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Empirical study of multifractal phase transitions in atmospheric turbulence

F. Schmitt[1], D. Schertzer[1], S. Lovejoy[2*] and Y. Brunet[3]

[1] Laboratoire de Météorologie Dynamique (CNRS), Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris Cedex 05, France
[2] Département de Physique, Université McGill, 3600 rue Université, Montréal (Québec), H3A 2T8, Canada
[*] present address L.M.D., Paris, France
[3] INRA Bioclimatologie, 71, Avenue Edouard-Bourleaux, BP 81, 33883 Villenave d'Ornon Cedex, France

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Abstract. We study atmospheric wind turbulence in the framework of universal multifractals, using several medium resolution (10 Hz) time series. We cut these original time series into 704 scale invariant realizations. We then compute the moment scaling exponent of the energy flux $K(q)$ for 4 and 704 realizations, in order to study qualitative differences between strong and weak events associated with multifractal phase transitions. We detect a first order multifractal phase transition of the energy flux at statistical moment of order $q_D \approx 2.4 \pm 0.2$: this means that when the number of realizations increases, moments order $q \geq q_D$ diverge. These results are confirmed by the study of probability distributions, and wind structure functions. A consequence of these findings is that it is no use to compare different cascade models in turbulence by using the high order wind structure functions, because a linear part will always be encountered for high enough order moments. Another important implication for multifractal studies of turbulence is that the asymptotic slope of the scaling moment function is purely a function of sample size and diverges with it; it implies the same for D_∞, which has often be considered as finite.

1 Introduction

Universal multifractals (Schertzer and Lovejoy, 1987, 1989, 1991; Fan, 1989; Lovejoy and Schertzer, 1990; Brax and Peschanski, 1991; Kida, 1991) have been shown to describe well the high variability of atmospheric and wind tunnel turbulence data (Schmitt et al., 1992b, 1992a, 1993; Schmitt, 1993). Here we focus on "multifractal phase transitions" (Schertzer and Lovejoy, 1992; Schertzer et al., 1993; Schertzer and Lovejoy, 1994c) which are predicted to occur with spatially or temporally averaged empirical data, and which are connected with (nonclassical) self-organized criticality (Bak et al., 1987). To do this, we compare the empirical scaling behaviour with the theoretical one for a large database, and we analyse the tails of the probability distributions.

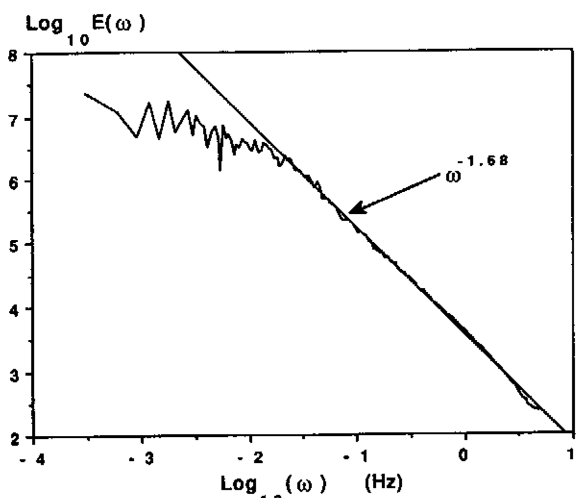

Fig. 1. The spectrum of wind velocity measurements sampled at 10 Hz. There is a scaling behaviour for frequencies from $\omega/1000 = 0.01$ to $\omega/2 = 5$ Hz: the data are scaling over a scale ratio of 500. The slope of the spectrum is close to Kolmogorov scaling: $E_v(\omega) \approx \omega^{-\beta}$ with $\beta \approx 1.68 \pm 0.02$.

2 Wind database and its transformation into energy flux

2.1 Turbulent wind velocity database

The turbulent velocity measurements we study here were obtained over a pine forest in south-west France, using a sonic anemometer located 25 m above ground, sampling at ω=10 Hz. Using the mean wind speed of 1.8 m/s, this acquisition rate corresponds to wind data averaged over scales of about 18 cm. Figure 1 shows the average energy spectrum of the velocity fluctuations of 22 profiles of duration 55 minutes each. It presents a scaling

Correspondence to: F. Schmitt

behaviour for frequencies from about $\omega/1000 = 0.01$ to $\omega/2 = 5$ Hz. In this "inertial" range, the energy spectrum follows a power-law behaviour:

$$E_v(\omega) \approx \omega^{-\beta} \quad (1)$$

where the slope β is close to the Kolmogorov value -5/3 (Kolmogorov, 1941; Obukhov, 1941): we obtain $\beta \approx 1.68 \pm 0.02$.

In order to obtain series whose smallest scale is within the scaling regime, we averaged and resampled pairs of data points. Finally we considered each consecutive section with 512 such points as separate realizations. This gave us 704 different realizations.

2.2 Transformation of the inertial range wind data into energy flux

The rate of energy transfer is, at dissipation scales (i.e. scales of the order of the Kolmogorov scale) equal to the dissipation (e.g. Monin and Yaglom, 1975):

$$\epsilon(x) = \frac{\nu}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 \quad (2)$$

where ν is the kinematic viscosity. This relation has often been used (e.g. Monin and Yaglom, 1975; Meneveau et al., 1990; Meneveau and Sreenivasan, 1991; Chhabra and Sreenivasan, 1991) to estimate (via the Taylor and isotropy hypotheses) the energy dissipation ϵ as $(\frac{\partial v}{\partial t})^2$. If this transformation can ever be justified for fluxes, it is for sampling frequencies high enough to reach dissipation scales. Since our resolution was two or three orders of magnitude lower than the dissipation scale, we proposed another way of estimating the energy flux (Schmitt et al., 1992b, 1992a, 1993; Schmitt, 1993), using the "refined similarity hypothesis" (Kolmogorov, 1941, 1962):

$$\Delta v_\lambda \approx (\epsilon_\lambda)^{1/3}\lambda^{-1/3} \quad (3)$$

where $\lambda = L/\ell$ is the scale ratio (L is the outer scale, and ℓ is the scale of interest) and $\Delta v_\lambda \equiv |v(x+\ell) - v(x)|$. Equation 3 has recently been questionned both on theoretical (Frisch, 1991) and on empirical grounds (Hosokawa and Yamamoto, 1992), but careful empirical studies have shown its validity (Chen et al., 1993; Grossmann and Lohse, 1993; Praskovsky, 1992; Stolovitzky et al., 1992).

In order to estimate the energy flux ϵ_λ we exploit Eq. 3 and perform a fractionnal integration of the wind field of order 1/3 (i.e. a power law filter of order $\omega^{1/3}$ in Fourier space), and then take the cube of the absolute value of the result. This removes the $\lambda^{-1/3}$ Kolmogorov scaling, and yields a conservative (and positive) quantity, which is an estimate of ϵ_λ. This transformation conserves the original scaling of the data, and yields a very intermittent field, a small portion of which is shown in Fig. 2. We now turn to the analysis of the scale invariant properties of this field, using powerful new multifractal analysis techniques.

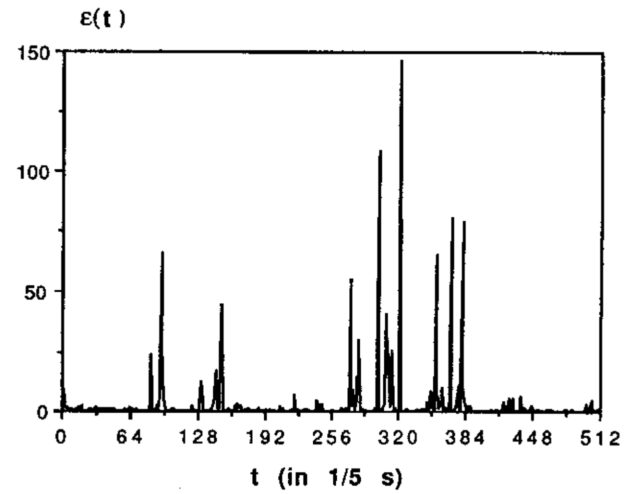

Fig. 2. A portion of the energy flux ϵ_λ, computed from the wind field as indicated in the text (normalized so that the mean=1 for all 704 realizations). This represents 1 realization consisting of 512 points and a total duration of about 1 mn 20 s. This field is very intermittent: on average over the 704 realizations, its value is 1, and some values reach here almost 150, while most of the time they are smaller than 1.

3 Multiscaling description of the turbulent fields: multifractals and universal multifractal relations

3.1 Multiplicative cascades as generic multifractal processes

Multiplicative cascade processes have been shown to generically lead to scale invariant multifractal measures (Schertzer and Lovejoy, 1987; Mandelbrot, 1991). When such cascades have proceeded over a scale ratio $\lambda = L/\ell$ (the ratio of the largest scale to the scale of interest) the density of the rate of energy ϵ_λ flowing from scale $\ell = L/\lambda$ to smaller scales has the singular behaviour (Schertzer and Lovejoy, 1987):

$$\epsilon_\lambda \propto \lambda^\gamma \quad (4)$$

$\gamma > 0$ is an order of singularity and $\gamma < 0$ is an "order of regularity", since as $\lambda \to \infty$ (or $\ell \to 0$), ϵ_λ will respectively diverge or converge to zero. The probability distribution of singularities whose order exceeds γ and the related statistical moments will have the following scaling behaviour (Schertzer and Lovejoy, 1987):

$$\Pr(\epsilon_\lambda \geq \lambda^\gamma) \approx \lambda^{-c(\gamma)} \iff \langle(\epsilon_\lambda)^q\rangle \approx \lambda^{K(q)} \quad (5)$$

where $c(\gamma)$ is the codimension function of the singularities, and is related by a Legendre transform (Parisi and Frisch, 1985) to the scaling exponent $K(q)$ associated with statistical moments ("$\langle\ .\ \rangle$" indicates ensemble averaging):

$$\left\{\begin{array}{l} K(q)+c(\gamma)=q\gamma \\ \gamma=K'(q) \end{array}\right. \quad or \quad \left\{\begin{array}{l} K(q)+c(\gamma)=q\gamma \\ q=c'(\gamma) \end{array}\right. \quad (6)$$

This gives a one-to-one relation between singularities and order of moments. Either the characterization of $K(q)$ or $c(\gamma)$ is enough to describe all the statistics of the multifractal field.

3.2 Universal multifractals and turbulence

When studying either the continuous limit of multiplicative processes or the limit of the multiplicative "mixing" of many processes, stable and attractive multiplicative cascade processes are obtained (Schertzer and Lovejoy, 1987, 1989, 1991; Lovejoy and Schertzer, 1990). Because the resulting statistics depend on few of the details of the elementary process, these are called the *universal multifractals*. For conservative universal multifractals, the functions $K(q)$ and $c(\gamma)$ depend only on two parameters:

$$K(q) = \frac{C_1}{\alpha - 1}\left(q^\alpha - q\right), \quad c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'} \quad (7)$$

with $0 \leq \alpha \leq 2$ (for $\alpha = 1$, we have $K(q) = C_1 q \ln(q)$ and $c(\gamma) = C_1 exp(\frac{\gamma}{C_1} - 1))$, $q > 0$ and $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. These parameters entirely determine the statistics of conservative fields (i.e. those whose mean is independant of the scale). The most important is the Levy index α of the generator of ϵ (see Lévy (1925, 1954) for universality of random processes under addition of random variables). In universal multifractals, it characterizes the degree of multifractality (Schertzer et al., 1991) of the field: it is proportional to the radius of curvature of the codimension function around the mean singularities (monofractals are linear). It is bounded and the extremes correspond to well-known models of turbulence: $\alpha = 0$ for the β-model (Novikov and Stewart, 1964; Mandelbrot, 1974; Frisch et al., 1978) and $\alpha = 2$ for the log-normal model (Kolmogorov, 1962; Obukhov, 1962; Yaglom, 1966). The second parameter C_1 is the codimension of the mean singularities of the field, and measures its mean fractal inhomogeneity. It is also bounded ($0 \leq C_1 \leq d$): $C_1 = 0$ for a homogeneous process, and $C_1 = d$ (the dimension of the embedding space) for a process whose mean intensity is extremely sparse.

3.3 Universal multifractals and empirical moments

Because it is conserved by the nonlinear terms of the Navier-Stokes equations, the turbulent energy flux is commonly considered as being conserved (on average) when an energy cascade is developed from larger scales to dissipation scales. We empiricaly show that the energy flux is indeed a conservative universal multifractal process.

To determine the indices α and C_1, we previously (Schmitt et al., 1992b, 1992a, 1993; Schmitt, 1993) applied a powerful and direct analysis technique – DTM – (double trace moment, Lavallée (1991)) to the energy flux data, which yielded the values: $\alpha \simeq 1.45 \pm 0.1$ and $C_1 \simeq 0.24 \pm 0.05$ for atmospheric, and $\alpha \simeq 1.3 \pm 0.1$ and $C_1 \simeq 0.25 \pm 0.05$ for wind tunnel turbulence. For a superficial comparison with other empirical results, we may calculate the standard intermittency parameter μ which is the autocorrelation exponent for ϵ: $\mu = K(2)$. For the lognormal model we have $\mu = 2C_1$, whereas for the β-model $\mu = C_1$. Here, with $\alpha = 1.45$, we obtain $\mu = C_1\left(2^\alpha - 2\right)/(\alpha - 1) \simeq 1.6C_1 \simeq 0.38 \pm 0.1$ which is close to various empirical estimates (Monin and Yaglom, 1975)[1].

To confirm and illustrate these results, we compute the empirical moment scaling exponent $K(q)$ which describes the scaling of the moments, and we compare it to the "universal fit" obtained for $\alpha = 1.45$ and $C_1 = 0.24$ in equation (7). As shown by Fig. 3, the two functions are in excellent agreement until the moment of order $q_{crit} \simeq 2.5$. What happens to the empirical fields at this critical moment is a "multifractal phase transition" and is studied in the next section.

4 Multifractal phase transitions

4.1 Multifractal phase transitions as statistical thermodynamics analogues

Various analogies can be formulated between multifractal formalisms (as a non-equilibrium flux dynamics) and statistical thermodynamics (Katzen and Procaccia, 1987; Feigenbaum et al., 1986; Schuster, 1988; Chhabra, 1989; Schertzer and Lovejoy, 1991; Schertzer et al., 1993; Schertzer and Lovejoy, 1994c, 1994b); for example the order of singularity γ is analogous to the energy and the codimension function $c(\gamma)$ to the entropy. This can be seen by recalling that in thermodynamics a Legendre transform associates the free energy F to the corresponding entropy S:

$$F(T) = E - TS(E) \quad (8)$$

here T is the temperature. Similarly, a Legendre transform expressed by Eq. (6) relates the scaling moment function $K(q)$ to the codimension function $c(\gamma)$ for multifractal processes. Following this analogy, the moment order q is associated with the inverse temperature β (Schuster, 1988), and the $K(q)$ function to the Massieu potential[2] $\Sigma(\beta) \equiv -\beta F$ (Schertzer et al., 1993; Schertzer and Lovejoy, 1994c, 1994b). Therefore, discontinuities in the first or second derivative of $K(q)$ are called first or second order "multifractal phase transitions", since this corresponds to discontinuities in the derivatives of thermodynamics potentials.

[1] Recently Sreenivasan and Kailasnath (1993) proposed a value of $\mu \simeq 0.25 \pm 0.05$. We believe that this slight difference may comes from their different way of estimating ϵ. We will discuss this question in detail in another paper.

[2] For a presentation of Massieu potential in thermodynamics, one can refer to Balian (1982).

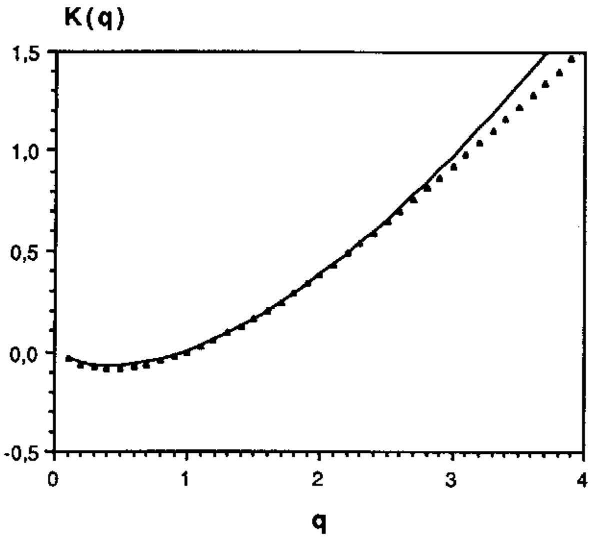

Fig. 3. The empirical moment scaling function (for 4 realizations) for the energy flux ϵ (triangular dots), compared to the theoretical one obtained for universal multifractals and $\alpha = 1.45$, $C_1 = 0.24$ in equation (7) (continuous line). The agreement is excellent until moment order $q_{crit} \simeq 2.5$, after while the empirical curve becomes linear.

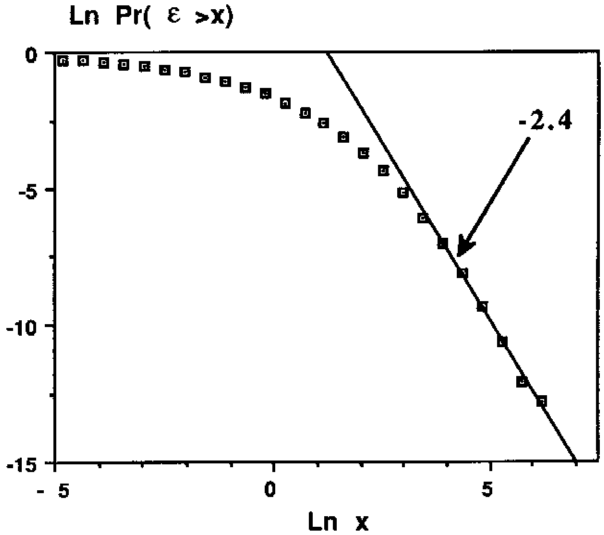

Fig. 4. The probability distribution of the dressed ϵ field at the finest accessible resolution, corresponding to 5 Hz. There is a "hyperbolic" tail: $\Pr(\epsilon > x) \approx x^{-2.4}$ for $x \gg 1$.

Discontinuities in the derivative of $K(q)$ are indeed expected when studying empirical scaling of multifractal fields. Such discontinuities can arise in several ways, that of particular interest here being due to a combination of finite sample size and spatial integration. In order to understand this it must be recalled that empirical data correspond to "dressed" quantities (Schertzer and Lovejoy, 1987), which are spatial (or temporal) averages of the "bare" small scale field; the bare field is the result of a multiplicative cascade developed from large to smaller scales. Below, the dressed scaling functions are denoted $K_d(q)$ and $c_d(\gamma)$, to distinguish them from the bare ones given by Eq. (7). Indeed, contrary to widespead opinion, bare and dressed quantities are often qualitatively different.

4.2 First order multifractal phase transition

When a bare multifractal process is averaged over a D-dimensional space, the resulting dressed quantities will display a divergence of moments order q_D given by (Schertzer and Lovejoy, 1987):

$$K(q_D) = (q_D - 1)\, D \tag{9}$$

where D is an "effective" dimension of dressing, and can be smaller than d, the dimension of the embedding space. Such a divergence is equivalent to having a "hyperbolic" (or algebraic) tail for the probability distribution of the dressed field (i.e. $\Pr(\epsilon > x) \approx x^{-q_D} \quad x \gg 1$). Taking the combination of scaling with hyperbolic probabilities as the defining feature of self-organized criticality (SOC) (Bak et al., 1987), this dressing provides a generic stochastic route to SOC (Schertzer et al., 1993; Schertzer and Lovejoy, 1994c). Therefore, some parallels can be made between the two behaviours: intermittency, dominant influence of larger events, etc.

The hyperbolic behaviour of the probability distribution leads to the following expression for the dressed codimension function (Schertzer and Lovejoy, 1994c):

$$c_d(\gamma) = \begin{cases} c(\gamma) & \gamma \leq \gamma_D \\ q_D(\gamma - \gamma_D) + c(\gamma_D) & \gamma_D \leq \gamma \leq \gamma_{s,d} \end{cases} \tag{10}$$

where $\gamma_D = K'(q_D)$ is the critical singularity associated to the critical order of divergence of moments, and $\gamma_{s,d}$ is the maximum reachable singularity for a finite number of realizations, which is given by (Schertzer and Lovejoy, 1989):

$$c_d(\gamma_{s,d}) = \Delta_s \tag{11}$$

where Δ_s is an effective dimension depending on the dimension of the observing space (d), and the number of realizations studied N_s: $\Delta_s = d + \log(N_s)/\log(\lambda)$. This gives, via the Legendre transform (Eq. (6)), the corresponding expression of the dressed "moment scaling" characteristic function for a finite number of realizations:

$$K_d(q) = \begin{cases} K(q) & q \leq q_D \\ \gamma_{s,d}\, q - \Delta_s & q > q_D \end{cases} \tag{12}$$

This result is a refinement of an earlier argument involving divergence of moments (Schertzer and Lovejoy, 1985) which was used to explain Anselmet et al. (1984)'s velocity structure functions. This last equation shows that:

- for $q \geq q_D$, $K_d(q)$ is linear;
- the slope $\gamma_{s,d}$ of this linear part increases with the number of realizations, and diverges with it;

- the first derivative of $K_d(q)$ is discontinuous at $q = q_D$: there is a *first order multifractal phase transition* at this point.

4.3 Second order multifractal phase transitions

Second order multifractal phase transitions are obtained because of sampling limitations, when the number of realizations studied is finite, but too small to obtain a first order phase transition (γ_D is too large to be observed with the available sample).

In this case, we still have a maximum reachable singularity γ_s given by:

$$c(\gamma_s) = \Delta_s \tag{13}$$

with the condition $\gamma_s < \gamma_D$ showing that the number of realizations is too low for the first order multifractal phase transition to be reached.

Then the dressed codimension function is simply :

$$c_d(\gamma) = c(\gamma) \quad \gamma \leq \gamma_s \tag{14}$$

The Legendre transform then gives the dressed characteristic function (Schertzer and Lovejoy, 1994c):

$$K_d(q) = \begin{cases} K(q) & q \leq q_s \\ \gamma_s q - \Delta_s & q > q_s \end{cases} \tag{15}$$

where the critical moment q_s (the upper bound of the moments giving the bare scaling exponent) is given by $q_s = c'(\gamma_s)$, which gives, in case of universality (with the help of Eq. (7)):

$$q_s = \left(\frac{\Delta_s}{C_1}\right)^{1/\alpha} \tag{16}$$

Equation (15) shows that:

- for $q \geq q_s$, $K_d(q)$ is linear, and follows the tangency of $K(q)$ for $q = q_s$ (the dressed curve is "outside" the bare one);
- q_s depends of the number of realizations, and increases with it, contrary to q_D, which is independent of the number of realizations;
- the second derivative of $K_d(q)$ is discontinuous for $q = q_s$: there is a *second order multifractal phase transition* at this point.

4.4 Relation to the strange attractor/dimension formalism

The codimension multifractal formalism used here is related to the dimension formalism (Halsey et al., 1986) developed to study strange attractors. In particular, we obtain

$$D_q = \frac{\tau(q)}{q-1} = d - \frac{K(q)}{q-1} \tag{17}$$

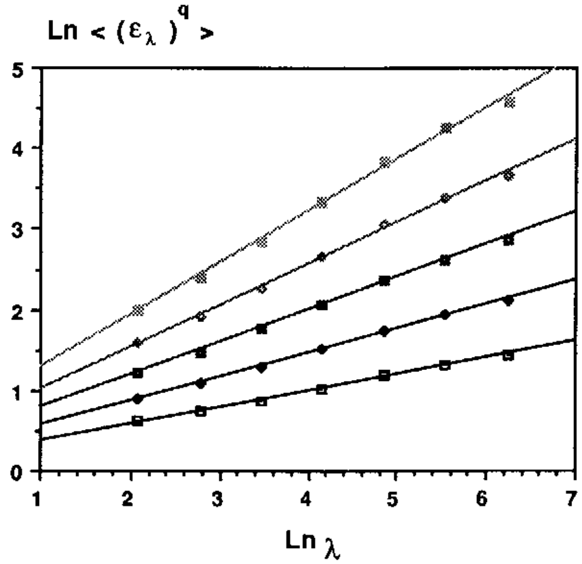

Fig. 5. Moments of the energy flux at different scales, in a log-log plot: the straight lines show the scaling behaviour, here for q=1.6, 1.8, 2, 2.2, 2.4 from bottom to top. The slopes of these lines give estimates of $K_d(q)$.

for the scaling exponent $\tau(q)$ and dimension D_q associated to the q^{th} order moment. This formalism has strong limitations, since the scaling exponents depend on the dimension d of the space in which the process is embedded. Furthermore, Eqs. (12), (15) and (17) show that

$$D_\infty = d - \gamma_{\max} \tag{18}$$

where $\gamma_{\max} = \gamma_{s,d}$ for a first order and $\gamma_{\max} = \gamma_s$ for a second order multifractal phase transition.

The value of D_∞ has been theoretically derived for various turbulence models (see e.g. Bershadskii and Tsinober (1992), and references therein). These models are microcanonical: the energy flux is conserved for each sample. So, there is a maximum singularity $\gamma_{\max}$ given by $\gamma_{\max} = d$, i.e. independent of sample size when $\gamma_s > d$. Nevertheless, when the number of realizations increases, the maximum singularity in Eq. (18) is not bounded, and D_∞ depends on the sample size and diverges with it to $-\infty$ (the existence of D_∞ was already discussed in Schertzer and Lovejoy (1985)). These results show that for general (canonical) multifractals, there is nothing fundamental about this quantity, because the codimensions are not bounded.

5 Empirical study of multifractal phase transitions

5.1 Probability distributions and moments

The empirical probability distribution of energy flux is shown in Fig. 4; there is evidence for a "hyperbolic" tail with an (absolute) slope of $q_D \simeq 2.4$. We analyse below

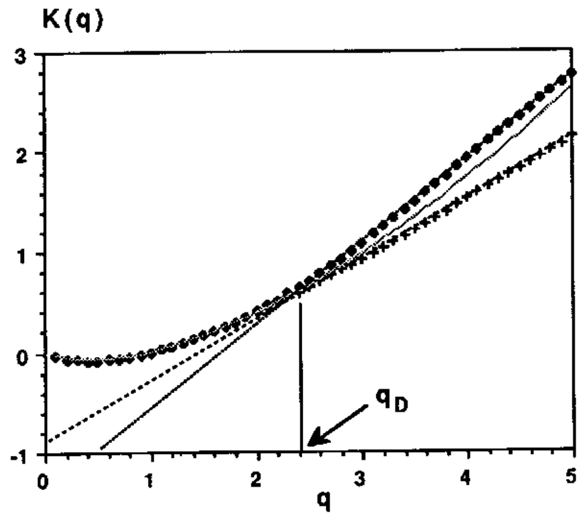

Fig. 6. The empirical scaling exponent moment function $K_d(q)$ estimated for 4 (crosses) and 704 realizations (dots), compared to the theoretical one computed from Eq. 7 and $\alpha = 1.45$ and $C_1 = 0.24$ (continuous line). The different curves are in excellent agreement until moment order $q \leq q_{crit} \simeq q_D \simeq 2.4$. For larger moments, the empirical estimates follow straight lines. For 704 realizations, the straight line is clearly "inside" the theoretical curve, which is only possible for a first order multifractal phase transition.

the behaviour of $K_d(q)$ when the number of realizations varies, in order to confirm this estimate.

We computed the scaling of the moments of the energy flux, as shown in Fig. 5: the statistical moments are shown as functions of the scale ratio, in a log-log plot. The straight lines show that Eq. (5) is valid over a wide range of scales. The slope of these straight lines give estimates of $K_d(q)$.

Figure 6 shows the plot of $K_d(q)$ for 4 and 704 realizations, compared to the universal fit obtained for $\alpha = 1.45$ and $C_1 = 0.24$ in Eq. (7). The different curves are in excellent agreement for moments order $q \leq q_{crit} \simeq q_D \simeq 2.4$. For larger moments, the empirical estimates of $K_d(q)$ follow straight lines, as predicted by Eqs. (12) and (15). We now analyse numerical values in order to distinguish between first and second order multifractal phase transitions.

5.2 Consistency of the numerical values

The straight lines obtained are of the form :

$$K_d(q) = \gamma_{\max} q - \Delta_s \tag{19}$$

with $\gamma_{\max} = \gamma_{s,d}$ for first order, and $\gamma_{\max} = \gamma_s$ for second order multifractal phase transition. Hence $\gamma_{\max}$ can be determined by the slope and Δ_s from the intercept of the asymptote with the K axis. Empirical values and computations using these relations are shown in Table 1, which presents the empirical values of Δ_s and $\gamma_{\max}$ for 4 and 704 realizations. We then compute

Table 1. Empirical values and computations showing:

1. that $q_D < q_s$ for 4 and 704 realizations (q_s estimated with $\alpha = 1.45$, $C_1 = 0.24$), hence that there is a first order multifractal phase transition in all cases;
2. the value of $\gamma_{s,d}$ computed from Δ_s and q_D is very close to the empirical $\gamma_{\max}$ in all cases;
3. $c_d\left(\gamma_{s,d}\right)$ computed from $\gamma_{s,d}$ and q_D is very close to the empirical estimate of Δ_s.

This forms a very consistent set of results.

N_s	4	704
Δ_s empirical	0.90	1.45
$\gamma_{\max}$ empirical	0.61	0.84
q_D	2.4	2.4
$\gamma_D = K'(q_D)$	0.59	0.59
$q_s = (\Delta_s/C_1)^{1/\alpha}$	2.57	3.61
$\gamma_s = K'(q_s)$	0.63	0.80
$\gamma_{s,d} = (\Delta_s + K(q_D))/q_D$	0.63	0.85
$c_d\left(\gamma_{s,d}\right) = q_D\left(\gamma_{s,d} - \gamma_D\right) + c(\gamma_D)$	0.87	1.43

the corresponding values of q_s and γ_s: in each cases, $q_s > q_D \simeq 2.4$ and $\gamma_s > \gamma_D \simeq 0.59$; this shows that the multifractal phase transition is first order. As a confirmation, we verify that $\gamma_{s,d} \simeq \gamma_{\max}$ and $c_d(\gamma_{s,d}) \simeq \Delta_s$. Considering that there are $N_s = 22$ independant realizations (because the 704 realizations come from 22 independant profiles), we can compute the theoretical sampling dimension $\Delta_s(22) \simeq d + \log N_S / \log \lambda \simeq 1 + \log 22/\log 512 \simeq 1.50$, which is in good agreement with the value 1.45 estimated in another way.

Previous relations (Eqs. (10), (11) and (12)) can also be used to quantify the divergence of moments of the dressed quantities: empirical estimates of the moments of order $q \geq q_D$ diverge as:

$$\lambda^{K_d(q)} \approx \lambda^{qK(q_D)/q_D} \left(\lambda^d N_s\right)^{\frac{q-q_D}{q_D}} \propto N_s^{\frac{q-q_D}{q_D}} \tag{20}$$

when $N_s \to \infty$, and for a fixed scale ratio λ.

Thus, for $q > q_D$ and a given scale ratio, the divergence of moments follows a power law on the number of realizations N_s. The above Eq. (20) was derived in a quite different way in Schertzer and Lovejoy (1984).

We have shown here a critical order of first order multifractal phase transition of $q_D \simeq 2.4 \pm 0.2$ for the energy flux. Below we directly analyse the wind data.

6 Velocity structure functions and probability distributions

6.1 Velocity structure functions

Because of Eq. (3), there is a link between the exponents of the wind structure functions and the function $K(q)$ for the energy flux:

$$\zeta(q) = \frac{q}{3} - K\left(\frac{q}{3}\right) \tag{21}$$

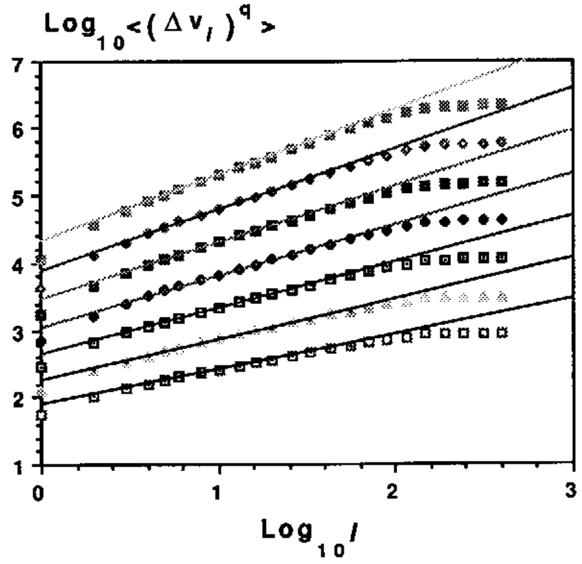

Fig. 7. The structure functions versus ℓ in a log-log plot for q=1.5, 1.75, 2, 2.25, 2.5, 2.75 and 3 (from bottom to top). The straight lines show the scaling ranges.

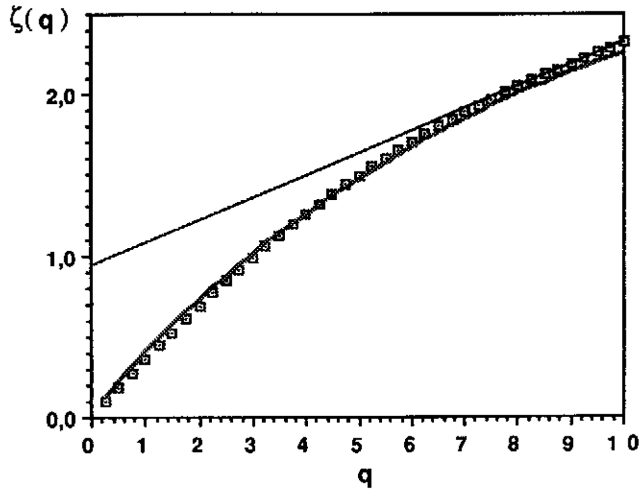

Fig. 8. The structure functions exponents $\zeta(q)$ for theoretical curve given by $\alpha = 1.45$ and $C_1 = 0.24$ in Eq. (21) (continuous line), compared to empirical estimates (rectangular dots). The two curves agree well until moment of order about 7. For larger moments, empirical curve is linear. This empirically confirms that Eq. (3) (Kolmogorov's "refined' similarity hypothesis") holds, at least for scaling behaviour.

where the moment scaling exponent $\zeta(q)$ is given by $\langle(\Delta v_\ell)^q\rangle \approx \ell^{\zeta(q)}$. Using Eq. (21) and the "bare" universal expression of $K(q)$ (Eq. (7)), we can compare empirical and theoretical wind structure function exponents. In Fig. 7, we represent the log of different structure functions versus $\log \ell$. The interpolation straight lines for certain values of ℓ indicate the scaling range. The slopes of these straight lines give estimates of the exponents $\zeta(q)$. These empirical estimates are represented in Fig. 8, and compared with the theoretical curve: the agreement is very good until moment of order about 7, which corresponds roughly to the value $3q_D = 3(2.4 \pm 0.2) \simeq 7.2 \pm 0.6$. For larger moments, the empirical $\zeta(q)$ function is linear. This confirms the validity (at least for the scaling behaviour) of the "refined similarity hypothesis" (Eq. (3)).

For a large enough database, we argue that this critical order of multifractal phase transition is ubiquitous. Therefore, to compare different theoretical cascade models or scale invariant models of turbulence[3], it is no use to compute moments of velocity structure function larger than this critical value. This fact brings into question certain attempts to determine which model of turbulence fits, using the scaling exponents of wind structure functions up to order 18 (e.g. Anselmet et al., 1984; Kida, 1991), and explains other empirical findings considering the large q linear behaviour of the structure functions, or the $K(q)$ function for the energy flux (Bershadskii and Tsinober, 1992).

[3] With the help of generalized scale invariance (Schertzer and Lovejoy, 1983, 1984, 1985, 1987; Pflug et al., 1993), we can talk about space-time multifractal processes: in this context, cascades provide dynamical multifractal models (Schertzer and Lovejoy, 1994a). Other authors (e.g. Parisi and Frisch, 1985) avoid discussion of specific cascade processes, considering purely geometric (and abstract) multifractal characterizations of the scale invariance.

6.2 Velocity probability distributions

The probability distribution of wind velocity shear (at the finest available resolution) is shown in Fig. 9: for sufficently large fluctuations, it displays a hyperbolic tail, whose slope is of the order $q_{D,v} \simeq 7.5 \pm 0.5$, which is very close to the previous estimate of the same critical moment (=3x2.4). Taking into account these two consistent estimates, we propose here the value:

$$q_{D,v} \simeq 7.0 \pm 1 \tag{22}$$

Early estimates corresponding to $q_{D,v} \simeq 5$ (Schertzer and Lovejoy, 1983, 1984), were obtained from vertical velocity measurements and extremely small ($\simeq$ 100 points) histogrammes from horizontal velocity measurements. No estimates of the temporal $q_{D,v}$ have been made up until now. See Chigirinskaya et al. (1994) and Lazarev et al. (1994) for a systematic comparison (with larger databases) between critical order of moment for horizontal and vertical velocity measurements, which tend to support the different vertical exponent ($\simeq 5$).

It should be underlined that previous studies of the probability density or probability distribution of turbulent wind fluctuations (e.g. Anselmet et al., 1984; Castaing et al., 1990; Gagne, 1991; Meneveau and Sreenivasan, 1991; She et al., 1991) have detected exponential tails rather than hyperbolic when the sample size is sufficient to detect a hyperbolic tail. One the one hand, such distributions are ad hoc since they could only hold

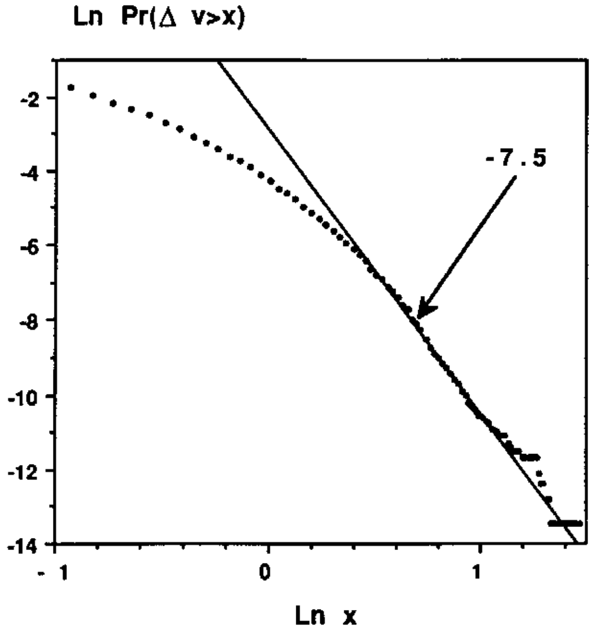

Fig. 9. The probability distribution of the wind shear, showing a hyperbolic tail, of slope $q_{D,v} \simeq 7.5 \pm 0.5$.

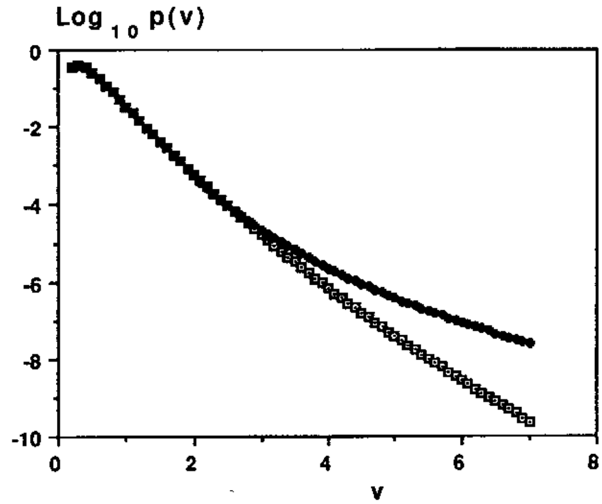

Fig. 10. The theoretical (bare and dressed) wind probability densities in a log-linear plot, with parameters $C_1 = 0.05$, $\alpha = 1.45$, $\lambda = 512$ in Eq. (7) for the bare velocity field (white squares) and $q_D = 7$ in Eq. (10) for the dressed velocity field (black diamonds). Using this log-linear representation the two tails are approximately linear, showing that we are not able to discernate between the two types of tails. Therefore our results look like being compatible with such (ad hoc) exponential regressions, whereas the (theoretical) dressed density is definitely hyperbolic! Note that empirical data can be seen as dressed quantities, except maybe when the sampling frequency is high enough to reach the dissipation scales: in this case they correspond to bare quantities.

exactly at a unique scale (they are incompatible with scaling), on the other hand this may be due to problems of precision of the representation of tails in log-linear or log-log plots. In Fig. 10 we display the theoretical probability density of the wind (in a log-linear plot) with parameters to[4] $C_1 = 0.05$, $\alpha = 1.45$, $\lambda = 512$ for the bare and dressed fields (Eqs. (7) and (10)): as can be seen, for both bare and dressed velocity fields, the tail is roughly linear, starting at $\Pr \simeq 10^{-3}$ to 10^{-7} (the typically available range). Over the observed probability range, our hyperbolic results are therefore visually compatible with (ad hoc) exponential regressions.

7 Conclusions

Analysing the empirical energy flux for an atmospheric wind turbulent database, we showed that a first order multifractal phase transition occurs for moments order $q_D \simeq 2.4 \pm 0.2$ of the "dressed" empirical energy flux. This shows that turbulence intermittency is very hard: the probability distribution of the extreme values of the energy flux is hyperbolic. This means that extreme values are much more frequent than for usual exponential tails; however, because of the very large databases needed the hyperbolic tail has not always been clearly visible. Indeed we develop a quantitative estimate of the necessary sample size.

The empirical scaling exponent function $K_d(q)$ is linear for moments larger than the relatively low critical value of q_D: this shows also that one must be careful when computing high order moments to compare different cascade models of turbulence. Furthermore, the estimates will depend strongly and diverge with the number of realizations in the sample studied, so it is no use to calculate moments of ϵ greater than 2.4 as is often tried, or to estimate moments of structure functions of the velocity fluctuations order greater than about 7. Finally, we have shown that the generalized dimension D_∞ is not a fundamental parameter because it depends on the maximum reachable singularity, which diverges with the number of realizations N_s.

We may underline here that the database we analysed is not so large (700,000 data points), but because of the sampling frequency of 10 Hz, the velocity data already correspond to dressed data. As a comparison, for usual studies of probability distributions of wind fluctuations, the sampling frequency of a few kHz (e.g. Castaing et al., 1990) gave bare velocity quantities (i.e. they reached dissipation quantities), whose fluctuations cannot show hyperbolic tails. It is only when averaged (dressed) sufficently, as in the present study, hyperbolic tails will appear. Furthermore, because of the low sampling frequency of our data, our database would correspond to a 700 million points database sampled at 10 kHz, which would be a much larger database than anal-

[4] As shown in Lavallée (1991), universal multifractals indices α and C_1 are transformed as follows when the field is raised to a power a: $\alpha \to \alpha$ and $C_1 \to C_1 a^\alpha$. Therefore these indices are for the wind field $\alpha \simeq 1.45 \pm 0.1$ and $C_1 = (0.24 \pm 0.05)/3^\alpha \simeq 0.05 \pm 0.01$.

ysed up until now. We also showed here that because of problems of precision of representation, hyperbolic and exponential tails are almost linear when represented in log-linear plots; therefore such plots are not useful in distinguishing the models – the scaling properties must be used.

Our results clearly do not support some recent speculations (e.g. Procaccia and Constantin, 1993; Grossmann and Lohse, 1994) saying that the intermittency corrections of the "trivial" scaling of the structure functions given by K41 theory are due to finite Reynolds number corrections. These speculations are based on a particular cascade model (see Chigirinskaya et al. (1994) for a discussion about models of this genre), and on the other hand, the procedures we followed here were self consistent in checking the scaling behaviour of the wind field for a range of scales between 0.01 and 5 Hz in the atmosphere 25 m above ground, which corresponds to Reynolds numbers of about 5 10^6.

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