Universal multifractal approach to intermittency in high energy physics

S.P. Ratti[1], G. Salvadori[1], G. Gianini[1], S. Lovejoy[2], D. Schertzer[3]

[1] Department of Nuclear and Theoretical Physics, University of Pavia and Sezione INFN, Pavia, Italy
[2] Physics Department, McGill University, Montreal, Quebec, Canada
[3] L.M.D., Université Pierre et Marie Curie, Paris, France

Abstract. A new stochastic approach to intermittency in high energy physics is proposed. It yields to intermittency exponents defined independently of phase-space dimensions; their role in the calculation of generalized moments is discussed. A straightforward application of universal multifractals is suggested and a new parametric technique for phase-space analysis is provided.

1 Introduction

The idea of scaling and self-similarity, used in the investigation of strange attractors and of turbulent phenomena since the late seventies [1–6], has recently found fruitful applications in the area of high energy physics. Although the concept of self-similarity in multiparticle production was not new [7–11], the papers by Bialas and Peschanski in 1986 [12] and 1988 [13] and by Hwa in 1990 [14], have provided practical tools to verify and survey the scaling properties of particle distributions in phase-space, raising a new experimental and theoretical interest in the subject. Both approaches attempted to seek the existence of power-law behaviours for statistical moments of the particle production process as already found to hold for multifractal cascades generated by multiplicative models of intermittency [1–6].

Bialas and Peschanski suggest that in the "spiky" pseudorapidity distributions observed in some high-energy experiments [15, 16] *statistical* fluctuations might hide *dynamical* intermittent patterns. Assuming the statistical noise to be Bernoullian or Poissonian [12, 13], they show how to operate a deconvolution. To get the true normalized statistical moments of a single event pseudorapidity distribution, they suggest measuring the space average of the normalized factorial moments of order q, given by:

$$F_q \equiv \frac{1}{\lambda^q} \sum_{m=1}^{\lambda} \frac{k_m(k_m-1)\cdots(k_m-q+1)}{N(N-1)\cdots(N-q+1)} \qquad (1)$$

λ (the resolution) being the number of bins in which the pseudorapidity interval is divided; k_m and N being the bin

multiplicity and the event multiplicity respectively. They suggest testing the power law:

$$F_q \propto \lambda^{f_q} \qquad (2)$$

predicted by a particular multiplicative cascade model [12] derived in analogy with existing models of turbulence [1–6].

The parameters f_q were called *intermittency exponents* and taken as a measure of the strength of the intermittency effect. The extension of this method, devised for a single event, to a sample of events of different multiplicities is done by simply averaging over the sample, with a slightly different normalization:

$$F_q \equiv \left\langle \frac{1}{\lambda^q} \sum_{m=1}^{\lambda} \frac{k_m(k_m-1)\cdots(k_m-q+1)}{\langle N \rangle^q} \right\rangle, \qquad (3)$$

where $\langle \ \rangle$ indicates the average over the whole sample (ensemble average).

Hwa on the other hand, tries to apply the strange attractor formalism to multiparticle production by introducing the generalized moments of order q, which, at resolution λ, are defined as:

$$G_q \equiv \left\langle \sum_{\text{non empty}} \left(\frac{k_m}{N} \right)^q \right\rangle \qquad (4)$$

where the sum runs *only over the non empty bins*. The author suggests testing the power law:

$$G_q \propto \lambda^{-\tau(q)} \qquad (5)$$

The strange attractor formalism can then be exploited to find a "spectrum" of the singularity dimensions $f(\alpha)$ and the moment dimension spectrum $D(q)$ defined in [5, 14]*. As we shall see, both factorial and generalized moment techniques are appropriate to a *geometrical* multifractals environment. In both techniques the influence of the event

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* α has nothing to do with the degree of multifractality introduced in Sect. 8. We used it here only in coherence with the notation of [14]. See Table 1 for the comparison of the two languages

sample size on the range of reliable estimates of the multifractal features is difficult to evaluate.

In this paper a more general approach to study intermittency of phase-space distributions is presented, adopting the *stochastic* formalism developed by Schertzer and Lovejoy [17] to deal with the onset of turbulence. The advantages offered by the new *stochastic* method over the *geometrical* multifractal approach are evaluated.

In certain cases, the concept of multifractality was introduced in a *geometrical* context because of the need to characterize, through the fractal dimension, the (possibly) different scaling relations of an inhomogeneus phenomenon. However, although embracing a wide class of multifractals, this *geometrical* connotation is limited. In fact, in a *geometrical* case, all fractal dimensions (and codimensions) do not exceed the euclidean dimension D of the support space. On the contrary, in a *stochastic* environment, infinite dimensional probability spaces are available, without limitations on the dimension spectrum and therefore on the orders of singularity. They allow for the existence of much more violent fluctuations and rarer events.

Problems with F_q (3) and G_q (4) arise from the fact that possible *strong* fluctuations at the highest inaccessible resolutions, as well as small scale resummation effects ("*hard*" behaviour) are not accounted for, while it is now clear that the description of the phenomenon at large scale cannot be divorced from the structure of the phenomenon at the microscopical scale. This corresponds to a first order phase transition analog [18], which will be discussed below in Sect. 6.

2 Multiscaling of probability distributions

Generalizing from a "pedagogical" multiplicative cascade model (called the "α-model") Schertzer and Lovejoy [17] derived the following probability scaling law for multifractal fields:

$$Pr(\varepsilon_\lambda \geq \lambda^\gamma) \propto \lambda^{-c(\gamma)}. \qquad (6)$$

Each value of the field ε_λ, at resolution λ, corresponds to a singularity of order γ and its probability distribution is fully determined by the codimension function $c(\gamma)$. The latter is independent of the dimension of the support space; it is a convex increasing function of γ (the larger the singularities, the rarer their occurrence), with a fixed point C_1: $c(C_1) = C_1$. In Fig. 1a an example of codimension function $c(\gamma)$ for conservative fields (see below) is shown.

The codimension function $c(\gamma)$ is a "statistical" exponent, nonetheless it may be given a *geometrical* interpretation whenever $c(\gamma)$ is less than the euclidean dimension D of the embedding space. In fact, one can then define the (positive) dimension function:

$$D(\gamma) = D - c(\gamma), \qquad (7)$$

$D(\gamma)$ corresponding to the fractal dimension of the support of the phenomenon which shows singularities greater than γ. This interpretation can, in principle, be useful in data analysis. Consider for instance the rapidity distribution of a sample of N_s events, regarded as statistically
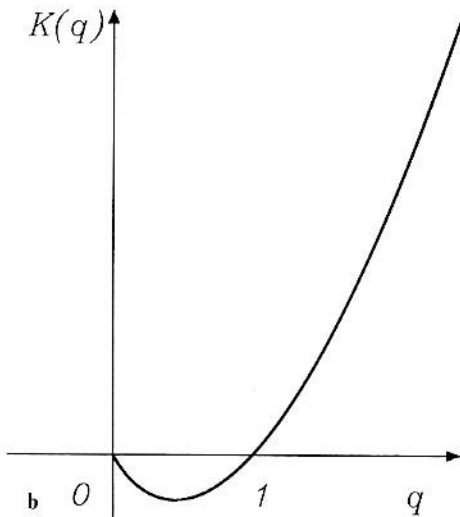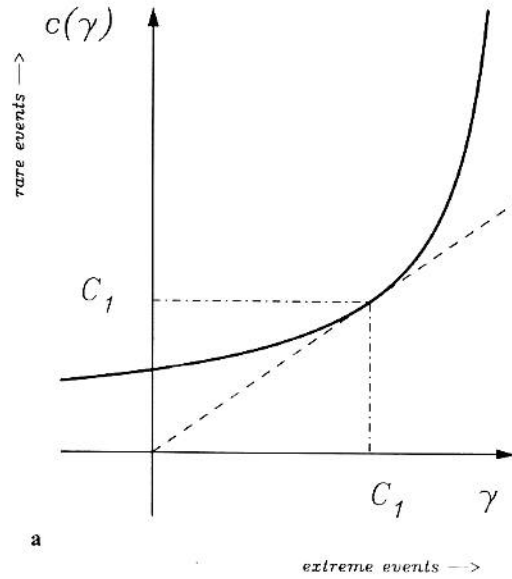

Fig. 1a, b. Example of codimension function **a** and moment scaling function **b**

independent "realizations" of the same *stochastic* process. A single event (dimension-D) will allow the exploration of structures having dimension $D(\gamma)$ between 0 and D; structures with $c(\gamma) > D$ (corresponding to impossible negative values of $D(\gamma)$) will be too sparse to be observed and almost certainly they will not be present in a single realization [19]. To investigate them, a "large" number of events (realizations) is needed. For reasons discussed in the next section, singularities below the dimension D of the space are called *calm*, singularities exceeding D are called *wild* [20].

The accessible range of singularities, given a sample of N_s events, can be easily estimated. Suppose we observe the process embedded in a space, dimension D, at some resolution λ; by using (6) to guess the maximum order of singularity γ_s, observed at least once, one can write:

$$1 \simeq N_s \lambda^D Pr(\varepsilon_\lambda = \lambda^{\gamma_s}) \simeq N_s \lambda^D \lambda^{-c(\gamma_s)}. \qquad (8)$$

By defining the *sampling dimension D_s*:

$$\lambda^{D_s} \equiv N_s, \quad D_s = \frac{\log N_s}{\log \lambda} \qquad (9)$$

we get the following simple relation:

$$c(\gamma_s) = D + D_s, \qquad (10)$$

Thus, the larger the sampling dimension D_s (i.e. the larger the number of independent realizations used), the wider the spectrum of accessible values of γ. Formula (10) provides a clear indication of the effect introduced by the sample-size on the upper bound of the singularities that can be spanned by a given analysis. For instance, using only the one-dimensional distribution of one event, say the pseudo-rapidity distribution of the JACEE event [15], we can reach at the most singularities of codimension one. By increasing the sample size we can go beyond this limit. It is worth noticing that the attainable limit of $c(\gamma)$ increases logarithmically with N_s. Furthermore, due to the convexity of $c(\gamma)$, γ_s grows more slowly than $\log N_s$.

A detailed description of a procedure, called *probability distribution multiple scaling* (PDMS) *technique*, to directly evaluate the codimension function $c(\gamma)$ for a sample of events can be found in [21–23]. An application to rapidity and pseudorapidity distributions of secondary particles is reported in [19], where ε_λ is defined as: k_m/N, N being the number of event used.

It is however "strictly" improper to apply PDMS to particle production since λ^γ cannot run continuously in the available range. In fact the number of particles per bin varies only by integer values. As a consequence λ^γ will show discontinuities.

However, in order to estimate $c(\gamma)$ the slope is needed, instead of the absolute value of Pr, therefore the discontinuities are somewhat ineffective.

3 Features of the singularities

There are singularities that cannot be handled by the traditional moments (3, 4), born within the framework of *geometrical* multifractals★.

In those multifractals, singularities are localized and their sparseness is limited by the dimension D of the support space; hence γ takes at most the value γ_g, solution of the equation:

$$c(\gamma_g) = D. \qquad (11)$$

On the contrary, in *microcanonical* multifractals (i.e. processes involving conservation of the field density in each individual realization of the process) the singularities are delocalized, although their sparseness is still limited by the dimension D of the embedding space. Considering the extreme *microcanonical* process (involving $(\lambda^D - 1)$ zeroes and a single value λ^{γ_m}), *microcanonical* conservation leads

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★ The following multifractal classification is discussed in more details in [20]
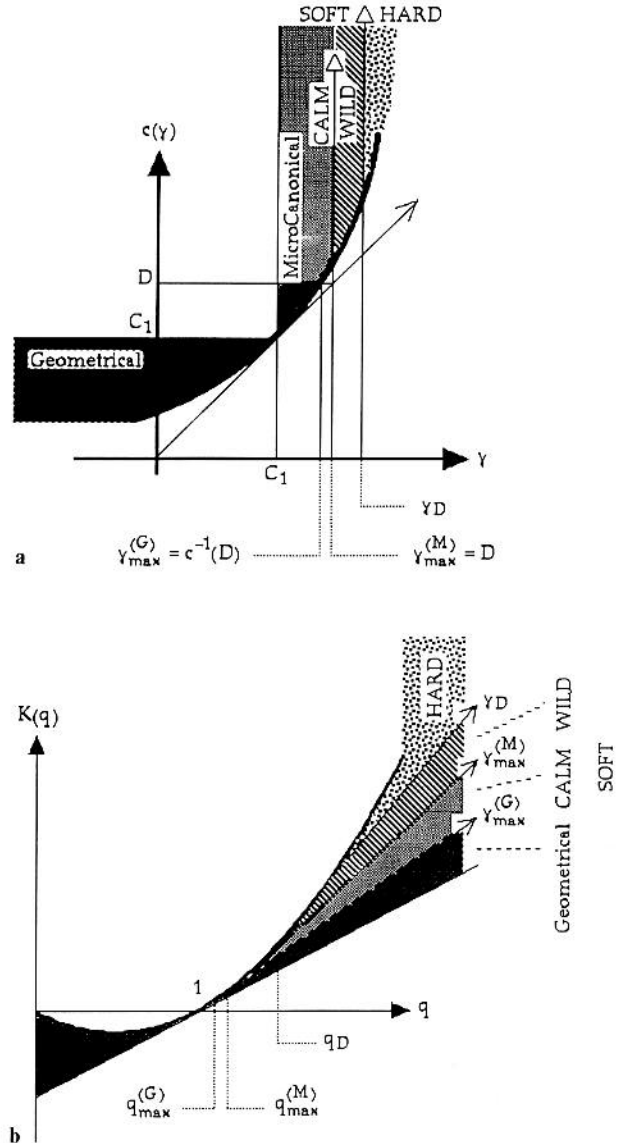

Fig. 2a, b. Schematic diagram showing the classification of (normalized) multifractal processes in terms of the codimension function behaviour **a** and in terms of the moment behaviour **b**. The superscript "G" and "M" indicate, respectively, "geometrical" multifractals and "microcanonical" cascades

to the condition $\lambda^{\gamma_m} \lambda^{-D} = 1$, and to an extreme singularity order:

$$\gamma_m = D. \qquad (12)$$

The limit is not on the value of the codimension function $c(\gamma)$, but rather on the value of its argument, i.e. the value of the singularity order. Hence *geometrical* and *microcanonical* multifractals involve only *calm* singularities.

The occurrence of *wild* singularities is possible in *canonical* multifractals in which conservation within a realization is replaced by the much less restrictive ensemble conservation of field density. The singularities with $\gamma > D$ are called *wild*, because they necessarily violate *microcanonical* conservation. It is also possible to obtain *grandcanonical* statistics by considering the properties of

canonical cascades on random (e.g. fractal) subspaces. A classification diagram for multifractal processes according to the behaviour of their codimension function is visualized in Fig. 2a.

4 Multiscaling of statistical moments

It is important at this point to investigate and discuss how the different moments might scale. Under fairly general conditions the knowledge of the probability distribution of a random variable is equivalent to the knowledge of all its moments. Letting q be their order, Schertzer and Lovejoy [17] introduced the moment scaling function $K(q)$ to characterize the scaling of the moments:

$$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)} \tag{13}$$

which is also independent of the dimension D of the support space since ε_λ is the density of a multifractal measure. It can be shown that $K(q)$ and $c(\gamma)$ are related by two Légendre Transforms [6]:

$$K(q) = \max_\gamma \{q\gamma - c(\gamma)\}$$
$$c(\gamma) = \max_q \{q\gamma - K(q)\} \tag{14}$$

which establish a one-to-one correspondence between γ and q. The order of moment q is strictly related to the order of singularity γ (and viceversa): $q = c'(\gamma)$; $\gamma = K'(q)$. Furthermore, it can be shown that $K(q)$ is convex. In addition $K(0) = K(1) = 0$.

An example of moment scaling function $K(q)$ is shown in Fig. 1b, while Fig. 2b displays the multifractal classification in terms of moment scaling function and translates into moment space the classification given in probability space (Fig. 2a).

The maximum order of moment q_s accessible with a sample of N_s events can be evaluated through $q_s = c'(\gamma_s)$. Since the maximum value γ_s depends upon sample size N_s, its calculation provides an estimate of the maximum order of moments q_s. This accounts only for "undersampling" effects but it is not enough. In order to discuss the role played by the moments in multifractal analysis we have to discuss how the integration performed at a finite resolution is affected by *hard* fluctuations occurring at finer resolutions: they lead to the divergence of statistical moments of larger scale, since they are too violent to be smoothed out by the integration (averaging) process.

It is significant to realize that the statistical properties of a multifractal field at a finite resolution λ – or equivalently the behaviour of $c(\gamma)$ or $K(q)$ – crucially depend upon the way in which the field has been generated. One can distinguish between *bare* and *dressed* properties [17]. *Bare* properties apply to multifractals obtained by developing a cascade down to resolution λ (i.e. no *resumming/integration* performed). *Dressed* properties apply to fields generated (e.g.) by integration, at finite resolution λ, of fully developed cascades ($\lambda \to \infty$). An intermediate class of *finitely-dressed* multifractals is obtained (e.g.) by integrating cascades developed down to resolution Λ at some resolution $\lambda < \Lambda$. Therefore, *bare* quantities are affected only by the behaviour of the field at resolution smaller than λ. It is exactly the *hard* behaviour of multifractals

fields at "small" scales (large resolution) that introduces a basic difference between *bare* and *dressed* properties.

A typical *dressed* quantity is the *flux* of a multifractal field ε through a subset B_λ (a *ball* or *cube* of volume λ^{-D}) of the D-dimensional support space defined as:

$$\Pi_{\Lambda;D}(B_\lambda) = \int_{B_\lambda} \varepsilon_\Lambda \, d^D x \tag{15}$$

($d^D x$ stands for the D-dimensional Hausdorff operator).

In the case of a rapidity distribution of secondary particles, considered on a grid having λ bins we may identify the flux through the m-th bin with the estimated particle density k_m/N.

It is possible to find an estimator which is able to give direct evidence for the divergence of the usual statistical moments. This is a new *dressed* quantity, the *trace moment* of order q proposed by Schertzer and Lovejoy in 1987 [17], which generalizes the "partition function approach" used in strange attractors by ensemble averaging (sometimes called "superaveraging" [17]).

5 Scaling of trace moments

Let us consider a (*bare*) field ε_Λ, resolution Λ, over a support set A covered by disjoint boxes B_λ, resolution λ. The trace moment (TM) of order q is obtained by summing, over each individual realization, the q-th power of the fluxes through the boxes B_λ's (*space average*) and then by averaging over all the available realizations (*ensemble average*), i.e.:

$$Tr_{A_\lambda}(\varepsilon_\Lambda)^q = \left\langle \sum_A [\Pi_\Lambda(B_\lambda)]^q \right\rangle. \tag{16}$$

The trace moments are defined for every positive q and can be shown to be smaller (larger) than the usual statistical moments $\langle \Pi_\lambda^q(A) \rangle$ when $q > 1$ ($q < 1$).

Furthermore, the following scaling relation [17] can easily be obtained:

$$Tr_{A_\lambda}(\varepsilon_\Lambda)^q = \lambda^{K(q)-(q-1)D}. \tag{17}$$

If we define the critical exponent q_D through the equation:

$$C(q_D) = D \tag{18}$$

and the *dual codimension function* $C(q) \equiv K(q)/(q-1)$ – an increasing function – formula (17) reads:

$$Tr_{A_\lambda}(\varepsilon_\Lambda)^q = \lambda^{(q-1)(C(q)-D)}. \tag{19}$$

Thus the trace moments diverge when $q > 1$, for $C(q) > D$ and when $q < 1$, for $C(q) < D$. This implies* that the usual statistical moments of fluxes may be not measurable for all orders.

When $q < q_D$ ($C(q_D) < D$) the statistical moments are determined at large scale and there is no significant difference between *bare* and *dressed* quantities; when $q > q_D$ ($C(q_D) > D$) moments of order q are strongly dependent

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* This is due to Jensen's inequality between the trace moment and the moment of the flux [17]. For $q > 1$: $\langle \Pi_\lambda^q(A) \rangle > Tr_{A_\lambda}(\varepsilon_\lambda)$

upon small scale phenomena and are no more macroscopically determined.

When the dual codimension function $C(q)$ has an upper limit, i.e. if $C(\infty) \equiv \lim_{q \to \infty} C(q) < +\infty$, multifractals are said to be conditionally hard, otherwise they are called *unconditionally hard*. With *conditionally hard* multifractals, all orders of moments will converge, as soon as $D > C(\infty)$. Dressed multifractals integrated over spaces with $D > C(\infty)$ are said to be *soft*.

If we consider the singularity corresponding to q_D (i.e. $\gamma_D = K'(q_D)$) we find that $\gamma_D > D$; hence the *hard* singularities $(\gamma > \gamma_D)$, responsible for the divergence of moments are also *wild*. Since, by definition, *geometrical* and *microcanonical* multifractals exclude *wild* singularities, they are always *soft*. For the more general *canonical* multifractals, *wild* singularities can be a problem and it is advisable to integrate the realization over a support of the largest available dimension.

6 Phase transitions in multifractal processes

It has recently been realized [18, 22] that phase transition-like discontinuities in the derivatives of the function $C(q) = K(q)/(q-1)$ can be wilder than previously expected (those phase transitions are produced in a generic scaling way, contrary to previous [24] hypotheses of scale breaking). Let us consider in fact the function $C(q)$ determined from a finite sample (note that $C(q)$ can be regarded as the "multifractal free energy" function, in analogy with thermodynamics). Since $q_s = c'(\gamma_s)$ is the maximum order of moment that can be reliably estimated due to finite sample-size, we obtain (via Légendre Transform) a "spurious" linear evaluation of K_L instead of the non linear K for $q > q_s$:

$$K_L(q) = \gamma_s(q - q_s) + K(q_s) \tag{20}$$

and hence there is a jump in the second derivative of $C(q)$ (i.e. a *second order* transition):

$$\Delta C''(q_s) = \frac{K_L''(q_s) - K''(q_s)}{q_s - 1} = \frac{-K''(q_s)}{q_s - 1}. \tag{21}$$

Furthermore, we can show that another more violent *first order* transition may occur when the sample size (i.e. the sampling dimension D_s) is large enough so that $q_s > q_D$, i.e. the "hidden" hard behaviour becomes visible. We obtain, by using definition (10):

$$C(q) \simeq \frac{(q - q_D)(\Delta_s/q_D)}{q - 1} \tag{22}$$

for $\Delta_s \simeq D_s \gg D$, which is an improvement with respect to previous papers [17]. Hence there is a jump in the first derivative of C roughly given by:

$$\Delta C'(q_D) \simeq \frac{D_s}{q_D(q_D - 1)} \tag{23}$$

i.e. a first order phase transition. We stress however that such effect will be visible only for extremely large samplesizes.

Table 1. Comparison between notations used in strange attractor formalism [6] and turbulence formalism [17]

Quantity	Strange attractors (measures)	Turbulence (density of measures)
Probability/flux	p	$\Pi(A) = \int_A \varepsilon \, d^D x$
Singularity order	α_D	$D - \gamma$
Singularity dimension	$f_D(\alpha_D)$	$D - c(\gamma)$
Moment order	q	q
Moment dimension	$D(q)$	$D - C(q)$
Scaling exponents	$-\tau_D(q)$	$K(q) - (q-1)D$

7 Comparison between strange attractor and turbulence formalisms

It is worth pointing out that the trace moments given by (16) coincide with the generalized moments (4) introduced by Hwa in 1990. In fact formula (16), due to definition (15), when applied to a rapidity distribution $(D = 1)$ simply leads to:

$$\text{Tr}_{A_\lambda}(\varepsilon_\lambda)^q = \left\langle \sum_{\text{non empty bins}} \left(\frac{k_m}{N} \right) \right\rangle. \tag{24}$$

Table 1 summarizes the glossary of the terms used in strange attractors and turbulence environment respectively. The "turbulence" quantities γ, $c(\gamma)$, $C(q)$, $K(q)$, are related to "measure densities" and are intrinsic to the process, whereas the strange attractors quantities also depend upon the dimension D of the observing space (hence the subscripts).

In conclusion, the trace moment technique allows the determination of the scaling component $K(q)$ or the dual codimension function $C(q)$ which characterizes the statistics of multifractal measures. However they are easily estimated only as long as $g < min (q_s, q_D)$. The same limitations apply obviously to the generalized moments (4). One needs a formalism which may point out the limits beyond which the traditional statistical moment calculation becomes ineffective.

8 Universal multifractals

In order to clarify the theoretical questions involved in our study we make a series of general comments on multifractals. It is clear at this stage that, in order to fully specify the multiple scaling of an arbitrary multifractal field, we need to know an infinite set of values of the scaling parameters, i.e. the whole function $c(\gamma)$ or the whole function $K(q)$.

This is a very disturbing inconvenience both from a theoretical and an experimental (numerical) point of view. It is worth recalling however that if a class of phenomena shows some universality properties then the large number of relevant parameters may eventually reduce to a reasonably small number (e.g. the familiar linear universality in drunkard's walk, where gaussian behaviour often occurs). Therefore the clue of the problem is to seek a class of multifractals exhibiting universal properties.

It has been shown [17, 20] that by appropriately re-normalizing the product of independent multifractal processes the resulting field exhibits universal properties, in spite of the possible huge complexity of the single process and the nonlinear relations between them. Here universality means that the statistical properties of the system depend upon a *finite* and usually *small* number of parameters. That is to say the estimate of a limited number of parameters may fully describe the statistics of the multifractal process.

Universality is obtained on a fixed scale ratio ($\Lambda < \infty$) by increasing the number of independent identical distributed interacting processes ($n \to \infty$). Characteristics of universality in the sense specified above are shown by a class of multifractal fields for which the codimension function $c(\gamma)$ and the moment function $K(q)$ are specified as a function of *only three* parameters [17, 23]. These are called *universal multifractals*.

The universal $c(\gamma)$ and $K(q)$ are (see Fig. 3a–b):

$$c(\gamma - H) = \begin{cases} C_1 \left(\dfrac{\gamma}{C_1 \alpha'} + \dfrac{1}{\alpha} \right)^{\alpha'} & \alpha \neq 1 \\ C_1 \exp\left(\dfrac{\gamma}{C_1} - 1 \right) & \alpha = 1 \end{cases} \tag{25}$$

$$K(q) + qH = \begin{cases} \dfrac{C_1}{\alpha - 1}(q^\alpha - q) & \alpha \neq 1 \\ C_1 q \log q & \alpha = 1 \end{cases} \quad (q \geq 0 \text{ for } \alpha < 2) \tag{26}$$

$$\frac{1}{\alpha} + \frac{1}{\alpha'} = 1. \tag{27}$$

The Lévy index α, also called *degree of multifractality*[*], is the most significant one. It may assume values ranging in the interval $0 \leq \alpha \leq 2$. It specifies the probability class of the process: $\alpha = 0$ defines monofractal processes; $\alpha = 2$ is the maximum degree of multifractality and it defines multifractals with gaussian generators; values $1 < \alpha < 2$ define processes with Lévy generators and unbound singularities; $\alpha = 1$ defines a process with Cauchy generator and $0 < \alpha < 1$ is a process with bound singularities.

The parameter C_1 is always the fixed point of the codimension function $c(\gamma)$ defined in Section 2 ($c(C_1) = C_1$, see Fig. 1a) and represents the codimension of the singularities contributing to the average intensity of the field.

Figure 3 shows an appropriate redrawing of Fig. 1 and Fig. 2 for the function $c(\gamma)$ (Fig. 3a) and $K(q)$ (Fig. 3b) provided by Universal Multifractals (for sake of simplicity $H = 0$). In Fig. 3a the dependence of $c(\gamma)$ from γ is displayed for different values of the degree of multifractality α. For sake of generality the curve is normalized to the value C_1 of the fixed point. The same normalization applies to Fig. 3b where the curve of $K(q)$ for the five main classes of universal multifractals is shown.

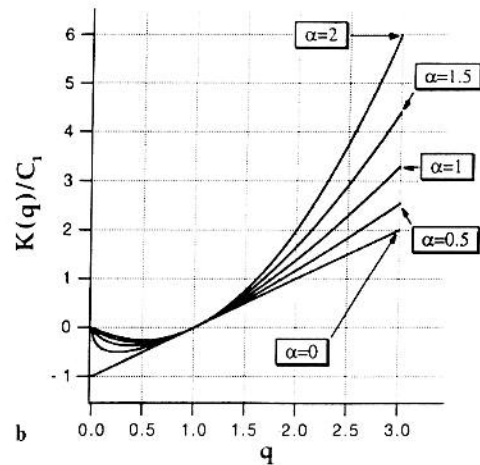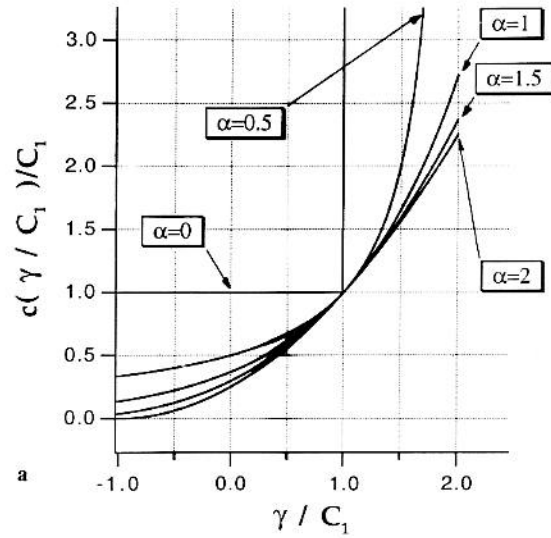The parameter H (called "degree of non conservation") arises from the fact that the processes may be not

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[*] This index α should not be confused with the one used in the strange attractor formalism (see Table 1) mentioned also in Sect. 1

Fig. 3a, b. Schematic diagram showing the typical behaviour of $c(\gamma)/C_1$ **a** and $K(q)/C_1$ **b** for each of the five main classes of universal multifractals

conservative. A conservative process ($H = 0$) is defined when the mean value of the field is constant for varying resolution; while for a non-conservative process ε_λ, the mean value changes with the resolution λ. For a multiplicative process, the conserved field ε_λ is obtained from a non conservative field ϕ_λ as [20]:

$$\phi_\lambda = \varepsilon_\lambda \lambda^{-H}. \tag{28}$$

Without entering into a detailed discussion of the different cases [17], we point out that three parameter specify processes ranging from a *geometrical* rigid monofractal to a lognormal multifractal. No values $\alpha > 2$ are possible. Thanks to the Universal Multifractal parameterization, the limitations due to undersampling can be better understood.

The maximum order of singularity and the maximum order of moments attainable with a given sample size and for a given value of α and C_1 can be proven [17, 20] to be

the following:

$$\gamma_s = \begin{cases} C_1 \alpha' \left[\left(\dfrac{D + D_S}{C_1} \right)^{1/\alpha'} - \dfrac{1}{\alpha} \right] & \alpha \neq 1 \\[3mm] C_1 \left[\log \left(\dfrac{D + D_S}{C_1} \right) + 1 \right] & \alpha = 1 \end{cases}$$

$$q_s = \frac{d\,c(\gamma)}{d\gamma} \bigg]_{\gamma = \gamma_{max}} = \left(\frac{D + D_S}{C_1} \right)^{1/\alpha}.$$

Furthermore the critical exponent q_D for the divergence of moments can be recovered [17, 20] by means of the following equations:

$$\begin{cases} \dfrac{q_D^2 - q_D}{q_D - 1} = \dfrac{D}{C_1}(\alpha - 1) & \alpha \neq 1 \\[3mm] \dfrac{q_D \log q_D}{q_D - 1} = \dfrac{D}{C_1} & \alpha = 1. \end{cases}$$

As long as the order q of the trace moments is lower than q_s and than q_D, meaningful moment estimates can be performed.

On the other side, after having adopted a universality hypothesis, our aim must no longer be the simple calculation of scaling exponents, but the determination of the universal multifractal parameters themselves.

9 The estimate of the universal multifractal parameters

The estimate of the universal parameters α, C_1 and H can be reached through several techniques. Some are mentioned in the present paragraph, together with the concerns that can arise from their use. The description of a more robust procedure is given in Sect. 10.

A very useful relation makes it easy to estimate the H parameter. It involves the application of the D-dimensional Fourier transform $\mathscr{F}_{\varepsilon_\lambda}(k)$ to the field under investigation. $E(k)$ obeys the scaling relation:

$$E(k) \simeq |\mathscr{F}_{\varepsilon_\lambda}(k)|^2 k^{D-1} \propto k^{-\beta}, \tag{29}$$

where $E(k)$ is the *spectrum* at wave number k and β is the *spectral slope*. Since the Fourier spectrum corresponds to the second order moment [2] the following relation holds:

$$\beta = 1 - K(2) \tag{30}$$

where $K(2)$ is the second order moments scaling function. The use of β in (26) gives immediately an estimate of H as a function of α and C_1.

Also the determination of α and C_1 can be achieved using numerical techniques, from the estimated $K(q)$. The parameter C_1, for instance, is given by the slope of $dK(1)/dq$. Using (26), a possibility [25, 26] to determine α from $K(q)$ (when $\alpha \neq 1$) is, for $H = 0$, to fix a value of q, say $\bar{q}$ and then to consider that the ratio $K(q)/K(\bar{q})$ is a function of q and of the α parameter only. The value of α can be estimated through a fit of the function:

$$K(q) = \frac{K(\bar{q})}{\bar{q}^\alpha - \bar{q}}(q^\alpha - q). \tag{31}$$

The reliabilities of these methods [25, 26], crucially depends upon a correct calculation of the scaling exponents $K(q)$, so that the problem remains of how to evaluate C_1 and α, properly taking into account the limitations introduced by *hard* singularities and by undersampling. Furthermore, if we first estimate $K(q)$ by mean of the trace moment technique in a region safe from moment divergence and with suitable statistics, and then perform a non linear regression to get α and C_1, the strong correlations might lead anyway to very poor parameter estimates [21].

10 The double trace moment technique

The *double trace moment* (DTM) technique [27] overcomes most of the problems of the more common universal parameter determination techniques, providing a robust and rather direct estimate of α and C_1.

Let us define the η-*flux*:

$$\Pi^\eta_{A;D}(B_\lambda) = \int_{B_\lambda} \varepsilon_A^\eta \, d^D x \tag{32}$$

that is the flux, resolution λ, of an arbitrary power η of the field ε_A, resolution $\Lambda > \lambda$, as a generalization of the flux (15). The double trace moments are defined as:

$$\mathrm{Tr}_{A_\lambda}(\Pi^\eta_{A;D}(B_\lambda)^q) = \left\langle \sum_A \Pi^\eta_{A;D}(B_\lambda)^q \right\rangle \tag{33}$$

that is a generalization of the trace moments (16) essentially by considering the field ε^η with $\eta \neq 1$. In analogy with (17) the scaling relation for the double trace moments reads:

$$\mathrm{Tr}_{A_\lambda}(\Pi^\eta_{A;D}(B_\lambda)^q) = \lambda^{K(q,\eta) - (q-1)D}. \tag{34}$$

For universal multifractals the scaling exponent allows the useful factorization:

$$K(q, \eta) = \eta^\alpha K(q, 1). \tag{35}$$

Hence, by keeping q fixed (but different from the special values of 0 and 1) and studying the scaling properties of the double trace moments for various values of η, one can determine the scaling exponent $K(q, \eta)$ as a function of η: it can be done through (34) written as:

$$\log \mathrm{Tr}_q^\eta = [K(q, \eta) - (q-1)D] \log \lambda. \tag{36}$$

Then, thanks to (35) written as:

$$\ln K(q, \eta) = \alpha \ln \eta + \ln K(q, 1), \tag{37}$$

the parameter α can be estimated from the slope of the $\log|K(q, \eta)|$ vs. $\log \eta$. The accuracy of these estimates can be verified by repeating the operation with various values of q. The double trace moment technique is therefore a powerful tool to determine the degree of multifractality α.

It is worth pointing out that formula (37) is true for the *bare* moments with $N_s \rightarrow \infty$; it will break down for finite sample sizes and *dressed* moments beyond the divergence threshold, the accurate criterion for validity being $\max(\eta, q\eta) < \min(q_D, q_s)$.

As for the estimate of the parameter C_1, by inverting (26), given α, one obtains, for conserved fields ($H=0$):

$$\begin{cases} C_1 = \dfrac{(\alpha-1)\,K(q)}{q^\alpha - q} & \alpha \neq 1 \\[2ex] C_1 = \dfrac{K(q)}{q\log q} & \alpha = 1. \end{cases}$$

It is important to notice that the accuracy on C_1 will depend upon the accuracy on α.

11 Application to experimental data

The universal multifractal analysis, using the DTM technique, to estimate the parameter α and C_1 for the conservative case ($H=0$) has been performed by Ratti et al. [19] on semiinclusive multiparticle production from hadron-hadron collisions at $\sqrt{s}=16.7$ GeV. The topological space used might be as large as $D=3$, i.e. rapidity $- p_\perp^2$ and azimuthal angle ϕ, but the analysis is limited to the rapidity space only. It is clear that events with different charged multiplicity may not have the same "sparseness" in rapidity (or pseudorapidity) space.

Applying the double trace moments technique to the semiinclusive rapidity distribution the authors of [19] have identified the range of values for η in which the log$-$log dependence is linear and where therefore the degree of multifractality α can be estimated: α and C_1 vary by about a factor two ranging from charged multiplicities $N_{\mathrm{ch}}=6$ to $N_{\mathrm{ch}}=20$. More precisely, for $h-h$ collisions at $\sqrt{s}=16.7$ GeV, $\alpha \approx 0.4$ $C_1 \approx 0.6$ for $N_{\mathrm{ch}}=6$; $\alpha \approx 0.8$ $C_1 \approx 0.3$ for $N_{\mathrm{ch}}=20$.

Other attempts to estimate the universal multifractal parameter α without the double trace moment technique have been made by Brax and Peschanksi [25] and Ochs [26]. The latter, following the procedure indicated by formula (31), performs a fit of the function:

$$K(q) = \frac{K(2)}{2^\alpha - 2}(q^\alpha - q) \tag{38}$$

obtaining the value $\alpha=1.6$. This estimate is not directly comparable to the values obtained in [19], mainly because Ochs uses the intermittency exponents, calculated through factorial moments, as an estimate of the scaling components $K(q)$, usually derived from trace moments. Furthermore in [19] a semiinclusive analysis of $h-h$ interaction is performed, while Ochs uses inclusive data.

We may finally note that Brax and Peschanski [25], using data from heavy ions collisions, obtain the value $\alpha \simeq 2.4 > 2.0$, which cannot correspond to any multifractal process originated by a Lévy generator.

While we claim that the universal multifractal approach provides a significant improvement in understanding intermittency phenomena, the most difficult problem is how to estimate the errors on the calculated parameters (the involved random variables may show for instance diverging variances).

12 Conclusions

In this paper the *stochastic* statistical approach to multifractal fields has been shown to possess a more general validity than geometrical approaches. The basic codimension functions $c(\gamma)$ and $C(q)$ are independent of the dimension D of the support space over which the process occurs.

The need of properly taking into account small scale *hard* fluctuations, that might affect the statistical fluctuation estimate at large scale, is pointed out. Failure in considering those effects may affect the reliability of the statistical description of the phenomenon. The same effects are neglected in the current use of both the factorial moments F_q and the generalized moments G_q.

The use of double Trace moments applied to universal multifractals introduces the possibility of explicitly estimating the range of validity of the statistical technique used. This is not possible otherwise. Any attempt to separate the contributions originated by different statistics is uninteresting.

The application of universal multifractals and of double trace moments to the high-energy data provides more information on the statistical properties of multiparticle production processes.

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