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Comment on "Are Rain Rate Processes Self-Similar?" by B. Kedem and L. S. Chiu

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Three years ago we discussed the question of scale invariance in the atmosphere concentrating our attention on the empirically accessible rain field [*Lovejoy and Schertzer*, 1985]. The main point of the paper was that the atmosphere was likely to be scaling (scale invariant) but could not be "self-similar"; hence the need for "generalized scale invariance" (as announced in the title and developed by *Schertzer and Lovejoy* [1985, 1987*a*, *b*]. In a recent paper [*Kedem and Chiu*, 1987] the authors equate scaling with self-similarity, in our opinion rendering the basic issues obscure. Furthermore, even within the framework of their definition of self-similarity, they confuse the properties of the rain field with those of its fluctuations in such a way that neither of their theorems are relevant to the problem of stochastic self-similar rain modeling. We would therefore like to take this opportunity for clarification.

The notion of "self-similarity" indicates that some aspect of a process or set is invariant under isotropic scale changing transformations such as simple dilations ("zooms"). The figures in *Mandelbrot*'s [1982] book are examples of self-similar (or at least asymptotically self-similar) fractals. The equality between the large and small scales of the fractal can be expressed in many ways. In the simplest case of geometric self-similar fractals, there is strict equality, in more interesting (random) cases, the equality is typically expressed in terms of probability distributions or via moments of different orders (e.g., "multiple scaling"; see below). Systematic study of scaling (fractal) systems (i.e., their quantitative rather than qualitative applications) is only just beginning and new forms will undoubtedly be discovered. The important point here is that the type of scaling has absolutely nothing to do with the (usual) scale change necessary to relate the small and large; it is to this latter element that the term self-similar naturally belongs. To be more precise, we will give examples of the two types of scaling most commonly encountered in physics, "simple" and "multiple" scaling [see *Schertzer and Lovejoy* [1987*a*, *b*] for more discussion), while simultaneously answering *Kedem and Chiu*'s [1987] important point about "the atom at zero."

The definition of scaling used by *Kedem and Chiu* [1987] (their equation (1); see below) if applied to the rain field itself (as assumed in their theorems) rather than to its fluctuations, might be called "very simple scaling"; it is simpler than the "simple scaling" discussed below, and we know of no applications of such scaling in physics, geophysics, or elsewhere. Indeed, their theorems show why this type of scaling is far too restrictive to be of interest in these contexts. However, if we apply their definition to fluctuations in the rain field (as in the works by *Waymire* [1985] and *Waymire and Gupta* [1989]), then it becomes equivalent to the following more interesting simple scaling or "scaling of the increments" (an example of which called the "fractal sums of pulses" process was used to illustrate nonself-similar scaling in the work by *Lovejoy and Schertzer* [1985]):

$$\Delta X(\mathbf{T}_\lambda \Delta \mathbf{r}) \overset{d}{=} \lambda^{-H} \Delta X(\Delta \mathbf{r}) \tag{1}$$

where $\Delta \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_\lambda$ is a (large scale) vector separating the points $\mathbf{r}_1$ and $\mathbf{r}_2$, and $\mathbf{T}_\lambda$ is an operator that reduces scales by a factor λ (more on this below for the moment, consider the isotropic, self-similar case $\mathbf{T}_\lambda = \lambda^{-1}\mathbf{1}$, where $\mathbf{1}$ is the identity). $\mathbf{T}_\lambda \Delta \mathbf{r} = \mathbf{r}_3 - \mathbf{r}_4$ is therefore the (small scale) separation of the points $\mathbf{r}_3$ and $\mathbf{r}_4$. Thus

$$\Delta X(\Delta \mathbf{r}) = X(\mathbf{r}_1) - X(\mathbf{r}_2)$$

$$\Delta X(\mathbf{T}_\lambda \Delta \mathbf{r}) = X(\mathbf{r}_3) - X(\mathbf{r}_4)$$

and the equality $\overset{d}{=}$ indicates equality of probability distributions (more precisely, the random variables a and b are equal in this sense if $Pr(a > q) = Pr(b > q)$ for all q, where Pr indicates "probability"). H is the (unique) scaling parameter. Simple scaling is exemplified by a coordinate of the motion of a Brownian particle. Such Brownian motion corresponds to the special case where (1) the probability distributions are gaussian, (2) $H = 1/2$, and (this is usually not stated explicitly), (3) $\mathbf{T}_\lambda = \lambda^{-1}\mathbf{1}$. Note that very simple scaling is obtained by replacing ΔX by X and $\Delta \mathbf{r}$ by $\mathbf{r}$ in (1) above.

Unfortunately for *Kedem and Chiu* [1987] if their scaling is applied to fluctuations in rain (hence that the rain rate follows simple scaling rather than very simple scaling), then neither of their theorems has much relevance. For example, theorem 1, which shows that scaling quantities cannot have probabilities of exceeding zero which grow with separation is easily satisfied by fluctuations (although not the rainfield itself). Similarly, theorem 2 shows that scaling processes cannot have stationary increments and remain positive for all values, which again, is a constraint for rain, but not for its fluctuations. Kedem and Chiu do, however, raise the interesting problem of the "atom zero," i.e., the necessity in rainfall modelling of having finite probability of zero rain rates. In simple scaling, it is clear that the process must not only be given a starting point (e.g., a value at the origin), but furthermore (as described by *Lovejoy and Schertzer* [1985]), negative values must be suppressed by some type of thresholding procedure. This thresholding is unphysical and constitutes a physical (not mathematical) criti-

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cism of all forms of simple scaling models of rain, hence the need for the more physical multiple scaling discussed below.

Returning to our discussion of scaling and self-similarity, we briefly indicate how simple scaling can be generalised by varying any (or each) of its three basic elements. For example in fractional Brownian motion, only 2 is modified ($H \neq \frac{1}{2}$) while in "stable Levy motion" 1 and (usually) 2 are modified. The variants (including their possible nonlinear transformations) are all produced by (weighted) sums of random variables, in accord with the single parameter H they are characterized by a single fractal dimension (D). In the works by *Lovejoy and Schertzer* [1985], *Schertzer and Lovejoy* [1984, 1985, unpublished manuscript, 1983], we explored generalizations which involve relaxing the self-similarity condition and allowing $\mathbf{T}_\lambda = \lambda^{-\mathbf{G}}$ (generalized scale invariance) where $\mathbf{G}$ is the generator of the (semi) group of scale changing operators. It is only in the very special case when $\mathbf{G} = \mathbf{1}$, that we obtain self-similarity. When $\mathbf{G} \neq \mathbf{1}$ but is linear (a matrix), then differential stratification and/or rotation follow (in the atmosphere this is necessary to account for respectively gravity and the Coriolis force). Nonlinear $\mathbf{G}$ are also possible and lead to even more general scale changing operators.

Two of the (many) drawbacks in using simple scaling models for rain are (1) they involve only a single fractal dimension and are thus very special (even empirically this is also too restrictive; see *Lovejoy et al.* [1987]) and (2) the increments rather than the process itself are stationary. The latter point is directly related to *Kedem and Chiu*'s [1987] basic point about the "atom at zero," or the fact that at any point, the rain process has positive probability of having no rain. This constraint, coupled with scaling implies stationarity of the process and is therefore incompatible with scaling increments.

A physically appealing way of simultaneously overcoming both drawbacks is to model rain using a totally different type of scaling called "multiple scaling," "multifractals," or "multiplicative chaos" (see the comments in section 7 of *Lovejoy and Schertzer* [1985] on the limits of monodimensional modeling based on additive processes and particularly *Schertzer and Lovejoy* [1987*a*, *b*] which includes a detailed discussion of the physical basis of multiplicative processes). (In the last few years, this type of scaling has generated considerable interest as well as a large literature (particularly in physics)). Rather than adding random elements at different scales, such processes involve multiplicative modulation of the small scales by the large and are exemplified by cascade processes. In this case, we require an entire scaling function rather than the single exponent H; we obtain an infinite hierarchy of fractal dimensions.

Once again, we define the resulting multiple scaling in a general manner which explicitly shows how self-similarity arises only as a very special case. Such multiple scaling can be expressed as

$$\lambda^{hD(A)}\left\langle\left(\int_{\mathbf{T}_\lambda A} X d^{D(A)}\mathbf{r}\right)^h\right\rangle = \lambda^{-p(h)}\left\langle\left(\int_A X d^{D(A)}\mathbf{r}\right)^h\right\rangle \qquad (2)$$

where A is a (large scale) integration set of dimension $D(A)$ (which can be a fractal); $\mathbf{T}_\lambda A$ is a (small scale, reduced by factor $\lambda \geq 1$) reduced set; angle brackets indicate statistical (ensemble) averages; and $p(h)$ is a (generally) nonlinear function characterizing the scaling of the various moments. Although multiple scaling was first discussed in the early 1960s [*Obukhov*, 1962; *Kolmogorov*, 1962]; *Mandelbrot* [1974] associated it with only a single fractal dimension (the dimension of the "carrier"). It was not until recently [*Hentschel and Procaccia*, 1983; *Grassberger*, 1983; Schertzer and Lovejoy, unpublished manuscript, 1983) that fractal dimensions ($D(h)$) were associated with each moment (h). This may be done via the equation

$$D(h) = d - \frac{p(h)}{h-1} \qquad (3)$$

where d = trace $\mathbf{G}$ (= the dimension of the underlying space when $\mathbf{G} = \mathbf{1}$) and is the "elliptical dimension" charcterizing the stratification when $\mathbf{G} \neq \mathbf{1}$.

Unlike additive scaling processes, the limiting behavior of multiplicative processes is very singular. This singular behavior solves the "atom at zero" problem in a somewhat unexpected but fundamental manner (rather than in the ad hoc way proposed by *Bell* [1987]), since in the limit, such processes are everywhere almost surely zero (see section 4.1 of *Schertzer and Lovejoy* [1987*b*] for details). Indeed, in the limit, all the rain is concentrated into singularities of various orders, each distributed over sparse fractal sets (the everywhere almost surely zero property is expressed by the fact that the corresponding dimensions are less than that of the embedding space). In such processes all observable quantities are spatial and/or temporal averages. Although, in reality, the cascade is eventually terminated at scales of the order of millimeters by viscosity, the properties of these observables (which are typically averages over scales much larger) will be approximated by the limiting behaviour. This singular behavior is therefore responsible for a fundamental difference between the physical quantities of interest (those involved at each step of the cascade), and the observables, and has important consequences for measurements (such as the divergence of high order statistical moments). See *Schertzer and Lovejoy* [1987*a*, *b*] for a complete discussion.

Using a variety of new analysis techniques (trace moments, functional box counting, elliptical dimensional sampling [*Schertzer and Lovejoy*, 1987*a*, *b*; *Lovejoy et al.*, 1987; *Gabriel et al.*, 1988] we have not only confirmed (2), over a wide range of scales in both radar rain data, and satellite ground and cloud radiance fields, but we have also shown (in rain) that $\mathbf{G} \neq \mathbf{1}$ and have estimated its trace (its "elliptical dimension") = 2.22 ± 0.07 which is considerably less than the value 3 that would hold in a self-similar (e.g., nonstratified) field.

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