PHYSICA A

Hard and soft multifractal processes

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We show that multifractal notions encompass a wider variety of phenomena than often believed. Ranked by increasing highest order of singularities we have geometric, then microcanonical and finally canonical multifractals. They are respectively localized and "calm", delocalized and "calm", and delocalized and "wild". Canonical multifractals may also involve rare violent ("hard") singularities which cause high order statistical moments to diverge. We demonstrate this classification in the framework of universal multifractals characterized by three fundamental exponents.

1. Introduction

In strange attractors [1, 2], turbulence [3, 4], statistical physics [5, 6], high energy physics [7], astrophysics and geophysics [8] we are confronted with infinite hierarchies of dimensions and multifractal statistics rather than unique fractal dimensions and fractal geometry. There is an increasingly widespread (and complacent) view that multifractals are so ubiquitous and have been so well studied that the unification of scaling notions has already been achieved: all that is needed is the (positive) fractal dimension (f) as a function of the order of singularity (α). Unfortunately, this standard theoretical framework and the corresponding empirical studies presuppose very restrictive calmness and regularity assumptions. It has therefore become a matter of some urgency to reveal the full diversity of multifractality. Below, we show that extremely wild but rare singularities exist. They can lead to a "hard" behavior, which is fundamental but nevertheless outside the scope of the usual discussions. In the course of the development these hard/soft, wild/calm notions will be made quite precise and will serve as a basis for classifying multifractals.

2. Basic properties of multifractal fields

We consider the (generic) case of stochastic multifractal processes (e.g. turbulent cascades) produced by random multiplicative modulation of larger

structures into smaller ones yielding a highly intermittent field ε (e.g. density of the energy flux in turbulence) on space/time. Note that each realization of such a field corresponds to a finite dimensional cut of a process in an infinite dimensional probability space. The multiple scaling behavior of this field ε_λ at scale ratio λ $(=L/l$, the ratio of the largest scale L to the scale $l)$ can be either characterized by its probability distribution or by the corresponding law for the statistical moments (via a Laplace transform):

$$\Pr(\varepsilon_\lambda \geqslant \lambda^\gamma) \approx \lambda^{-c(\gamma)} , \tag{1}$$

$$\langle \varepsilon_\lambda^q \rangle \approx \lambda^{K(q)} \approx \int \lambda^{q\gamma}\lambda^{-c(\gamma)} \, \mathrm{d}c(\gamma) . \tag{2}$$

Here and below the $\approx$ sign means equality within slowly varying and constant factors (negative $\mathrm{d}c/\mathrm{d}\gamma$ correspond to reverse inequality in eq. (1)). In eq. (1) the statistical exponent $c(\gamma)$ is a codimension since the probability measures the fraction of the probability space occupied by the singularities exceeding the order γ. At scale ratio λ, it can be estimated as the ratio of the number $(N_\lambda(\gamma))$ of corresponding structures to the total number of structures (N_γ):

$$\Pr(\varepsilon_\lambda \geqslant \lambda^\gamma) \approx N_\lambda(\gamma)/N_\lambda . \tag{3}$$

$c(\gamma)$ also has a geometrical interpretation over a D-dimensional observing set A whenever $D \geqslant c(\gamma)$. In this case, not only $N_\lambda \approx \lambda^D$, but also, on almost any realization, $N_\lambda(\gamma) \approx \lambda^{D(\gamma)}$, with a positive $D(\gamma)$ which is then a usual geometric fractal dimension. This geometric interpretation corresponds to the (restrictive) starting point of the considerations of Parisi and Frisch [4]. "Wild" singularities, as defined below, are outside of this geometrical framework. Now, we can consider D-dimensional integration over A:

$$\Pi_\lambda(\mathrm{A}) = \int_\mathrm{A} \varepsilon_\lambda \, \mathrm{d}^D x , \tag{4}$$

corresponding to the energy flux (turbulence) or the multifractal probability (strange attractors). Π_λ also has multiple scaling behavior corresponding to the usual multifractal measure exponents α, f, τ, but there is a D dependence (rendered explicit by the subscript "D")

$$\Pr(\Pi_\lambda(\mathrm{B}_\lambda) \geqslant \lambda^{-\alpha_D}) \approx \lambda^{f_D(\alpha_D)}/\lambda^D , \qquad \alpha_D = D - \gamma , \qquad f_D(\alpha_D) = D - c(\gamma) , \tag{5}$$

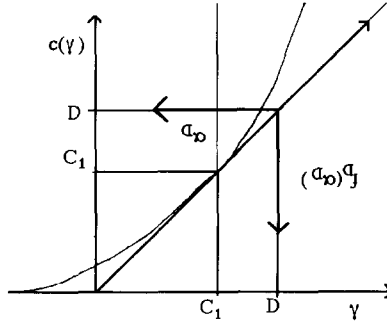on a ball B_λ of size L/λ. The D-dimensional integration merely smooths the

Fig. 1. A schematic comparison of the intrinsic (absolute) representation (γ, c) of the singularities of the density ε vs. their (relative) D-dimensional counterparts (α_D, f_D) for the D-dimensional integral Π, obtained by turning the diagram upside down and shifting the origin along the bisectrix as shown.

singularities of the density ε_λ, linearly shifting their order (see fig. 1). Generalizing the partition function used in strange attractors, the "trace moment" [9] combines ensemble with spatial averaging,

$$\mathrm{Tr}_A \, \varepsilon_\lambda^q = \left\langle \sum_A (\varepsilon_\lambda \lambda^{-D})^q \right\rangle \approx \left\langle \sum_i \Pi_\lambda (\mathrm{B}_{\lambda,i})^q \right\rangle , \tag{6}$$

where the sum over A is done at resolution λ, i.e. on a (more or less optimal) covering of $N_\lambda(A) \approx \lambda^D$ balls $\mathrm{B}_{\lambda,i}$:

$$\mathrm{Tr}_A \, \varepsilon \lambda^q \approx \lambda^{-\tau_D(q)} , \qquad \tau_D(q) = (q-1)D - K(q) . \tag{7}$$

While the intrinsic quantities γ, $c(\gamma)$, $K(q)$ are independent of the dimension D of the observing space A, unfortunately α_D, $f_D(\alpha_D)$, $\tau_D(q)$ diverge as we increasingly explore the infinite dimensional probability space $(D \to \infty)$. One may note that this turbulence notation avoids also artificial problems of negative dimensions ("latent dimensions" [10]), which occur for "wild" singularities (see below). Just as $f(\alpha)$ is the Legendre transform [4, 2] of $\tau(q)$, so $c(\gamma)$ is the transform of $K(q)$:

$$K(q) = \max_\gamma \left[q\gamma - c(\gamma) \right] , \qquad c(\gamma) = \max_q \left[q\gamma - K(q) \right] . \tag{8}$$

These relations establish a one to one correspondence between orders of singularities and moments $(q = c'(\gamma), \gamma = K'(q))$. The fact that Laplace transform (between probabilities and moments) reduces to the Legendre transform (for the exponents) is a consequence of the saddle point approximation, generally valid in the large λ limit. However, the Legendre transform only

yields the exponential part of the probability distribution, missing logarithmic factors particularly important for universal multifractals (see below) with parameter $\alpha < 2$.

3. Classification of multifractal fields

Since convexity is preserved by Legendre transforms $c(\gamma)$ is convex as is $K(q)$ (which is the base λ Laplace second characteristic function [11, 12]) of $\log \varepsilon_\lambda$. $c(\gamma)$ admits a fixed point ($\gamma = C_1$) simultaneously corresponding to the mean singularities and their codimension, since $C_1 = c(C_1)$ corresponds to $q = c'(C_1) = 1$, which shows also that $C_1 = K'(1)$. If $C_1 > D$, the mean is so sparse that on the observing space, the process is almost surely almost everywhere zero, it is "degenerate", hence we require $C_1 \le D$. These features can be seen in fig. 2.

As $(\Sigma_l x_l^q)^{1/q}$ is a decreasing function of q we have a Jensen inequality [9]

$$\mathrm{Tr}_A \ \varepsilon_\lambda^q \le \langle \Pi_\lambda(A)^q \rangle , \qquad q \ge 1 ; \tag{9}$$

hence, using eq. (7) we may define the critical order of moment q_D by $K(q_D) = D(q_D - 1)$: $\langle \Pi_\lambda(A)^q \rangle \to \infty$ for all $q \ge q_D$ (>1). Introducing the dual
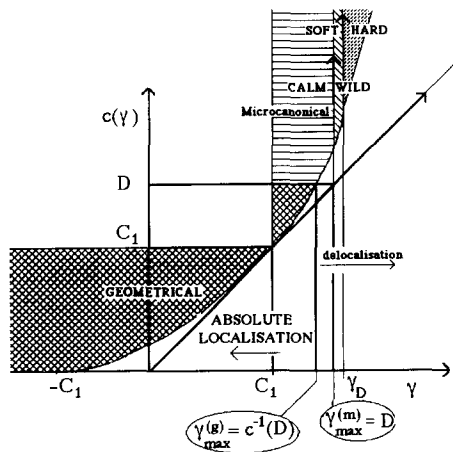

Fig. 2. Phase diagram showing the attainable singularities for a multifractal observed on a space dimension D. It shows the mean singularity ($C_1 = c(C_1)$), the maximum singularities of geometrical multifractals $\gamma_{max}^{(g)}$ ($=c^{-1}(D)$), and microcanonical multifractals $\gamma_{max}^{(m)}$ ($=D$). Wild singularities ($\gamma \ge \gamma_{max}^{(m)}$) violate microcanonical conservation. Hard singularities ($\gamma > \gamma_D > \gamma_{max}^{(m)}$) cause divergence of high order statistical moments. The bonding curves are the two extreme universal cases ($\alpha = 0, 2$).

codimension $C(q) = K(q)/(q-1)$, which is the slope of the chord connecting the points $(1, 0)$ and $(q, K(q))$, q_D corresponds to the (critical) chord having slope $C(q_D) = D$, qth order moments diverge as soon as $C(q) \geq D$. Note that the general relation between τ_D and K (eq. (7)) no longer holds as soon as $q \geq q_D$.

This "hard" behavior is at first sight surprising because if we consider $\varepsilon_{\lambda, D} = \Pi_\infty(B_\lambda)/\mathrm{Vol}(B_\lambda)$ as an estimate of ε_λ over the ball B_λ; the two will therefore have totally different statistical properties:

$$\langle \varepsilon_\lambda^q \rangle < \infty, \quad \text{all } q, \qquad \langle \varepsilon_{\lambda, D}^q \rangle = \infty, \quad q \geq q_D \ (>1). \tag{10}$$

This difference between the properties of the cascade constructed down to scale λ, and that of a coupled cascade integrated (smoothed) over the same scale, prompted us [9] to denote the two quantities "bare" ε_λ and "dressed" $\varepsilon_{\lambda, D}$ respectively. One may note that the singular statistics (of dressed quantities) has been taken as a basic feature of self-organized criticality [13]. Not too surprisingly, the "hard" singularities, which are responsible for this divergence, also break the microcanonical conservation of the flux (Π) (i.e. conservation on individual realizations). Indeed, D is the upper bound of the "calm" singularities ($\gamma \leq \gamma_{\max}^{(m)} = D$, the superscript m corresponding to "microcanonical") respecting the microcanonical conservation. This bound must satisfy: $\Pi_\lambda \geq \lambda^{-D}\lambda^{\gamma_{\max}^{(m)}}\Pi_1$. It is reached only for the extreme case in which for each step all the density of the flux is concentrated on a single subeddy (of volume λ^{-D}). $\gamma_{\max}^{(m)}$ is smaller than the critical order γ_D ($= K'(q_D)$) of hard singularities ($\gamma \geq \gamma_D$), since $\gamma_D \geq C(q_D) = D$ due to the convexity of $K(q)$ (at point $(q_D, K(q_D))$ the tangent is steeper than the chord connecting it to the point $(1, 0)$). The width of the interval of 'soft' ($\gamma < \gamma_D$) but "wild" singularities ($\gamma > \gamma_{\max}^{(m)}$) is: $\gamma_D - \gamma_{\max}^{(m)} = K'(q_D) - D = (q_D - 1)C'(q_D) > D$ (since $q_D > 1$, $C' > 0$).

Conversely, if for any reason, the orders of singularities are bounded above by $\gamma_{\max}$ ($c(\gamma) = \infty$, $\gamma > \gamma_{\max}$) the divergence will be suppressed (except when $\gamma_D < \gamma_{\max}$), and the corresponding multifractals will be "soft". For $\gamma_{\max} < \gamma_{\max}^{(m)} = D$, they will be soft and calm. This is exactly the situation in the two most popular varieties of multifractals: the microcanonical multifractals [14–16] just discussed, and the geometric multifractals [4]. In the latter, there is no stochastic process, no probability space, the singularities are merely distributed over geometrical sets whose largest codimension is therefore equal to that of the embedding space (geometric sets cannot have negative dimensions). We therefore obtain $\gamma_{\max}^{(g)} = c^{-1}(D) < \gamma_{\max}^{(m)}$ (fig. 2), confirming that microcanonical variability can be more extreme.

4. Phase transitions and delocalization

A consequence of bounded singularities is the existence of a (possibly infinite) moment order q_{max} $(=c'(\gamma_{max}))$, for which the Legendre transform indicates that $K(q)$ becomes linear: $K(q) = q\gamma_{max} - c(\gamma_{max})$ for $q > q_{max}$ and $C(q) \leq C(\infty) = \gamma_{max} < \infty$. One may note that an analogous behavior occurs for multifractals with singularities bounded below [17]. The (non-analytical) transition to straight line behavior for $q > q_{max}$ (when finite) can be considered a phase transition [6, 7, 18, 19] (also ref. [20] for a "flux dynamics" formulation of turbulent cascades): the multifractal free-energy analogue $C(q)$ remains "frozen" (a second order transition). The spurious scaling arising from under-sampling [19, 21] may lead to underestimate the order of occurrence of the phase transition-like discontinuities. Indeed a (finite) number of samples (N_s) may introduce an effective bound of singularity $\gamma_s < \gamma_{max}$ and a corresponding $q_s < q_{max}$. With help of the "sampling dimension" [20, 21] D_s, $N_s \approx \lambda^{-D_s}$, we can estimate γ_s: $\gamma_s \approx c^{-1}(D + D_s)$. However, harder spurious scaling arises also from divergence of moments [3] for unbounded singularities ($\gamma_{max} = \infty$). This latter case corresponds to a harder phase transition: $C(q)$ does not remain frozen, $C'(q)$ has jumps (a first order transition) whose amplitudes depend on the sample size. As developed and discussed elsewhere, the jump is roughly: $\Delta C'(q_D) \approx D_s/q_D(q_D - 1)$.

Another distinctive feature of random multifractal processes is that contrary to geometric multifractals, their singularities are generally not localized. Consider an "incipient" singularity about a point x: $\gamma_\lambda(x) = \log \varepsilon_\lambda(x)/\log \lambda$, complete localization is obtained when $\gamma(x) = \lim_{\lambda \to \infty} \gamma_\lambda(x)$ is well defined. Localization is usually simply assumed without any justification; this is the source of many difficulties [20] with existing multifractal analysis techniques (such as wavelet analysis). In random processes, $\gamma_\lambda(x)$ will follow a random walk as λ is increased not converging (generally) to any limit. Nonetheless, at any scale λ the (non-local) histogram [9] of the incipient singularities, eq. (1), still holds. Obviously, harder and harder singularities are more and more delocalized.

5. Hard and soft universal multifractals

An important type of multifractals are the universal multifractals [9] obtained by densification – or more generally by mixing – of identical independent multiplicative processes [22]. When the mean flux is conserved, they are prescribed by only two basic exponents, which define the infinite hierarchies of singularities and dimensions: the mean singularities and their codimension C_1 measures the mean inhomogeneity, α $(0 < \alpha < 2)$ measures the deviation from

monofractality (the curvature radius of $c(\gamma)$ at $\gamma = C_1$ is $R_c = 2^{2/3}\alpha$) and is precisely the Lévy index of the generator of the process. The corresponding universal $K(q)$ and $c(\gamma)$ are (only for $c' > 0$ when $\alpha < 2$)

$$K(q) = \frac{C_1 \alpha'}{\alpha}(q^\alpha - q), \quad \alpha \neq 1, \qquad K(q) = C_1 q \log(q), \quad \alpha = 1, \quad (11)$$

$$c(\gamma) = C_1\left(\frac{\gamma}{C_1 \alpha'} + \frac{1}{\alpha}\right)\alpha', \quad \alpha \neq 1, \qquad c(\gamma) = C_1 \exp\left(\frac{\gamma}{C_1} - 1\right), \quad \alpha = 1,$$
$$(12)$$

with $1/\alpha + 1/\alpha' = 1$.

The two extreme universal cases are the monofractal β-model ($\alpha = 0$) and lognormal model ($\alpha = 2$), which bound (an application of eq. (11) has led to a questionable value $\alpha \approx 2.3$ ($>2!$) for hadron jets [23], more recently ref. [24] points out $0.4 \leqslant \alpha \leqslant 0.75$) the attainable singularities in fig. 2. Eqs. (11), (12) show that for $\alpha \geqslant 1$, $C(\infty) = \gamma_{max} = \infty$; hence the processes are "unconditionally hard" (hard for any D), whereas for $\alpha < 1$, $C(\infty) = \gamma_{max} = C_1/(1 - \alpha) < \infty$ (although $q_{max} = \infty$), hence these multifractals are "conditionally soft" (soft for $D > \gamma_{max}$).

6. Conclusions

The wide variety of multifractals processes can be classified from very soft to hard. In the case of universal multifractal processes, this characterization is particularly powerful since it is entirely deduced from the behavior near the mean. This is already the basis of the robust analysis techniques [25], as well as of simulations [9, 22, 26]. For instance, evidence of the divergence of various moments of turbulent atmospheric quantities has suggested for several years that turbulence really is hard; recent works shows it is unconditionally so with $\alpha \approx 1.3$ [27]. On the contrary rain time series exhibit conditionally soft behavior with $\alpha \approx 0.5$ [28].

These results have fundamental implications for the understanding of non-linear processes (e.g. fully developed turbulence) since it characterizes the entire class of the admissible solutions of the corresponding nonlinear equations.

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