Laplace characteristic function.[34,47] Their empirical estimates, which were obtained without absolute values, are therefore subject to caution. Furthermore, they claim to obtain–and it can be seen on their figure—that the empirical values of $\zeta(q)$ for turbulence and financial data are close to each other. This cannot be the case, because of the different values of H (in our notation): $H \simeq 0{\cdot}4$ in turbulence, and $H = 0{\cdot}6$ for financial data: this discrepancy is important because the linear drift of $\zeta(q)$ is contained in the value of H, and due to the relative smallness of C_1, for small q this term dominates.

This last point has been justifiably critized by Arneodo *et al.*[21] and Mantegna and Stanley,[17] but these authors argue that the fluctuations are not correlated because of a exactly -2 slope of their power spectrum. First, their slope is probably not exactly -2, as empirically shown in several studies.[11,12,14] Secondly, in any case even if the slope is exactly -2 (which means $\zeta(2) = 1$) this would indicate that there is no correlation for the first moment, but it wouldn't necessarily generalize to other moments. In short, $\zeta(2) = 1$ does not imply that the process has independent increments. Independence necessitates an analogous behaviour for other moments, leading to $\zeta(q) = q\zeta(1) = qH$. Because the other moments are nonlinear in q, we conclude that the data are incompatible with process with independent increments.

We must mention also the AutoRegressive Conditional Heteroskedastic (ARCH) family,[54] which are discrete processes for which we have

$$X(t) = \sum_{i=1}^{t} \varepsilon_i \tag{18}$$

$$\varepsilon_t = \sigma_t u_t \tag{19}$$

where u_t is a random variable with zero mean and unit variance, which is typically taken as Gaussian, and σ_t depends on the information of the system at time $t-1$. For the ARCH(q) model proposed by Engle[54] we have simply

$$(\sigma_t)^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 \tag{20}$$

where $\omega > 0$ and $\alpha_i \geqslant 0$. This model was designed mainly to obtain volatility clustering. It was later generalized in the following way, giving GARCH (p, q) models:[55]

$$(\sigma_t)^2 = \omega + \sum_{i=1}^{q} \alpha_i \in_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \tag{21}$$

where the different parameters must obbey some conditions to have a well-defined process (see Reference 56 for a review). The model GARCH (1, 1) is often considered to give a good approximation to data, with values verifying $\alpha + \beta < 1$, but close to 1. For some parameters, GARCH models can also display fat-tails and be linked to stable processes.[57–59] These properties are close to the data, but despite this, some other properties of these models do not seem realistic; first, the models are built for discrete times; a continuous limit has been described only for GARCH (1, 1).[60] Second, the GARCH models possess a fixed and finite regression range, giving a finite memory. As shown recently in Muller *et al.*[61] this finite memory is in contradiction with long memory of volatility.[62,63] A HARCH model (heterogeneous interval, autoregressive, conditional heteroskedasticity) was then proposed by these authors,[61] containing a long memory through a recursive regression linking different scales to each other. This gives the following expression for the HARCH(p) model:

$$(\sigma_f)^2 = \omega + \sum_{j=1}^{p} c_j \left(\sum_{i=1}^{j} \varepsilon_{t-i} \right)^2 \tag{22}$$

We have here a linear combination of the squares of aggregated returns.

In order to test these models, we simulated several of them belonging to these three classes (ARCH, GARCH and HARCH), and tested the result using the structure function scaling analysis. We simulated for each test 10^5 datapoints, tested the scaling of the moments, and then plotted the resulting $\zeta(q)$ functions. We did this for ARCH(1) with $\omega = 10^{-3}$ and $\alpha = 0{\cdot}5$; GARCH (1, 1) with $\omega = 10^{-3}$, $\alpha = 0{\cdot}1$ and $\beta = 0{\cdot}8$; and finally HARCH (7) with the values of the parameters proposed in Reference 61. In all these cases the scaling of equation (3) is excellent for all orders of moments (from $q = 0{\cdot}1$ to 4). The resulting $\zeta(q)$ functions are all linear, and very close to $q/2$ which corresponds to Brownian motion; see Figure 10 for the functions corresponding to GARCH (1, 1) and HARCH (7). This means that, even if they are built to reproduce a clustering of volatility and interrelations between scales, these models quickly do not differ significantly from the Brownian motion case for their scaling properties. Furthermore, these models fail to reproduce the nonlinear shape of the $\zeta(q)$ functions which is empirically observed. We may note that Mantegna and Stanley[64] also recently compared some scaling properties of ARCH(1) and GARCH(1, 1) models to empirical data, concluding that they fail for short time horizons. These considerations and criticisms explain why we consider that our proposal is the closest to the data.

As shown by Muller *et al.*,[61] large-scale volatility predicts small-scale volatility much better than the other way around; they have interpreted this as an influence of long-term traders on short-term traders. Their HARCH model already discussed above was inspired by this fact. As

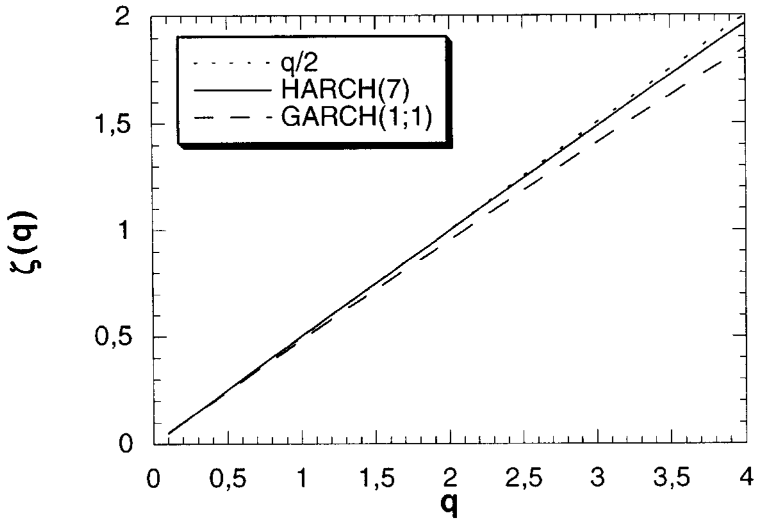


Figure 10. The $\zeta(q)$ function for GARCH(1, 1) model (broken line) with parameters $\omega = 10^{-3}$, $\alpha = 0{\cdot}1$ and $\beta = 0{\cdot}8$ and for HARCH(7) model (continuous line) with the parameters proposed by Muller et al. [61]. These curves are linear, with slopes respectively 0·46 and 0·49, very close to the Brownian motion case (dotted line) for which the slope is 1/2

discussed by Ghashghaie *et al.*,[20] this behaviour can be compared to the energy flux in hydrodynamic turbulence, which cascade from large scales to smaller ones, until viscosity at small transforms mechanical energy into heat. This cascade of energy flux is often modelled with multiplicative cascades, which give rise to multifractal fields. This means that the findings of Muller *et al.*,[61] as well as the nonlinear multifractal shape of $\zeta(q)$ give a coherent picture of the financial process. The usual comparison between finance and hydrodynamic turbulence would then well be more than a qualitative comparison (it is quantitative in term of H, C_1, α). For practical considerations, a multiplicative cascade with the parameters we have estimated here can easily be simulated using multifractal simulation techniques.[65] The predictability of such processes has also been explored in the turbulent framework;[66] to what extent it can be adapted to the financial frame is an interesting question.

4. CONCLUSION

We have shown that the nonlinearity of the moment scaling function $\zeta(q)$ is not compatible with the additive scaling models proposed to date for financial fluctuations: Brownian, fractional Brownian, Lévy, truncated Lévy and fractional Lévy models. It is neither compatible with several popular models belonging to the ARCH family. As an alternative, we proposed a universal multifractal framework to characterize the fluctuations of five foreign exchange data series, at all scales, and for all intensities. Universal multifractals are appealing because they are believed to be the generic outcome of high numbers of degrees of freedom nonlinear processes which respect scaling symmetries. The statistics of the data are fully described with four parameters, taking into account two complementary aspects of financial time series: the multiple scaling and the hyperbolic (Pareto) probability distributions, which is a generic feature of multifractal processes. This model is, we believe, the closest available to the data. We have shown that analysing directly the price or the log of the price, makes no significant difference for the scaling exponents of the fluctuations.

Some implications of our findings should be underlined: as already noticed by Belkacem *et al.*[15] and Bouchaud and Sornette,[67] many financial models are developed in a Gaussian framework, assuming that the price fluctuations follow Brownian motion; but these studies suffer from the same flaw, because they propose, without a strong empirical basis, an alternative but restricted model using Lévy (or truncated Lévy) motions. To be compatible with our results, models should be multiplicative rather than additive. The 'multifractal corrections' which arise are certainly much more important than second order, because quasi-Gaussian fluctuations are not intermittent whereas the multifractal fluctuations are so intermittent that they give rise to divergence of moments. We hope to stimulate new theoretical developments to explain this multifractal behaviour and to explore their consequences.

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APPENDIX
STRUCTURE FUNCTIONS FOR LÉVY MOTION AND FRACTIONNAL LÉVY MOTION FOR N_R REALIZATIONS

We establish the structure function exponents $\zeta(q)$ first for Lévy motion for one realization, then for fractional Lévy motion in the general case of N_r independent realizations (each of them of length N datapoints). We denote Y the Lévy noise of index $\alpha (0 \leqslant \alpha \leqslant 2)$ and h the order of fractional integration of this noise ($h = 1$ for the standard integration giving Lévy motion).

Because of the stability property of the Lévy-stable random variables, which can be written[34]

$$\sum_{i=1}^{n} Y_i =^{\mathrm{d}} n^{1/\alpha} Y \tag{23}$$

where $=^{\mathrm{d}}$ means equality of probability distributions, and Y_i and Y belong to the same Lévy-stable distribution, of index α, we have the following result for the structure functions of the Lévy motion:

$$\langle (\Delta X_\tau(t))^q \rangle = < \left(\sum_{i=t}^{t+\tau} Y_i \right)^q > \sim \tau^{q/\alpha} \langle Y^q \rangle_\tau \tag{24}$$

where $\langle Y^q \rangle_\tau$ indicates that the moment is estimated for non-overlapping intervals of amplitude τ. This shows that, for low order of moments ($q < \alpha$), the expression $\langle Y^q \rangle_\tau$ converges, and the process X has the same exponent as a fractional Brownian motion of order $H = 1/\alpha$: $\zeta(q) = q/\alpha$. For moments of order $q \geqslant \alpha$, there is a divergence of $\langle Y^q \rangle_\tau$ which depends on the number of points taken into account: for an infinite sampling, this expression diverges; but for a finite sampling (N data points), the ensemble average is estimated for N/τ different realizations of the variable. Let us recall[68] how to quantify this divergence for a given number of independent realizations (N/τ here):

$$\langle Y^q \rangle_\tau \sim \left(\frac{N}{\tau}\right)^{(q-\alpha)/\alpha} \sim \tau^{1-q/\alpha} \tag{25}$$

This gives finally the following $\zeta(q)$ function in case of a Lévy walk (for one realization):

$$\zeta(q) = \frac{q}{\alpha}, \quad q < \alpha \tag{26}$$

$$\zeta(q) = 1, \quad q \geqslant \alpha \tag{27}$$

In the more general case of fractional Lévy motion for N_r independent realizations, we have the following decomposition:[32]

$$\langle (\Delta X_\tau(t))^q \rangle \sim \tau^{q(h-1+1/\alpha)} \langle Y^q \rangle_\tau \tag{28}$$

As is done for equation (23), we have for N_r realizations (NN_r data points):

$$\langle Y^q \rangle_\tau \sim \left(\frac{NN_r}{\tau}\right)^{(q-\alpha)/\alpha} \sim (\tau^{-1-D_s})^{(q-\alpha)/\alpha} \tag{29}$$

where N is still a constant, and we denote $N_r = \tau^{-D_s}$: D_s is then called a 'sampling dimension'. This finally gives

$$\zeta(q) = q\left(h - 1 + \frac{1}{\alpha}\right), \quad q < \alpha \tag{30}$$

$$\zeta(q) = q\left(h - 1 - \frac{D_s}{\alpha}\right) + 1 + Ds, \quad q \geqslant \alpha \tag{31}$$

The change of slope between the two straight lines is $(1 + D_s)/\alpha$. The single realization corresponds to $D_s = 0$; for an infinite sampling, $D_s \to \infty$ and thus $\zeta(q) \to \infty$ for $q \geqslant \alpha$. The Brownian and fractional Brownian motions are recovered for $\alpha = 2$ and, respectively, $h = 1$ and $h \neq 1$ (there is only one regime in these cases); the Lévy motion corresponds to $h = 1$.

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