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# The Dimension and Intermittency of Atmospheric Dynamics

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## Abstract

We question the existence of a dimensional transition separating quasi-two dimensional and quasi-three dimensional atmospheric motions, i.e. large- and small-scale dynamics. We insist upon the fact that no matter how this transition should occur, it would have drastic consequences for atmospheric dynamics, consequences which have not been observed in spite of many recent experiments.

An alternative simpler hypothesis is proposed: that small scale structures are continuously deformed – flattened – at larger and larger scales by a scale invariant process. This continuous deformation may be characterised by defining an intermediate fractal dimension  $D_{el}$  that we call an elliptical dimension. We show both theoretically and empirically that  $D_{el} = 23/9 \sim 2.56$ . Atmospheric structures are therefore never “flat” ( $D_{el} = 2$ ), nor isotropic ( $D_{el} = 3$ ), but always display aspects of both. Larger structures are on the average, more stratified as a result of a well-defined stochastic process.

In this scheme, the intermittency must be quite strong in order to produce the well-known meteorological “animals” such as storms, fronts, etc. We propose that intermittency is characterised by hyperbolic probability distributions with exponents  $\alpha$ . This possibility was first suggested by Mandelbrot (1974a) for the rate of turbulent energy transfer ( $\varepsilon$ ). We investigate intermittency for the wind ( $v$ ) the potential temperature ( $\theta$ ), and  $\varepsilon$  in terms of this hyperbolic intermittency. In particular, we find  $\alpha_v = 5$ ,  $\alpha_\varepsilon = 5/3$ ,  $\alpha_{\ln \theta} = 10/3$ , which show that the fifth moment of  $v$ , the second moment of  $\varepsilon$ , and the fourth moment of  $\ln \theta$  diverge.

We re-examine Mandelbrot’s model of intermittency and generalize it for anisotropic turbulence. We stress that it cannot be characterised by a single parameter, the dimension of the support of turbulence: we show that, except in a trivial case, several fractal dimensions intervene. We exhibit, for instance, a two-parameter model depending on the fractal dimension of the very active regions.

Finally, we sketch a direction for future work to assess this 23/9 dimensional scheme of atmospheric dynamics with hyperbolic intermittency.

## 1. Introduction

The classical approach to the analysis of atmospheric motions (e.g. Monin 1972), considers the large scale as two-dimensional, and the small scale as three-dimensional. In this view, a transition, which for obvious reasons we call a “dimensional transition”, is expected to occur in the meso-scale, possibly in association with a “meso-scale gap” (Van der Hoven 1957). This scheme favours the simplistic idea that at planetary scales the atmosphere looks like a thin envelope, whereas at human scales, it looks more like an isotropic volume.

A dimensional transition, if it were to occur, would be likely to have fairly drastic consequences because of the significant qualitative difference of turbulence in two and three dimensions (Fjortoft 1953; Kraichnan 1967; Batchelor 1969). Two-dimensional turbulence is very special since in the vorticity equation, there is no source term, and it is therefore conserved. Mathematically, it introduces a second quadratic invariant (the enstrophy, or mean square vorticity), and physically, the all important stretching and folding of vortex tubes cannot occur. Since the 50’s, there has been a wide debate over the effective dimension of atmospheric turbulence, due in particular to the extension of two-dimensional results to the case of quasi-geostrophy (Charney 1971; Herring 1980).

Although, a dimensional transition should be readily observable, experiments over the last 15 years have failed to detect it (see Table 1 for a summary). Attempts to explain this fact without abandoning the idea of a transition (e.g. Gage 1979), require a series of ad hoc hypotheses about the distribution of energy sources and sinks, hypotheses which are probably no longer tenable.

At the same time, Vinnechenko (1969), Endlich et al. (1969) questioned the existence of a break in the spectrum and the relevance of the notion of the meso-scale. Lovejoy (1982) turned this argument around by suggesting that the failure of experiments to find clear evidence of a length scale characterising a transition, was in itself positive evidence of the scaling (fractal) nature of the atmosphere – at least up to distances of  $\sim 1000$  km. The primary purpose of this paper is to extend this idea considerably: based on radiosonde, aircraft, and other data, we propose that the atmosphere is never isotropic (three dimensional), nor completely “flat” (two dimensional), but is anisotropic and scaling<sup>1</sup> throughout, a fact that can be characterised by the intermediate “elliptical” dimension  $D_{el} = 23/9 \sim 2.56$ . The expression “elliptical” may be understood by the fact that vertical cross-sections of large eddies appear as flattened ellipses.

This dimension is greater than the dimension of the support of turbulence ( $D_s$ ) which must take into account the effects of intermittency. This question will also be investigated.

In Section 2, we examine in some detail the theoretical and empirical reasons for doubting the existence of a dimensional transition, and for suspecting that the atmosphere is characterised by uniform scaling laws. In Section 3, the special role of the vertical structure is examined: in particular the significance of the buoyancy force and the vertical shear. Section 4 is devoted to a review of the empirical data used to establish the vertical scaling and intermittency of the wind, potential temperature and rate of turbulent energy transfer fields. The vertical scaling is found to be quite different from that known to hold in the horizontal, (and in the case of the velocity field, can be obtained by a dimensional argument similar to that advanced by Obukhov (1959) and Bogliano (1959)). Another significant result is the very large tails of these probability distributions, a feature associated with intermittency.

An important result is that the fifth (and higher) moments of the velocity field diverge. In Section 5, we present a preliminary discussion, and finally, in Section 6 we develop a central idea of the paper: that the anisotropic scaling nature of the atmosphere may be characterised by an elliptical dimension. Section 7 examines an immediate consequence: stochastic stratification. Section 8 is devoted to a preliminary analysis of hyperbolic intermittency: extreme fluctuations are ruled by algebraic fall-off of the probability distributions (a precise definition is provided in Section 3). Such a behaviour has been proposed for the rate of turbulent energy transfer by Mandelbrot (1974 a) and observed for the flux of rain by Lovejoy (1981). As pointed out by Mandelbrot (1974 a) this is connected with the question of the dimension of the support of turbulence, but (as we stress) is not fully characterised by it. Indeed we show that, except in a trivial case, several dimensions intervene. For instance, a two-parameter model is discussed, which depends mainly on the fractal dimension of the very active regions.

Section 9 contains some theoretical developments of anisotropic scaling and hyperbolic intermittency and combines them in order to study relations between various fields, phenomenology and renormalisation procedures. Section 10 summarizes our results and indicates a prognosis for the future.

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1 Wherever it appears, the term “scaling” (or scale invariance) is used in the sense commonly accepted in physics, the phenomenon does not depend on the scale of observation. The most familiar consequence of scaling is undoubtedly the power law spectra of scaling objects (i.e.  $E(k) \sim k^{-\beta}$ , where  $k$  is the wavenumber,  $E$  the energy,  $\beta$  the spectral exponent). A precise definition of scaling is provided in Section 3

**Table 1.** Empirical evidence for horizontal scaling

Investigators	Type of data	Analysis technique	Range of scales studied
Richardson (1926)	Smoke, motes, volcanic ash, balloons	Atmospheric diffusion	1 cm to 1000 km
Stewart and Townsend (1951)	hot-wire anemometry	Energy spectra	10 cm to 10 m
Vinnichencko (1969)	Records of wind data from 1 sec to 5 years converted to equivalent distances	Energy spectra	10 m to 1500 km
Morel and Larchevêque (1974)	Constant altitude balloons	Atmospheric diffusion	100 km to 1000 km
MacPherson and Isaac (1977)	Wind data from flights through clouds	Energy spectra	10 m to 2.5 km
Gage (1979)	VHF Doppler radar wind data (with time to space conversion)	Structure function	4 km to 400 km
Brown and Robinson (1979)	radiosonde	correlation function	500–2500 km
Gilet et al. (1980)	Doppler radar wind data in rain	Energy spectra	1.6 km to 25 km
Lovejoy (1982)	Satellite cloud and radar rain pictures	Area-perimeter relation	1 km to 1000 km
Lilly and Peterson (1983)	Aircraft wind data	Energy spectra	1 km to 250 km
Nostrum and Gage (1983)	Aircraft wind data	Energy spectra	1 km to 700 km
Boer and Shepherd (1983)	radiosonde	Energy spectra	1000–4000 km

## 2. Why a Dimensional Transition is Unlikely

### 2.1 Empirical Reasons

There is a growing body of evidence, going back to Richardson's (1926) famous atmospheric diffusion experiment, which is consistent with a uniform scaling from at least several centimetres (the dissipation length) to perhaps 4000 km. (Boer and Sheperd (1983) find this for planetary wavenumbers greater than 8–10). Table 1 summarizes different experimental results. Note that in recent years, all the results over this range are consistent with a  $k^{-5/3}$  energy spectrum- with the possible exception of Morel and Larchevêque (1974). They analysed the dispersion of constant altitude balloons over length scales in the region of 100 km to 1000 km. Although they found no evidence of a characteristic length scale in this region, their data was compatible with the existence of an exponential diffusion law which is normally associated with two-dimensional turbulence (and by implication a  $k^{-3}$  spec-

trum). A question outside the scope of the present paper, is to show whether or not this diffusion law is compatible with the 2.56 dimensional model proposed here.

The failure of so many investigators to find a characteristic length characterising a transition is significant because most of these experiments were designed to detect such a length scale.

## 2.2 Theoretical Reasons

Scaling is now a widespread notion in physics. It is a basic tool in the theory of the Renormalization Group (e.g. Wilson 1975) and in the construction of fractal objects (Mandelbrot 1975, 1982). The prevalence of scaling is directly connected with the fact that many non-linear equations do not introduce a characteristic length, and thus admit scaling, hence fractal, solutions. In the case of gravitation, Mandelbrot (1982) has traced this idea back to Laplace.

The prototypical example of such an equation is the Navier-Stokes equation – which is also the basic equation of fluid dynamics, and hence of meteorology. In detail it is far from understood and is still the object of numerous investigations, both theoretical and numerical. Recently the existence of scaling solutions has been verified by direct numerical simulation even in three-dimensional turbulence: Brachet et al. (1983) have succeeded in exhibiting an inertial range in pseudo-256<sup>3</sup> simulations of the Taylor-Green vortex. Chorin (1981) has even given strong evidence of the scaling behaviour of the set of singularities, i.e. that the intermittency is also scaling, this is necessary if the solution is to be truly scaling. This will be discussed in Section 8 devoted to intermittency.

The scaling is expected to hold down from some large “outer” scale – the exact value of which depends on the energy injection and other details – to a small dissipation scale. This dissipation length is necessary to prevent divergence of the enstrophy. In the atmosphere, there is general agreement that this occurs on centimeter scales. As for the “outer” length, Richardson (1926) doubted it “occurred” within the atmosphere, and as Table 1 indicates, it may be at least 4000 km.

The dissipation length and the “outer” scale are the basic length scales. Others length scales are based on various statistics of the flow, e.g. the “mean” velocity, and are therefore not fundamental in the same sense.

In principle, this scaling could be broken by the boundary conditions if they had a well-defined length scale. One kind of boundary condition often used by modelers to break the scaling is orographic forcing or topography. However, Mandelbrot (1977, 1982) has found evidence that the topography is scaling up to planetary scales. (The recent theoretical studies by Holloway and Hendershot (1974) and Herring (1977) of the possible effects of a spectral peak in a random topography does not address the question of whether or not the topography does in fact have such a peak.)

Another way in which the scaling could be broken is if the “forcing” of the atmosphere (the sources and sinks of diabatic heating) had characteristic lengths. In the case of diabatic heating via solar radiation, Gautier (1982, personal communication) has analysed satellite data and found a scaling behaviour for distances of up to at least hundreds of kilometers. Diabatic heating via the release of latent heat, is also unlikely to create structures with well defined size, as was shown by an analysis by Lovejoy (1982) of rain and cloud areas from 1 km<sup>2</sup> to 1.2 × 10<sup>6</sup> km<sup>2</sup>. Finally, Schertzer and Simonin (1982) and especially Simonin (1982) have shown that diabatic cooling due to radiative dissipation is likely to be scaling – at least up to hundreds of kilometers.

Because of the non-linear couplings between the different meteorological fields, the existence of a length scale in one, is likely to manifest itself in the others. It is therefore likely

that over a given range, that all fields are either scaling or not scaling. The results above are all consistent with this.

In the light of these results, we consider as unlikely, the idea proposed by Gage (1979) that severe storms with well defined sizes could account for the observed scaling by providing an energy source for a reverse two-dimensional energy cascade. In fact, the existence of a reverse cascade of this sort for the energy, is unlikely to be accompanied by a reverse cascade of a passive scalar. Indeed, contrary to earlier conjectures (e.g. Lesieur et al. 1981), we believe that such a cascade could not exist at all. This may best be understood by recalling that Kraichnan's (1967) theoretical derivation of the reverse energy cascade in two dimensions relies on the existence of two quadratic invariants for the velocity, (see also Fox and Orszag (1973), Basdevant and Sadourny (1975), Carnevale et al. 1981).

This is convincingly established by the use of statistical mechanical arguments for non-dissipative systems. The situation is entirely different in the case of passive scalars, since in any dimension there exists only one quadratic invariant quantity. In this case, the same arguments show that an inverse cascade is impossible, a conclusion supported by numerical calculations on the test field model (Lesieur and Herring 1984). On this basis, we would expect a reverse energy cascade to be accompanied by a peak in the spectrum of passive scalar quantities at a wavenumber corresponding to the scale of the energy injection, a phenomenon which is not observed anywhere near the  $\sim 100$  km scale proposed by Gage (1979) (the clearest evidence for this is the Morel and Larchevêque (1974) experiment).

We feel that the effort to locate sources and sinks at specific length scales is often misplaced, because the Kolmogorov scaling may be expected to hold under quite general conditions, not only for a true inertial range (i.e. a range without a source). The only necessary assumption is that of a constant flux of energy through the range. This could be obtained either by the usual hypothesis which locates the forcing at large scales, or by forcing which is the same at all scales. Indeed, the only theoretical derivation (based on renormalisation group techniques) of the Kolmogorov scaling, has been obtained under the latter hypothesis (De Dominicis and Martin 1979 and Martin and De Dominicis 1978).

Another theoretical reason which makes it unlikely that the atmosphere has a well defined dimensional transition, lies in our current understanding of the nature of intermittency (for example Batchelor 1969 or Curry et al. 1982). Intermittency is now viewed as the frequent switch from quiescence to chaos, and this makes the existence of a well-defined transition doubtful.

### 3. The Vertical Structure

Perhaps the most serious objection to the hypothesis of a scaling behaviour of atmospheric motions arises from the special role of the vertical axis. Indeed, there has been a deluge of papers based on non-scaling techniques which implicitly reject *a priori* any possibility of vertical scaling (for example, the "second order models" and "one point closures"). In what follows, it will be apparent that this rejection has had unfortunate consequences.

The vertical direction plays a key role for the following reasons:

- i) The gravity field defines a direction at every point.
- ii) The atmosphere is globally stratified.
- iii) It has a well defined thickness (exponential decrease of the mean pressure).
- iv) The fundamental sources of disturbances are the vertical shear and the buoyancy force (e.g. the Kelvin-Helmoltz instability).



In the following, we examine the possibility that the atmosphere is in fact scaling in the vertical as well as in the horizontal direction. To do so, we examine empirically the wind and temperature fields, attempting to capture two basic and conceptually distinct properties of these fields.

*Scaling:* The scaling provides the relationship between fluctuations of a field  $X$  ( $\Delta X(\Delta z) = X(z_0 + \Delta z) - X(z_0)$ , is the fluctuation between  $z_0$  and  $z_0 + \Delta z$ ), for large scales  $\lambda \Delta z$  ( $\lambda \gg 1$ ) to the small scales ( $\Delta z$ ) by:

$$\Delta X(\lambda \Delta z) \stackrel{d}{=} \lambda^H \Delta X(\Delta z)$$

where  $H$  is the scaling parameter and “ $\stackrel{d}{=}$ ” means equality in probability distributions ( $X \stackrel{d}{=} Y$  means  $\Pr(X > q) = \Pr(Y > q)$  for all  $q$  and  $\Pr$  denotes probability). Note that the exponent of the corresponding power spectrum ( $-\beta$ ) is related to  $H$  by  $\beta = 2H + 1$ , in the case of finite variance.

### *Hyperbolic Intermittency*

Intermittency is directly connected with the probability law. One is particularly interested in the tail of this law, since it controls the relative frequency of extreme (intermittent) behaviour. For example, if the distribution has an algebraic fall-off at large fluctuations ( $\Delta X$ ), then the degree of intermittency can be characterised by the exponent  $\alpha$  (the hyperbolicity) defined as follows:

$$\Pr(\Delta X' > \Delta X) \propto \Delta X^{-\alpha}$$

Random variables are distinguished from non-random ones by primes or by the use of upper case symbols for random, and lower case for non-random variables. The above equation therefore reads “the probability of a random fluctuation  $\Delta X'$  exceeding a fixed  $\Delta X$  is proportional to  $\Delta X^{-\alpha}$ ”.

Behaviour of this sort was first predicted by Mandelbrot (1974 a), who also showed that it was related to the problem of the dimension of the support of turbulence  $D_s$ , a question, we pursue in Section 8 in the context of anisotropy. Hyperbolic distributions have been invoked in other fields of physics (the “Holtmark distribution”, Feller (1971), or see Mandelbrot (1982) for other examples). Note that all moments of order  $\alpha$  or higher diverge, a fact that has important consequences<sup>2</sup>. Levy (1937) and Feller (1971) are standard texts, in the case  $\alpha < 2$ , (the Levy-stable laws which form a convolution semi-group). We term this kind of intermittency “hyperbolic intermittency”. In Section 9 we will discuss the connection with previous experimental results of this kind obtained for the rain field (Lovejoy 1981).

We shall primarily be interested in the vertical fluctuations of the horizontal velocity field ( $\Delta v$ ) and in the buoyancy force per unit mass acting across a layer of thickness  $dz$ :  $df = gd \ln \theta$ , where  $\theta$  is the potential temperature, and  $g$  the acceleration of gravity. These quantities are related to two fundamental frequencies: that of the vertical shear ( $s$ ) and the Brunt-Vaisala frequency ( $n$ ):

$$s = dv/dz \quad \text{and} \quad n^2 = gd \ln \theta/dz = df/dz.$$

The ratio of their squares defines the dimensionless Richardson number:

$$\text{Ri} = n^2/s^2.$$

<sup>2</sup> E.g. the experimental estimation increases without limit with the sample size (cf. Schertzer and Lovejoy 1983 b)

The shear frequency characterises the dynamical processes, and the Brunt-Vaisala frequency, the stability (and gravity waves). The dominant process has the highest frequency. Note that the above are defined locally (in time and space) and hence are often called "turbulent" quantities. Unfortunately, one is used to considering them in averaged forms.

To determine their scaling regime, Fourier analysis could be used. Here, we analyse directly the scaling of the probability law by measuring quantities across atmospheric layers of thickness  $\Delta z$ . This has the advantage that it enables the scaling parameters ( $H$ 's) and the hyperbolicities ( $\alpha$ 's) to be obtained simultaneously. We therefore define:

$$s^2(\Delta z) = \Delta v^2(\Delta z)/\Delta z^2; \quad n^2(\Delta z) = g \Delta \ln |\theta|/\Delta z$$

$$\text{Ri}(\Delta z) = n^2(\Delta z)/s^2(\Delta z).$$

## 4. Data Analysis

### 4.1 Radiosonde Analysis

$s$ ,  $n$ ,  $\text{Ri}$  were evaluated from the high resolution radiosonde data obtained in the 1975 experiment in Landes, France. The quantities  $\theta$ ,  $v$  and the humidity were obtained every second ( $\sim 3$  m in the vertical) and processed to yield low noise data every 5 s ( $\sim 15$ – $20$  m) (see Tardieu (1979) for more details). The velocity values were somewhat noisier than the  $\theta$  values and were therefore smoothed, yielding low noise estimates every 50 m. All estimates of  $n$ ,  $s$ ,  $\text{Ri}$  were thus made over layers at least 50 m thick, and from the ground up to a somewhat arbitrary height of 6 km. Since the number of points in each layer varied somewhat (averaging  $\sim 3$  for each 50 m layer), mean  $\theta$ ,  $v$  were computed and  $\Delta \ln \theta$ ,  $\Delta v^2$  were then determined. These are required to calculate  $s$  and  $n$ . The data examined are from 80 consecutive soundings taken at 3-hour intervals at Landes.

From the log-log plots shown in Figs. 1, 2, 3 it can easily be verified that the probability distributions of  $\Delta v$ ,  $\Delta f$  and  $\text{Ri}$  exhibit both scaling and hyperbolic behaviour. Perhaps the easiest way to see this is to recall that for hyperbolic distributions:

$$\text{Pr}(\Delta X' > \Delta X) = (\Delta X/\Delta X^*)^{-\alpha} \quad \text{for large } \Delta X$$

$\Delta X^*$  is the "width" of the distribution, or the amplitude of the fluctuations. Scaling implies then that the width grows with the separation as:

$$\Delta X^*(\Delta z) = (\Delta z/\Delta z_0)^H \Delta X^*(\Delta z_0) \quad \Delta z_0 \text{ is a constant.}$$

This is shown on the graph by the constant shift  $H \log 2$  for each doubling of the separation  $\Delta z$ . The value  $(-\alpha)$  is the slope of the straight line asymptote.

We obtain:

$$H_v = 3/5 \quad \alpha_v = 5$$

$$H_{\ln \theta} = 9/10 \quad \alpha_{\ln \theta} = 10/3$$

$$H_{\text{Ri}} = 1 \quad \alpha_{\text{Ri}} = 1.$$

The  $H$ 's and  $\alpha$ 's are given rational expressions, since, as explained below, they can often be deduced by dimensional considerations.

In a humid atmosphere, the buoyancy is determined by the wet bulb potential temperature (yielding  $n_w$ ). We also plotted the distribution of  $n_w$  and found it to be virtually



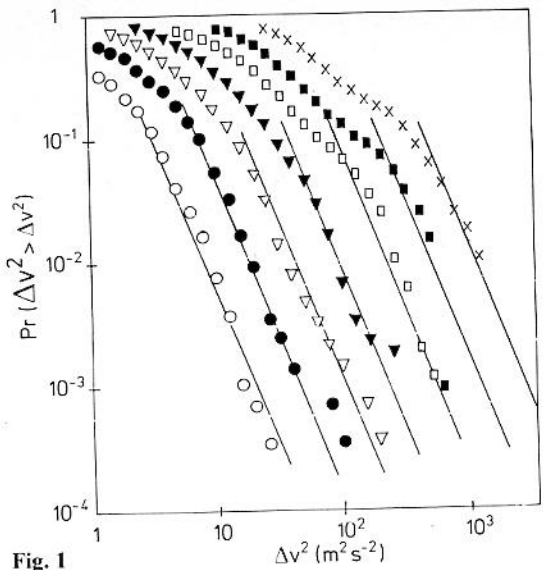


Fig. 1

Fig. 1. The probability distribution of fluctuations in the quantity  $\Delta v^2(\Delta z)$  where  $v$  is the horizontal velocity for different layers as follows:

- :  $\Delta z = 50$  m,
- :  $\Delta z = 100$  m,
- ▽:  $\Delta z = 200$  m,
- ▼:  $\Delta z = 400$  m,
- :  $\Delta z = 800$  m,
- :  $\Delta z = 1600$  m,
- ×:  $\Delta z = 3200$  m.

The straight lines follow the equations in the text

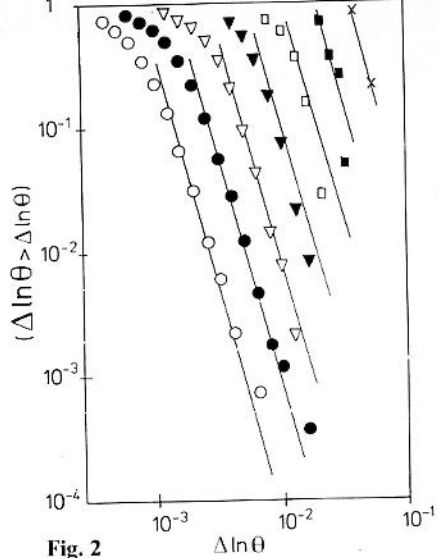


Fig. 2

Fig. 2. The probability distribution of fluctuations in the quantity  $\Delta \ln \theta$ ,  $\theta$  being the potential temperature. The symbols are the same as in Fig. 1. The straight lines follow the equation given in the text

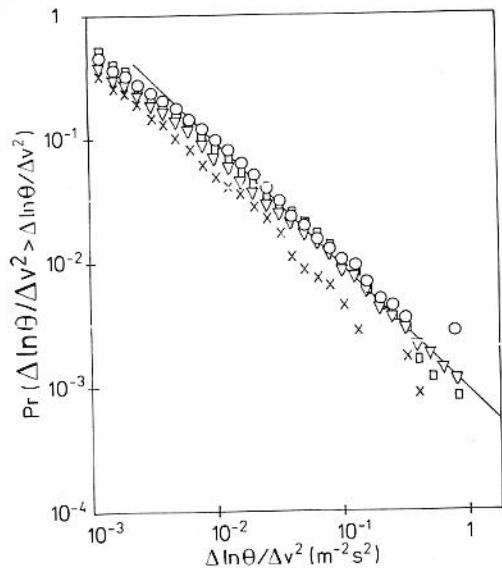


Fig. 3. The probability distribution of fluctuations in the quantity  $\Delta \ln \theta / \Delta v^2 = Ri \cdot \Delta z^{-1}$ ,  $Ri$  being the Richardson number. The symbols are the same as in the previous figures. The curves for 100, 400, 1600 m have been suppressed for clarity. (The 50 m curve probably falls below the others because of the effect of noise in the  $\Delta v^2$  measurement in this value). The straight lines follow the equations given in the text.

indistinguishable from that of  $n$ . This is probably because Landes was quite dry – the relative humidity rarely exceeded 50%.

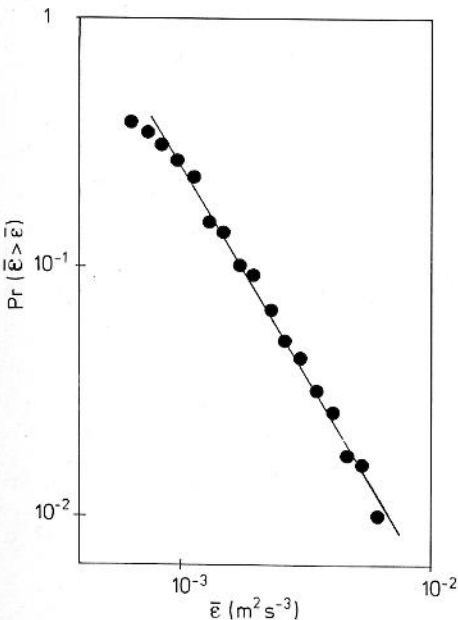
These exponents are quite robust i.e. the difference between the individual sounding is the width of the distributions, not the exponents, this is supported by the comparison of four subsamples of 20 soundings each. Although the most unstable samples had a width of the Ri distribution twice as small as the most stable sample, the exponents were unaffected (this was also true of  $s$  and  $n$ ). These results are fully consistent with the pooled results shown in Figs. 1, 2, 3 if it is remembered that pooling data from hyperbolic distributions with similar exponents only changes the width of the distribution. Furthermore, comparisons of data from above and below 2 km show that the hyperbolic form is not due to the pooling of data from different altitudes.

#### 4.2 Merceret's (1976) Aircraft Data

Another quantity of interest is the rate of turbulent energy transfer ( $\bar{\epsilon}$ ). This is the fundamental dynamical quantity in the turbulent cascade of energy from large to small scales. In this case, we obtained only a probability distribution by replottting Merceret's aircraft data. The result, which is shown in Fig. 4 can be represented by the following formula:

$$\text{Pr}(\bar{\epsilon}' > \bar{\epsilon}) \sim \bar{\epsilon}^{-\alpha_\epsilon}; \quad \alpha_\epsilon \sim 5/3.$$

Merceret obtained his value of  $\bar{\epsilon}$  by calculating the spectrum of horizontal wind fluctuations every second ( $\sim 100$  m – actually the 1 s spectrum is the average of literally thousands of spectra taken on millisecond time scales), from a research aircraft. Note that this 5/3 exponent is conceptually quite different from the famous Kolmogorov 5/3 exponent. Mandelbrot (1974a) points out that the value of  $\alpha_\epsilon$  depends on the dimension of the region over which it is averaged. We assume this average to have been taken over a straight line of dimension 1. In the following, we use the bar notation to indicate a spatial average and angular brackets  $\langle \cdot \rangle$  for ensemble averages.



**Fig. 4.** The probability distribution of  $\bar{\epsilon}$  as replotted from Merceret (1976) Fig. 14. (These values are averages over 100 m in the horizontal, which are taken to be made over a line, thus  $D_A = 1$ . Linear regression yielded a correlation coefficient of 0.998, and a slope of  $-1.691$ , which implies  $\alpha = 1.691 \sim 5/3$ )

## 5. Preliminary Discussion

### 5.1 Obukhov (1959) and Bogliano (1959) Predict $H_v = 3/5$

One of the most significant implications of the present study is that  $H_v = 3/5$ . A very similar result was predicted almost 25 years ago for the directionally averaged spectrum in the so-called "buoyancy subrange", by Obukhov (1959), and independently by Bogliano (1959). To our knowledge, Endlich et al. (1969) were the first to test this prediction empirically by performing spectral analyses of Jimsphere vertical soundings of the horizontal wind field. After considerable interpolation of the data, and smoothing of the power spectrum (both of which increase the spectral exponent  $\beta$ ), they found  $\beta = 5/2$  and no evidence of a characteristic vertical length scales up to distances of at least 10 km. However, Adelfang (1971) applied a structure function analysis to raw data obtained from another Jimsphere experiment (up to 16 km) and obtained  $H_v = 0.60$ , implying  $\beta = 2H_v + 1 = 2.2$ , in agreement with our results. He also found no evidence for a vertical length scale.

We therefore believe that non-linear data processing may account for the discrepancy between the result of Endlich et al. (1969) and that found here. The same comment applies to Van Zandt's (1982) analysis of the Adelfang data (he found  $\beta = 2.4$ ). It may be noted that the method of analysis adopted here is statistically quite robust since it depends only on the scaling of the asymptotic part of the probability distribution which is relatively noise free.

To derive  $H_v = 3/5$  Obukhov and Bogliano considered the fundamental parameter determining the vertical structure to be the flux of temperature variance (in analogy with the flux of kinetic energy  $\bar{\epsilon}$  in Kolmogorov's derivation of  $H_h = 1/3$ ). The only other quantity of interest is the "buoyancy parameter"  $g/\bar{T}$  (where  $\bar{T}$  is the average temperature), a quantity that characterises the strength of the coupling between the temperature fluctuations and the dynamics in the Boussinesq approximation. The result  $H_v = 3/5$  follows immediately from dimensional considerations.

Instead of applying this derivation to fluctuations averaged over all directions, we suppose that it applies to fluctuations along the vertical. This derivation can even be slightly improved by considering directly the physically meaningful quantity  $\bar{\phi}$  the flux of buoyancy force variance (this has the advantage that it does not depend on the Boussinesq approximation):

$$\bar{\phi}(\Delta z) = \tau^{-1}(\Delta z) \Delta f^2(\Delta z)$$

where  $\tau(\Delta z)$  is a characteristic time for the transfer process. Dimensional analysis yields:

$$\Delta v(\Delta z) \stackrel{d}{=} \bar{\phi}(\Delta z)^{1/5} \Delta z^{3/5}. \quad (1)$$

While this scaling holds in the vertical, the quite different Kolmogorov scaling holds in the horizontal:

$$\Delta v(\Delta x) \stackrel{d}{=} \bar{\epsilon}(\Delta x)^{1/3} \Delta x^{1/3}. \quad (2)$$

These formulae show that characteristic lengths arise neither in the horizontal, nor in the vertical directions.

The fact that the  $H$ 's can be deduced from dimensional considerations justifies the use of rational approximations to the exponents.

See Section 9 for similar arguments in the case of the  $\alpha$ 's.

## 5.2 Persistence and Coherence of Vertical Structures

$H_v$  and  $H_h$  are separated by the value  $1/2$  which is the value of  $H$  that would obtain if fluctuations were uncorrelated (cf. the coordinate of a Brownian particle).  $H < 1/2$  therefore implies that the fluctuations cancel more quickly than in the independent case, a phenomenon called "antipersistence" by Mandelbrot and Wallis (1968). Similarly,  $H > 1/2$  implies that fluctuations tend to add together in the same direction, "persistent" behaviour.

We therefore interpret  $H_v = 3/5$  as reflecting the fact that  $v$  tends to behave in a coherent way over atmospheric layers, whereas, it tends to be anti-coherent in the horizontal.

## 5.3 The Turbulent Richardson Number

From Fig. 3 we obtained  $\text{Pr}(\text{Ri}' > \text{Ri}) \propto \text{Ri}^{-1} \Delta z$ . This distribution has such large fluctuations ( $\alpha_{\text{Ri}} = 1$ ) that even the mean may not converge (if  $\alpha_{\text{Ri}} \leq 1$ ). The large fluctuations in this quantity have long been recognised, and a series of modified Richardson numbers (based on the ratio of two statistics) have been introduced to reduce the fluctuations to manageable proportions (e.g. the flux Richardson number  $R_f$ ).

Distributions of this type (e.g. Cauchy distributions) arise naturally when the ratio of two independent random variables is taken (on condition that the denominator has a non-zero probability density at the origin) – e.g. the Student distribution. In this case, the correlation between  $s$  and  $n$  was found to be  $0.048 \pm 0.018$ , suggesting that  $s$ ,  $n$  are weakly dependent, thus explaining  $\alpha_{\text{Ri}} \sim 1$  (note that both  $\text{Ri}$  and  $1/\text{Ri}$  were found to have  $\alpha$ 's approximately equal to one). This hypothesis was confirmed by randomising the experimental pairs  $s$ ,  $n$  and calculating the new distribution of the ratio  $n^2/s^2$ . The resulting distribution was indeed the same as the unrandomised one, supporting the hypothesis. The value of  $\text{Ri}$  is expected to be related to the onset of turbulence, and therefore its erratic nature is directly related to the phenomenon of intermittency.

This fact calls for cautious use of methods for solving the equations via limited expansions in  $\text{Ri}^{-1}$  ( $= /\text{Fr}^2$ ,  $\text{Fr}$  being the Froude number). In particular, the proposal in Riley et al. (1981) to use this sort of expansion to establish a separation between a 2-D horizontal turbulent regime and a wave dominated regime must be placed in a stochastic context. The same applies to the idea proposed by Lilly (1983), of "wave collapse". There is no doubt that some meteorological events correspond to  $\text{Ri}^{-1} \ll 1$ , so that limited expansions should shed some light on the mechanisms involved. However, the statistics of  $\text{Ri}^{-1}$  indicated here imply that these events occur in a very erratic manner. More work is therefore needed to understand this dual nature of  $\text{Ri}^{-1}$ , but the present results show that there is no way to separate waves and turbulence in the atmosphere, except perhaps in special situations.

## 5.4 The Divergence of the Fifth Moment of the Velocity

Assigning the value  $\alpha_v = 5$  which implies the divergence of the fifth and higher moments of the velocity field, requires some justification.

Assuming Eqs. (1) and (2), and assuming that  $\alpha_v = 5$  holds in both vertical and horizontal directions leads to two conclusions. First,  $\alpha_\phi = 1$  (since  $\overline{\phi^d} = \Delta v^5 (\Delta z) / \Delta z^3$ ) and second  $\alpha_\epsilon = 5/3$  (since  $\epsilon = \Delta v^3 (\Delta x) / \Delta x$ ).

Although  $\alpha_\phi = 1$  has not been independently verified (it would require an understanding of the correct  $\tau(\Delta z)$ ), Fig. 4 independently shows that  $\alpha_\epsilon \sim 5/3$ .

The fact that radiosonde data of velocity fluctuations in the vertical, and aircraft observations of horizontal energy transfer give such close agreement is strong evidence in favour of the correctness of both exponents. Other evidence for divergence of high moments of the

velocity come from the wind tunnel experiments of Anselmet et al. (1983). Schertzer and Lovejoy (1983b) argue that these results are consistent with the universality of  $\alpha_v \sim 5$  in fully developed turbulence.

The divergence of the higher moments of  $\bar{\epsilon}$  was first proposed by Mandelbrot (1974a) who suggested  $\alpha_\epsilon \sim 4$  on the basis of the evidence then available.

## 6. Dimension and Scaling Laws

### 6.1 Anisotropic Scaling and the Elliptical Dimension $D_{el} = 23/9$

We have examined the theoretical and empirical evidence in favour of the horizontal and vertical scaling of meteorological quantities. We now turn to examining some of the consequences of these findings.

Objects which scale in the same way in all directions are called self-similar fractals because the large scale can be simply viewed as a magnification of the small scale. In the atmosphere, we have argued that scaling, although present in all directions, and over a wide range of lengths, is quite different in the vertical and horizontal. Large-scale structures can no longer be simply regarded as large-scale copies of smaller ones. In order to transform from small to large scales, in addition to magnification, we must also stretch.

To make this magnification and stretching procedure precise, we express the horizontal and vertical scaling of the horizontal velocity fluctuation  $\Delta v$  as follows:

$$\Delta v(\lambda \Delta x) \stackrel{d}{=} \lambda^{H_h} \Delta v(\Delta x)$$

$$\Delta v(\lambda \Delta y) \stackrel{d}{=} \lambda^{H_h} \Delta v(\Delta y)$$

$$\Delta v(\lambda \Delta z) \stackrel{d}{=} \lambda^{H_v} \Delta v(\Delta z).$$

These equations describe how  $\Delta v$  varies with different scales of ratio  $\lambda$ . In order to examine how actual structures vary, we rewrite these equations in the following form:

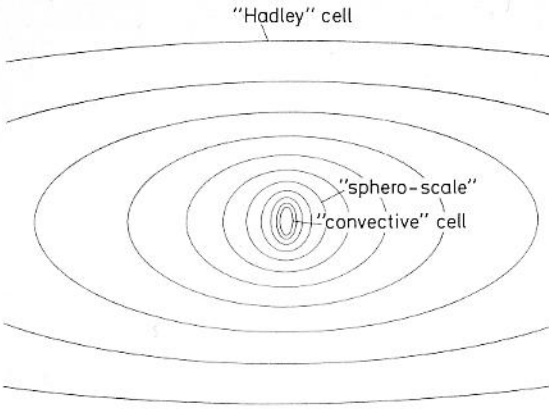
$$\Delta v(\underline{T}_\lambda \Delta r) \stackrel{d}{=} \lambda^{H_h} \Delta v(\Delta r).$$

where  $\underline{T}_\lambda$  is written as:

$$\underline{T}_\lambda = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{H_z} \end{bmatrix} \quad \text{with } H_z = H_h/H_v.$$

The action of this matrix may be seen to consist of an overall magnification by  $\lambda$  coupled with a stretching in the  $z$  direction. Note that  $\underline{T}$  transforms the *statistical* properties of the atmosphere, from one scale to another because the equalities in the above apply to probability distributions. The existence of  $\underline{T}$  permits the introduction of an elliptical geometry (see Schertzer et al. (1984)), necessary because of the different horizontal and vertical scaling.

Figure 5 shows how a small vertically oriented average eddy is transformed at larger scales. The magnification and stretching process transforms the vertically oriented "convective" type eddy into a large horizontally oriented "Hadley" type eddy. The transformation increases the volume of the eddy by the factor  $\lambda \cdot \lambda \cdot \lambda^{H_z} = \lambda^{D_{el}}$  with  $D_{el} = 2 + H_z =$



**Fig. 5.** This figure shows the shape of the vertical cross-section of averaged eddies at different scales. Note that at small scales, the eddies are vertically oriented (“convective” type), at intermediate scales, they are circular (the “sphero-scale”), and at very large scales, they are again elliptical, but elongated in the horizontal rather than the vertical (“Hadley” type)

$23/9 \sim 2.56$ . In an isotropic three-dimensional atmosphere,  $H_z = 1 \Rightarrow D_{e1} = 3$ , and in the isotropic two-dimensional case,  $H_z = 0 \Rightarrow D_{e1} = 2$ . It is therefore natural to regard  $D_{e1}$  as the fractal dimension of this anisotropic space.

Writing the above relationship in the form:  $\text{Det } \tilde{T} = \lambda^{D_{e1}}$ , we are led to the following definition of  $D_{e1}$ :

$$D_{e1} = \text{Tr}(d\tilde{T}/d\lambda)_{\lambda=1}$$

which is the elementary mapping between averaged eddies of different scales  $d\lambda$ .

The number  $n(l)$  of eddies of horizontal scale  $l$  may now be written:

$$N(l) \propto l^{-D_{e1}}$$

$D_{e1}$  is not the same as the oft cited dimension of the support of turbulence ( $D_s$ ) which expresses the fact that because of intermittency, eddies do not fill all the space available to them. This point will be discussed further in Section 8.

An example of an anisotropic fractal is the graph of the coordinate of a particle subjected to Brownian motion ( $X(t)$  as a function of time ( $t$ )). Applying the above definition, it is easy to see that  $D_{e1} = 1.5$  which is the same as the fractal dimension suggested by Mandelbrot (recall that in this case  $X$  is scaling, with  $H = 0.5$ ). Mandelbrot calls this kind of fractal “self-affine”. In Schertzer et al. (1984), the term “elliptical” fractals is used, to stress the elliptic metric involved. In any case, this dimension does not correspond to a isotropic Hausdorff dimension, as fractal dimensions usually do.

## 6.2. The “Sphero-Scale”

The distinction between isotropy and anisotropy is profound. Consider a stretched and folded vortex tube of scale  $l$  in the anisotropic case. A horizontal cross-section will have an area proportional to  $l^2$  whereas a vertical cross-section will have an area proportional to  $l^{1+H_z}$  and thus their ratio, which gives a quantitative measure of the “verticalness” of the vortex is proportional to  $l^{1-H_z}$ . There therefore exists a scale at which the ratio is 1, and the vortex appears isotropic. The nature of the sphero-scale may be understood by considering it to be defined as the scale for which the amplitude of  $\Delta v$  (as measured by  $\langle \Delta v^2 \rangle^{1/2}$  for example), is the same in both horizontal and vertical directions, (for the same length  $\Delta z = \Delta x$ ). This scale will depend on the relative amplitudes of  $\bar{\epsilon}$  and  $\bar{\phi}$  both of which show



large fluctuations. The size of the sphero-scale may therefore vary in an extremely erratic manner.

Another feature of the sphero-scale is that its effect is very gentle in as much as it may be viewed as the point beyond which one power law ( $l$ ) dominates over another ( $l^{H_z}$ ). This is quite unlike the usual case in which the behaviour suddenly changes at the scale in question. The sphero-scale is therefore quite unlike an ordinary length scale.

## 7. Stochastic Stratification

When  $T_\lambda$  acts on a small vertically oriented eddy, it clearly tends to flatten it out in the horizontal. At sufficiently large scales, any structure will be quite flat i.e. stratified. Figure 6a shows an isotropic gaussian field with  $H = 1/2$  and  $D_{el} = 2$ , Fig. 6b shows the corresponding field for  $D_{el} = 3/2$  (the same white noise field was used in both cases to generate via Fourier transforms, the random fields). Figure 6b yields a clear stratification, Fig. 6a does not since the resulting field is isotropic. Stratification may be seen as resulting from the anisotropic scaling law, thus as a result of a stochastic process. See Section 9.3 and Fig. 11 for an example of stratification in a hyperbolic context.

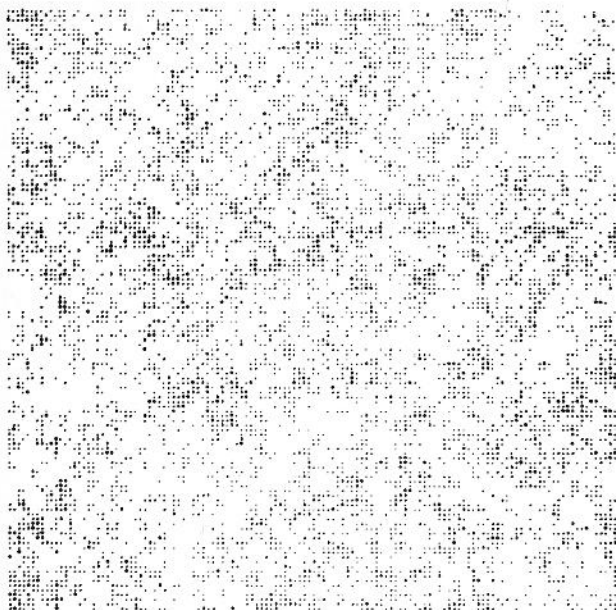
The notion of stochastic stratification is a necessary consequence of our anisotropic fractal space, and needs considerable development. It may be that it is even compatible with the exponentially decreasing mean pressure field, in this case, the physically significant quantity is  $\log p$ , and the hydrostatic relation is recovered by assuming that the vertical scaling of  $\log p$  has the parameter  $H_{v(\log p)} = 1$ , a fact that was verified here using the radiosonde data described in Section 4. This implies that the horizontal scaling has the value:  $H_{h(\log p)} = 1 \cdot H_z = 5/9 \sim 0.56$ . Isotropic homogeneous turbulence in an incompressible medium can be shown theoretically to have the value  $H_{h(p)} = 2/3$  since in an incompressible fluid, fluctuations in pressure are proportional to the square of velocity fluctuations (see also Batchelor (1953), Monin and Yaglom (1975)). In a compressible fluid this relationship no longer holds, and presumably the effect of compressibility is to lower the scaling exponent – possibly to the value 5/9. In any case, to our knowledge, this value has never been adequately determined empirically. A prediction of the stochastic stratification, is that the horizontal scaling of the potential temperature is  $H_{h(\ln \theta)} = 9/10 \cdot H_z = 1/2$ . The value of 1/2, if true, would mean that the inhomogeneities in the horizontal potential temperature field could be regarded as independent (cf. Section 5.2).

## 8. Hyperbolic Intermittency

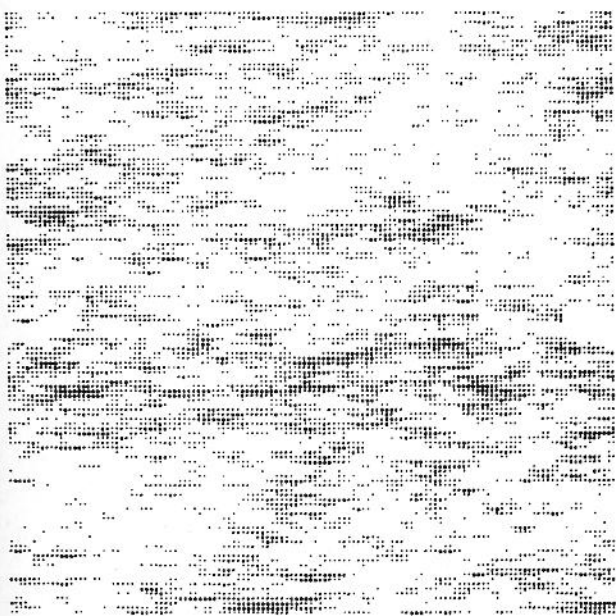
### 8.1 Introduction

Intermittency expresses the fact, that, roughly speaking, the turbulence doesn't fill all the volume of the space available to it. This is a fundamental aspect of turbulent flows, which, in spite of much work, is still far from completely understood. The first theoretical work on the subject, seems to be Leray's (1933) conjecture on the existence of a set of singularities of the Euler and Navier-Stokes-Equations. Von Neumann (1949) was aware of this work and stressed that the characterisation of these singularities is a fundamental task.

In the case of turbulence, the basic singularities are related to the distribution of the density of "active" or turbulent regions, that do not fill all of the volume. Other authors, such as Batchelor (1951) related this to the growing experimental evidence of the "spottiness" of turbulence (see also Batchelor and Townsend (1949)).



**Fig. 6a.** A random isotropic field with  $H = 1/2$ , the intensity of the field is proportional to the shade of the grey



**Fig. 6b.** An example of stochastic stratification. This figure is the same as Fig. 6a, except that  $D_{el} = 1.5$  instead of 2. In one direction,  $H = 1$ , and in the direction perpendicular,  $H = 1/2$ . This stratification is strong since  $H_z = 1/2$ . Note that the "sphero-scale" has a value of  $\sim 1\%$  of the length of one side of the figure

Landau and Lifshitz (1959) also questioned the universality of Kolmogorov's spectrum, since it had been derived under the assumption that the active regions were uniformly distributed in space. Kolmogorov (1962) and Yaglom (1966) presented a corrected spectrum that took into account intermittency, by assuming a log-normal distribution of  $\bar{\epsilon}$ . Since then, Orsag (1970) and Mandelbrot (1974a) have pointed out several theoretical and physical difficulties with this hypothesis. In particular, Mandelbrot, building upon an earlier, explicit model for the "spottiness" (Novikov and Stewart 1964), showed that log-normality may only

be expected to hold under rather special conditions, whereas hyperbolic behaviour was more likely. This latter possibility, which involves the divergence of higher moments of  $\bar{\epsilon}$ , was unfortunately dropped in Kraichnan (1974) and Frisch et al. (1978), and subsequent works.

Mandelbrot's model is in fact fairly general; however, the basic features may be understood by considering two special cases. The first is "curdling"; it divides all eddies strictly into either completely "dead" or uniformly "active" regions, and is often referred to in the literature as the " $\beta$  model". This model is fully characterised by the dimension of the support of turbulence,  $D_s$ . This is the only case where no divergence of moments occurs. The second model is "weighted curdling", it divides eddies into either "weak" or "very active", and permits in addition to the study of  $D_s$ , that of the divergence of the moments, characterised

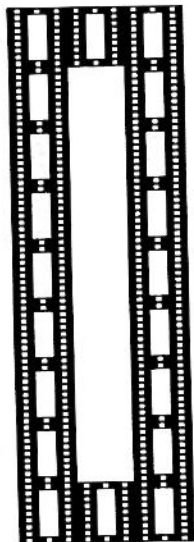


Fig. 7. This shows an example of deterministic curdling on an anisotropic grid ("elliptical Sierpinsky carpet"). At each stage (generation) of the process, the horizontal is divided into 9 parts, and the vertical into 3, yielding  $D_{cl} = \log 27 / \log 9 = 1.5$ . The curdling is deterministic because there is a fixed "generator" whose outlines can still be fairly discerned (i.e. the edges). In this case,  $D_s = \log 20 / \log 9 = 1.36 \dots$

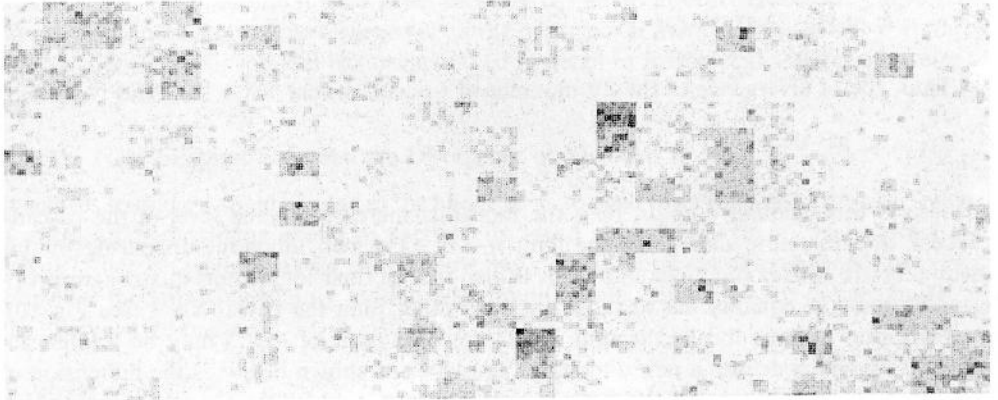


Fig. 8. A random curdling on an anisotropic grid, with  $D_{cl} = \log 20 / \log 5 = 1.86 \dots$ . All eddies are either "dead" or "active", such that the dimension of the support is:  $D_s = 1.70$

by the exponent  $\alpha$ . This latter subcase of the Mandelbrot model, could be called the “ $\alpha$  model”, because of the hyperbolic exponent it introduces.

Figure 7 presents the result of deterministic curdling, i.e. with a fixed “generator” but anisotropic (it could be called an “elliptical Sierpinsky carpet” according to Mandelbrot’s terminology). Figure 8 presents the result of a random “curdling” (“ $\beta$  model”) and Fig. 9 the one of a “weighted curdling” (“ $\alpha$  model”), in the latter case the resulting active “eddies” are of very different intensities as indicates by the grey.

In the following, we re-examine Mandelbrot’s model, placing particular emphasis on the importance of  $\alpha$ , and the dimension over which the spatial average of  $\varepsilon$  is taken. More details, in the isotropic case, may be found in Mandelbrot (1974a, b), Peyrière (1974) and Kahane (1974). We also demonstrate the utility of  $D_{el}$  by generalising Mandelbrot’s model to the



**Fig. 9.** A “weighted curdling”,  $D_{el} = \log 20 / \log 5 = 1.86$ ,  $D_s = 1.83$  and  $D_\infty = 1.5$ . The log of the intensities of the eddies is denoted by the intensity of grey

anisotropic case. By counting the number of eddies of a given size, we establish that this generalisation may be accomplished by simply replacing different isotropic dimensions by “elliptical” ones.

The rest of Section 8 is somewhat technical. Readers who are willing to accept that weighted curdling generally leads to the divergence of moments may skip to the next section.

## 8.2 A Summary of the Mandelbrot Model

The basic ingredient of this model is the random variable  $W$  which distributes the rate of turbulent energy transfer  $\varepsilon$  from an eddy of size  $l_{n-1}$  to its sub-eddies of size  $l_n$ , along a cascade of typical length scales:  $l_n = l_0 \lambda^{-n}$ ,  $\lambda$  being the constant ratio of cascade sizes. Specifically,

$$\varepsilon_{l_n}(x_i) = W_i \varepsilon_{l_{n-1}}$$

where  $x_i$  is the center of the  $i^{\text{th}}$  sub-eddy. The different realisations  $W_i$  of  $W$  are all independent. In what is called a “canonical” cascade,  $W$  is subject only to the condition of conservation of the average flux, i.e.:

$$\langle W \rangle = 1.$$

The spatially averaged flux  $\bar{\epsilon}_{l_n}$ , which is the flux of typical scales  $l_n$ , averaged over a subspace  $A$  of dimension  $D_A$  (e.g. a line or a plane), can be defined as follows:

$$\bar{\epsilon}_{l_n} = \sum_{(N_A(l_n))^{-1}} \epsilon_{l_{n-1}}$$

where  $N_A(l_n)$  is the number of eddies (empty or not) which intersect  $A$ :

$$N_A(l_n) = (l_n/l_0)^{-D_A} = \lambda^{D_A}$$

A straightforward calculation, (Mandelbrot 1974a), shows that to the leading term:

$$\begin{aligned} \langle \bar{\epsilon}_{l_n}^h \rangle &\sim \sum_{N_A(l_n)} \langle W^h \rangle \langle \bar{\epsilon}_{l_{n-1}}^h \rangle / N_A(l_n)^h \\ &\sim N_A(l_n)^{1-h} \langle W^h \rangle \langle \bar{\epsilon}_{l_{n-1}}^h \rangle. \end{aligned}$$

Thus, the of divergence of the  $h^{\text{th}}$  moment of  $\bar{\epsilon}$  occurs when:

$$\langle W^h \rangle > \lambda^{(h-1)D_A}$$

because, if this inequality holds, then the moments increase at each level of the cascade process, and therefore diverge in the limit  $n \rightarrow \infty$ . The two functions appearing in this inequality are convex functions of  $h$ . This is also true of their logarithms (a stronger property), so that the equality has at most one root other than the trivial  $h = 1$  (see Fig. 10).

Note that at least locally, around  $h = 1$ , the behaviour of  $\langle W^h \rangle$  may be written as:  $\lambda^{(h-1)(D-D_s)}$  which defines a new dimension  $D_s$ , which as shown below, is the dimension of the support of turbulence.  $D$  is the dimension of the space in which the cascade occurs, which Mandelbrot took to be 3. In the case of anisotropic cascades,  $D$  should be replaced by  $D_{e1}$ .

This leads to:

$$(D - D_s) = d(\log_\lambda \langle W^h \rangle) / dh |_{h=1}$$

or

$$(D - D_s) = \langle W \log_\lambda W \rangle > \langle W \rangle \log_\lambda \langle W \rangle = 0.$$

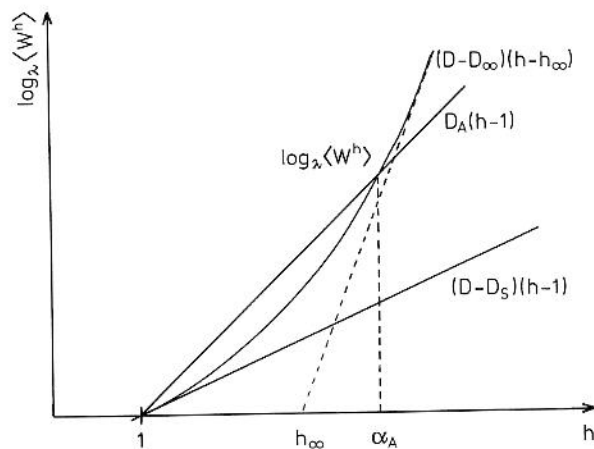


Fig. 10. An illustration of the Mandelbrot model of turbulent intermittency showing the functions discussed in the text

If  $D_A > D - D_s$  then we have a non-divergent behaviour of  $\bar{\varepsilon}^h$  for  $h \rightarrow 1$ . We therefore must average over a space of at least dimension  $D_A > D - D_s$  in order for  $\langle \bar{\varepsilon} \rangle$  to be finite. This gives a simple interpretation of  $D_s$  as the dimension of the support of turbulence.

### 8.3 The Simplest Case of the Mandelbrot Model

The simplest case is obtained when the local behaviour of  $\langle W^h \rangle$  near  $h = 1$ , holds for all  $h$  (the slope in Fig. 10 is constant). In this case, we have:

$$\langle W^h \rangle = \lambda^{(h-1)(D-D_s)}$$

which corresponds to the Bernoulli process:  $\Pr(W = 0) = 1 - \lambda^{-(D-D_s)}$  and  $\Pr(W = \lambda^{D-D_s}) = \lambda^{-(D-D_s)}$ . Clearly, the only parameter is  $D_s$  which uniquely determines the number of "active" eddies at each generation:  $N_a(l_n) = \lambda^{D_s}$  all other eddies are considered "dead". This model gives convergence of all the moments of  $\bar{\varepsilon}$  since  $\langle W^h \rangle < \lambda^{(h-1)D_A}$  for all  $h$  if  $D_A > D - D_s$ . This model has been studied in detail in Frisch et al. (1978), who refer to it as the " $\beta$  model".

### 8.4 The Next Simplest Case and the Divergence of Moments of $\bar{\varepsilon}$

It suffices to make a slight change in the above so as to yield divergent moments of  $\bar{\varepsilon}$ . Indeed, if rather than classifying eddies as "active" or "dead", we classify them "very active" and "weak", then, in general, as we show below, high order moments of  $\bar{\varepsilon}$  no longer converge. Consider the following slightly modified Bernoulli model defined by the dimension  $D_\infty$  of the "very active regions":

$$\begin{aligned} \Pr(W = \lambda^{(D-D_\infty)}) &= \lambda^{-(D-D_\infty)/h_\infty}, \quad h_\infty > 1 \\ \Pr(W = a_1) &= 1 - \lambda^{-(D-D_\infty)h_\infty} \end{aligned}$$

(note this reduces to the preceding case for  $h_\infty = 1$ ).

Due to the linear asymptotic behaviour of  $\log_\lambda \langle W^h \rangle$  for large  $h$ , higher moments of  $\bar{\varepsilon}$ , averaged over  $A$ , diverge when:

$$D - D_s < D_A < (D - D_\infty).$$

In concluding of this section, we point out that, in the general case, as  $h$  is increased the dimension of the support of  $\varepsilon^h$ ,  $D_s(h)$ , decreases. In the case of the " $\alpha$  model" studied above, these dimensions were bounded below by  $D_\infty$  which is the dimension of the more extreme active regions ( $D_\infty = D_s(\infty)$ ). The fact that intermittency is characterised by many fractal dimensions has been already discussed in Schertzer and Lovejoy (1983a)<sup>3</sup>. Recently, Parisi and Frisch (1984) have developed the idea of many fractal dimensions in the context of a somewhat different model that they call "multifractals".

## 9. The Phenomenology of Scaling, Hyperbolically Intermittent Fields

### 9.1 Introduction

If the scaling and intermittency described here hold under quite general conditions, then they must be compatible with the rich diversity of meteorological fields. Below we examine two

3 The exact formula is:  $D_s(h) = D - \log_\lambda \langle W^h \rangle / (h - 1)$



aspects of phenomenology and explain why we believe they can be better understood by using the scaling and intermittency described here.

## 9.2 Algebraic Formulae Relating Various Fields

One goal of phenomenology is to relate different meteorological fields. If scaling laws hold, and intermittency is expressed via hyperbolic distributions, then we may expect the form of the probability distributions (in particular the hyperbolic exponents) of different fields to be related by fairly simple algebraic equations. For example, if we define an "eddy" as a region surrounded by lines of constant horizontal velocity, then for a given  $\varepsilon$  and  $\phi$ , the horizontal and vertical extents ( $\Delta x$ ,  $\Delta z$ ) of the eddy may be obtained by equating the  $\Delta v$ 's in Eqs. (1), (2):

$$\bar{\varepsilon} \Delta x \stackrel{d}{=} \bar{\phi}^{3/5} \Delta z^{9/5}$$

hence  $\alpha_\varepsilon = 5\alpha_\phi/3$  as required by dimensional analysis (from Fig. 4  $\alpha_\varepsilon \sim 5/3$ ,  $\alpha_\phi \sim 1$ ). Note that although  $\bar{\varepsilon}$  and  $\bar{\phi}$  are related in this way, they may or may not be dependent on each other. The sphero-scale, which can be defined by  $\Delta x = \Delta z$  will therefore be a random variable unless  $\bar{\phi}$  and  $\bar{\varepsilon}$  are completely dependent.

One area where the investigation of algebraic relations is perhaps particularly urgent is in the case of the rainfield. This is a case where numerical modeling is notoriously difficult, but in which simple algebraic relations may exist. In particular, Lovejoy (1981) showed that  $\alpha_R \sim 5/3$  in diverse areas of the world. (Note that this result applies to changes in the flux of rain ( $R$ ) from isolated storms which is a 2-D lagrangian statistic. It is therefore unlikely to be trivially connected to the  $\varepsilon$  field whose value  $\alpha_\varepsilon \sim 5/3$  is a 1-D eulerian statistic.)

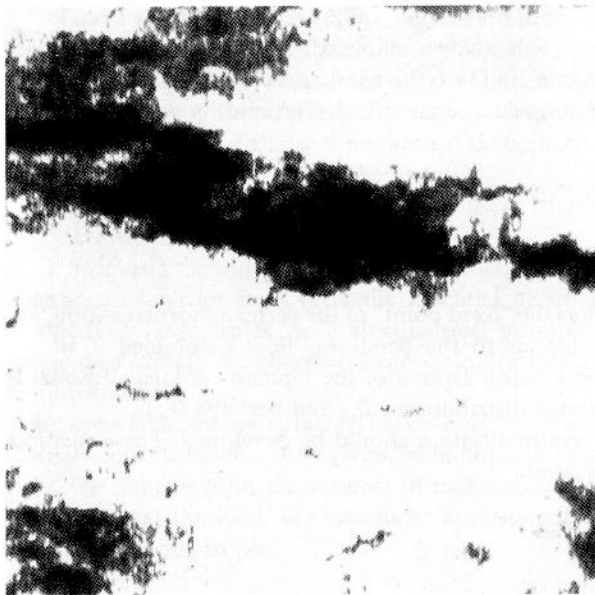
## 9.3 Shapes, Structure and "Animals"

Another goal of phenomenology is the classification and understanding of the shapes and structures of individual meteorological fields. In fluid mechanics, the shapes and structures are so diverse that popular jargon refers to the different morphologies as fluid "animals", with the phenomenologist playing the role of a zoologist. In percolation theory the shapes encountered are so bizarre that the expression "lattice animals" has gained official acceptance. Below, we argue that the use of scaling and hyperbolic distributions may be important in taming the different meteorological animals including "fronts", "cells" and others. In this case, the animals are clearly fractals because of the scaling.

In appreciation of the significance of hyperbolic distributions (particularly for  $\alpha < 2$ ), Mandelbrot and Wallis (1968) pointed out that they would have the effect of causing the largest of a set of such fluctuations to dominate over the others. In reference to the extreme meteorological fluctuation responsible for The Flood, they coined the expression "Noah effect" to describe this phenomenon. It is ironic that it has only recently been applied to actual meteorology (Lovejoy 1981).

Figure 11 shows a "hyperbolic fractal animal" obtained by Monte Carlo simulations in an anisotropic space. In contrast to the example of stochastic stratification with gaussian fluctuations (Fig. 6), here, the elementary fluctuations are hyperbolically distributed, hence the field is dominated by one main "animal" although other smaller ones are visible. In the extreme gaussian case, the morphology is entirely determined by the scaling because no matter what elementary fluctuations are used, they all have approximately the same intensity and thus their details tend to cancel.

In the case of hyperbolic fluctuations, on the contrary, the largest dominates and imposes its geometry, although it does so in an indirect fashion. Here, specifying  $H$ ,  $\alpha$  is insufficient



**Fig. 11.** An example of a hyperbolic fractal animal on an anisotropic space, with  $D_{el} = 1.80$ . The log intensities are indicated by the intensity of the grey. Note that in distinction to the gaussian case (Fig. 6 b), one large “animal” dominates image, although others are visible, a manifestation of the “Noah” effect. (This model is on an  $800 \times 800$  point grid, and the sphero-scale has the value 30 pixels. The fine structure is therefore oriented perpendicularly to the overall shape)

to determine the morphology – changes in the underlying metric (Section 6), or of the geometry of the elementary fluctuations strongly influence the observed shapes and structures.

Figure 11 is an anisotropic version of the fractal rain model discussed in (Lovejoy and Schertzer (1984); Lovejoy and Mandelbrot (1984)) which uses annuli (with intensities distributed with  $\alpha \sim 5/3$ ) as a basic shapes. There it is argued that the existence of hyperbolic distributions is directly related to a hierarchy of straight-line structures, and hence to rain “fronts”.

#### 9.4 Hyperbolic Renormalisation

The phenomenological model of intermittency discussed in Section 8.2 may be placed in a dynamical context since it results from the fact that the characteristic time  $\tau$  of transfer of energy from one scale  $l_n$  is:

$$\tau(l_n)^{-1} = W \Delta v(l_n)/l_n.$$

Hence, due to the presence of the random variable  $W$ ,  $\tau$  is far from having a characteristic value as in usual phenomenological models ( $W = 1$ ). This can be understood as a phenomenological “renormalisation” of the vertex for the following reasons.

Renormalisation of the Navier-Stokes-Equations – as described for instance in Forster et al. (1978) – consists of systematically taking into account the direct interactions in a “decimation” process from high to low wave numbers. The basic problem encountered is the calculation of the effect of the indirect interactions of the low wavenumbers via the eliminated ones, which modify the interaction of the conserved wavenumbers. Renormalisation has therefore only been applied successfully in the case of low wavenumbers, where indirect interactions do not need to be taken into account (due to Gallilean invariance De Dominicis and Martin 1979). The indirect interactions seem however to be extremely important in the inertial range, as evidenced for example in the failure of the Direct Interaction Approximation in the inertial range (Kraichnan 1958). Until now, this failure has only been cured by *ad hoc* procedures (e.g. Herring et al. 1983).

We propose that repeated application of the decimation process, proceeding from low to high wave numbers in the inertial range, will create a renormalisation of the intensity of the vertex converging to the random variable  $W$ . Thus, the whole process of the application of the renormalisation group (with rescaling) can be described symbolically as:

$$\begin{aligned} l &\rightarrow \lambda l \\ v &\rightarrow \lambda^{H_v} v \\ P &\rightarrow W \lambda^{-1} P \quad \text{with } \langle W \rangle = 1 \end{aligned}$$

where  $P$  is the vertex intensity.  $W$  is thus the "fixed point" of the vertex renormalisation, and the flux conserving condition is analogous to the condition  $W = 1$  obtained if  $W$  was deterministic. We may therefore expect many aspects of the phenomenological model to follow from the dynamics (e.g. hyperbolic distributions,  $D_s$ , and perhaps  $D_\infty$ ).

In our view, specific methods of renormalisation should be developed. These methods could be called hyperbolic renormalisation.

## 10. Conclusions

### 10.1 Summary of Principle Results

The object of this paper has been to elucidate the relationship between small and large scale structures in the atmosphere. The vertical shear and buoyancy force play a key role in the genesis of inhomogeneities, and were therefore investigated in detail using high resolution radiosonde and aircraft data. Two aspects of the data were investigated: the scaling exponents ( $H$ ), and the hyperbolic exponents which characterise the intermittency ( $\alpha$ ). These parameters were analysed over atmospheric layers between 50 m and 3.2 km thick.

The principal results are as follows:

a)  $H_v = 3/5$ : This empirical value was also deduced by a dimensional argument similar to that first proposed by Obukhov (1959) and Bogliano (1959). Our derivation is analogous to the dimensional argument (based on  $\varepsilon$ ) used to establish the horizontal (Kolmogorov) scaling ( $H_h = 1/3$ ). It supposes that the fundamental dimensional quantity is  $\phi$ , the flux of buoyancy force variance. Our result agrees with Adelfang (1971).

b)  $\alpha_v = 5$ : This empirical value was corroborated by a re-analysis of Merceret's (1976) measurements of  $\bar{\varepsilon}$  from which we obtained  $\alpha_\varepsilon = 5/3$ . Simple physical arguments confirm  $\alpha_v = 3\alpha_\varepsilon$ . An immediate consequence is the divergence of the fifth and higher order moments of the velocity field, and the second and higher moments of  $\bar{\varepsilon}$ . The divergence of high moments of  $\bar{\varepsilon}$  was first proposed by Mandelbrot (1974 a).

c)  $\alpha_{Ri} = 1$ : This shows that the Richardson number has such extreme fluctuations that even the mean may not exist. Monte Carlo methods showed that this result could be explained as the result of dividing two weakly correlated random variables. This points to the need to re-examine the validity of theories based on limited expansions of the equations in  $Ri$ .

d)  $H_{1n\theta} = 9/10$ ,  $\alpha_{1n\theta} = 10/3$ : The potential temperature was also found to be scaling in the vertical and to be extremely intermittent-fourth and higher moments diverge.

e)  $D_{el} = 23/9$ : To account for the different horizontal and vertical scaling, in the wind field, we introduced a special kind of fractal dimension ( $D_{el}$ ) called an elliptical dimension because it expresses the fact that small eddies with circular cross-sections in the vertical appear as flattened ellipses at larger scales. Our definition of  $D_{el}$  leads to the equation

$D_{el} = 2 + H_h/H_v = 23/9 \sim 2.56$  (a convenient mnemonic to remember this result is “two plus Kolmogorov over Obukhov”).

f) *The sphero-scale*: The velocity scales differently in the horizontal and vertical directions, and therefore there will be a distance at which the magnitude of the fluctuations will be equal. We call this distance the sphero-scale because at this scale, eddies are isotropic, and therefore, averaged eddies are spherical.

The sphero-scale is quite unlike an ordinary length scale – for example, it may vary in a very erratic manner.

g) *Stochastic Stratification*: A consequence of our anisotropic scaling space is that any structure, no matter how vertically oriented at small scales, will appear quite flat (i.e. stratified) at large scales. Several numerical examples were given. This concept may be sufficient to explain the observed large-scale stratification of meteorological fields, including the pressure field.

h) *hyperbolic intermittency*: The outlines of Mandelbrot’s (1974a) model of fractally homogeneous turbulence were given, including a discussion of the relation between intermittency, the dimension of the support of turbulence ( $D_s$ ), and  $D_{el}$ , and the various observed hyperbolic distributions. We conclude: atmospheric turbulence is fractally homogeneous and fractally anisotropic<sup>4</sup>.

## 10.2 A New Direction for Meteorology?

The hypothesis that atmospheric motions are qualitatively quite different at small and large scales can be traced back to at least the 19<sup>th</sup> century. At first, this idea could only have had an *ad hoc*, phenomenological basis, since the equations of fluid mechanics and thermodynamics contain no length scales, and our knowledge of motions on scales less than hundreds of kilometers was inadequate. In recent years, theorists such as Charney (1971) have attempted to use turbulence theory to give this distinction a solid basis by arguing that large scale motions are “quasi-two dimensional”, and are separated by a dimensional transition from small scale “quasi-three dimensional” motions. Experimentalists, from Pinus (1968) onwards, using high resolution satellite, aircraft, radiosonde and other data have searched for empirical evidence of a transition, apparently, without success in the range 10 cm to 4000 km.

We believe that the hypothesis of a dimensional transition is no longer tenable either theoretically, or empirically. The observed structure of the atmosphere can be explained by a simpler hypothesis: that it is anisotropic and scaling throughout, a fact that can be characterised by the intermediate “elliptical” dimension  $D_{el} = 23/9 = 2.56$ . If this hypothesis is true, then the atmospheric motions are never “flat” (two-dimensional), nor isotropic (three-dimensional), but always display aspects of both, according to a well defined fractal geometry.

In spite of the existence of an entire programme of theoretical and empirical research, the hypothesis of a dimensional transition is on more tenuous ground than ever. The alternative, anisotropic scaling hypothesis proposed here will undoubtedly also require a large research program to substantiate, and cannot be said to be definitively established. For instance, the effects of the Coriolis force have not been considered. It is quite possible that this could be

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<sup>4</sup> In a celebrated poem, Richardson summarised atmospheric dynamics: “Big whorls have little whorls that feed on their velocity, and little whorls have smaller whorls and so on to viscosity – in the molecular sense.” Anisotropic scaling adds a second stanza: “Flutter whorls have rounder whorls that feed on instability, and roundish whorls have rounder whorls, and so on to sphericity – in the statistical sense.”

accounted for by a non-diagonal  $T$ , but *a priori*, there is no reason to suppose this to break the scaling. It is therefore possible that baroclinic instabilities will also be scaling.

This alternative, does offer several advantages, not the least of which is simplicity. If it is correct, then, small scale structures differ from large scale ones only in a continuous transformation in which they are progressively "flattened" or stratified at larger and larger scales. This transformation depends mainly on the action of two groups, the scaling group  $T$ , and the convolution group of stable Levy distribution (which characterise both the scaling and the intermittency respectively). Thus, atmospheric motions can be seen to be anisotropic, but scaling and intermittent at all scales.

One immediate practical consequence of this work is its impact on what we call the "stochastic coherence" of numerical weather forecasting procedures (see Schertzer et al. (1983)). This means that we verify that the scaling of the meteorological fields are not artificially modified through the different stages of the forecast procedure e.g. during the data assimilation, initialisation and numerical weather-prediction integrations model. This may prove an invaluable and general way of testing the validity of numerical forecasts because the capacity of modern computers is sufficient to allow scaling regimes to be obtained.

### 10.3 Theoretical Implications

The results of this paper may be expected to have an impact on at least two different areas of theoretical work.

The first is turbulence in anisotropic media, which should be approached in the fractally anisotropic framework proposed here (i.e. without separating the mean and fluctuating fields). Indeed, this framework may be essential in overcoming difficulties where so far only formal manipulations of renormalisation schemes have succeeded (e.g. Kraichnan (1964 a, b), Carnevale and Martin (1982)). The second way in which this work may have significant theoretical implications is in resolving difficulties of isotropic turbulence. In particular, this paper shows the need to develop methods appropriate for renormalising situations far from the usual one of "quasi-gaussian" distributions – methods that should perhaps be called "hyperbolic renormalisation".

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