

MULTIFRACTAL FLUCTUATIONS IN FINANCE

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We consider the structure functions $S^{(q)}(\tau)$, i.e. the moments of order q of the increments $X(t + \tau) - X(t)$ of the Foreign Exchange rate $X(t)$ which give clear evidence of scaling ($S^{(q)}(\tau) \propto \tau^{\zeta(q)}$). We demonstrate that the nonlinearity of the observed scaling exponent $\zeta(q)$ is incompatible with monofractal additive stochastic models usually introduced in finance: Brownian motion, Levy processes and their truncated versions. This nonlinearity correspond to multifractal intermittency yielded by multiplicative processes. The non-analyticity of $\zeta(q)$ corresponds to universal multifractals, which are furthermore able to produce “hyperbolic” pdf tails with an exponent $q_D > 2$. We argue that it is necessary to introduce stochastic evolution equations which are compatible with this multifractal behaviour.

1. The use of structure functions to discriminate models

Financial markets display some common properties with fluid turbulence, and their fluctuations are often characterized as being “turbulent”. Indeed, as for fluid turbulent fluctuations, financial fluctuations display intermittency at all scales. In fluid turbulence, a cascade of energy flux is known to occur from the large scale of injection to the small scales of dissipation. Since the 1980’s, this cascade is mainly modeled by multiplicative cascades,¹ generically leading to multifractal fields.^{2,3}

In finance, the picture of a cascade of information flowing from large-scale investors to small scale ones has been proposed,^{4–6} and several authors showed empirically that the fluctuations of various financial time series possess multifractal statistics.^{6–9} This corresponds to abandoning the classical Brownian motion picture, and even all other models based on additive processes: fractionnal Brownian

motion, Lévy and truncated Lévy processes. Here we show how structure function analysis is a simple yet powerful tool in comparing the different models.

Assuming statistical time translational invariance, the structure function $S^{(q)}(\tau)$, i.e. the statistical moments of the increment of the Foreign Exchange rate $X(t)$ will depend only on the time lag τ , and according to a power law if the process is scaling:

$$S^{(q)}(\tau) = \langle |X(t+\tau) - X(t)|^q \rangle \sim S^{(q)}(T) \left(\frac{\tau}{T}\right)^{\zeta(q)} \quad (1.1)$$

where T is the fixed largest time scale of the system, $\langle \cdot \rangle$ denotes statistical average (for non-overlapping increments of length τ), q is the order of the moment (we take here $q > 0$), and $\zeta(q)$ is the scale invariant structure function exponent. Structure function analysis corresponds in fact to studying “generalized” average volatilities at scale τ , since only moments of order 1 or 2 are usually used to define the volatility. Furthermore, the present analysis consists in analysing this generalized volatility for all time scales.

The average of the fluctuations correspond to $q = 1$, and $H = \zeta(1)$ is the so-called “Hurst” exponent characterizing the scaling non-conservation of the mean. The second moment is linked to the slope β of the Fourier power spectrum: $\beta = 1 + \zeta(2)$. The main property of a multifractal processes is that it is characterized by a nonlinear $\zeta(q)$ function.¹⁰ This function is convex, being proportionnal to the second Laplace characteristic function of the generator of the cascade.^{1,3} Multifractals are the generic result of multiplicative cascades. A continuous-scale limit of such processes leads to the family of log-infinitely divisible distributions, among which are the universal multifractals,¹ which have a normal or Levy generator, and for which:

$$\zeta(q) = qH - \frac{C_1}{\alpha - 1}(q^\alpha - q) \quad (1.2)$$

where $C_1 \leq d$ is an intermittency parameter, d is the dimension of the space (here thus $d = 1$) and $0 < \alpha \leq 2$ is the basic parameter which characterizes the process; $\alpha = 2$ corresponds to the log-normal distribution (a normal generator).

On the other hand, additive models correspond to a linear or bilinear $\zeta(q)$. Indeed, for Brownian motion (Bm) $\zeta(q) = q/2$, and for fractionnal Brownian motion (fBm) $\zeta(q) = qH$, for a fractionnal integration of order $H + 1/2$ of a Gaussian noise. Thus a purely linear $\zeta(q)$ function indicates Bm or fBm. We showed numerically in Ref.6 that several ARCH and GARCH models quickly converge to giving $\zeta(q) = q/2$. We obtained also a bilinear expression for the quite popular Lévy-stable and truncated Lévy-stable processes¹¹⁻¹³: $\zeta(q) = qH$ for $q < 1/H$ and $= 1$ for $q \geq 1/H$, where $H = 1/\alpha$, and α is the Lévy index ($0 \leq \alpha \leq 2$). For $q \geq 1/H$ the above expression is valid for one realization; when the number of realization increases, because of the divergence of moments of order α of Lévy processes, $\zeta(q)$ diverges (but its estimate on finite samples is always finite^{1,3}).

It is important to note that multiplicative cascades generically produce also hyperbolic tails leading to divergence of moments of order q_D which could be larger than 2,¹ whereas additive processes are bounded to $q_D < 2$. Equation (1.2) is a

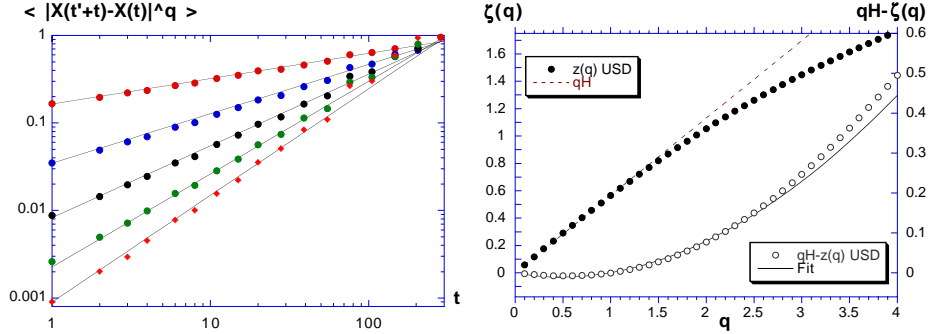


Figure 1: Left: Scaling of the structure functions in log-log plot for moments of order 0.5, 1., 1.5, 2. and 2.5 (from top to bottom); Right: the resulting scaling exponent $\zeta(q)$ which is clearly nonlinear (full dots), compared to the straight line $q\zeta(1)$ (dotted line). Also shown is the convex function $q\zeta(1) - \zeta(q)$ (open dots), with a universal multifractal fit (continuous line).

nonlinear behaviour obtained for $q \leq q_D$, whereas for $q > q_D$, $\zeta(q)$ is linear, with a slope depending on the number of realizations studied.²

2. Multifractal data analysis: an example

We show here as a case study the analysis of a daily US Dollar/French Franc exchange rate from 1 January 1979 to 30 November 1993. This corresponds to 3680 data points, and a scaling of nearly three orders of magnitude (but the analysis of intraday data showed that the scaling of the financial fluctuations can go from several minutes to several years).

In Fig. 1a we show the structure functions in log-log plot for different orders of moments. The straight lines show that the scaling of Eq.(1) is very well respected; we repeated this for moments up to 4.0, with a 0.1 increment; only for moments larger than about 4.0, the scaling begins to be broken because of the insufficient amount of data analyzed. The resulting $\zeta(q)$ function is shown in Fig. 1b: there is a clear nonlinearity; we also directly estimated the scaling exponent of the nonlinear term $\tau^{qH} / \langle (\Delta X_\tau)^q \rangle$, which is a convex function plotted on the same graph. We obtain the following values for several currencies (slightly different values for each currency were obtained, but nevertheless in each case the resulting $\zeta(q)$ function was nonlinear): $H = .58 \pm 0.03$ and $\zeta(2) = 1.06 \pm 0.05$, and using specific analysis techniques, $C_1 = .05 \pm .03$ and $\alpha = 1.5 \pm .3$. We also obtained $q_D = 3.0 \pm .5$, a value which is confirmed by an analysis on a much larger dataset.¹⁴ This seems to indicate that, in general, FX data are characterized by multifractal processes with a hyperbolic slope of 3; a structure function analysis does indeed display a nonlinear curve up to the third order moment, and then a straight line with a slope linear in $-\ln N$, where N is the number of datapoints used for the statistical estimates.⁶

The main application of this new approach is predictability: past and present

values of the time series can be exploited in order to provide an optimal forecast. This contradicts the Efficient Market Hypothesis, which relies on memoryless models. As with all symmetry principles, in the absence of specific, strong scale breaking mechanisms, we must assume that the scaling is unbroken and that the small empirical deviations are due to poor statistics. The observed nonlinearity of $\zeta(q)$ thus demonstrates that it is multifractal. Nevertheless, proposals have been made to either intentionally¹⁵ or inadvertently¹⁶ drop the scaling assumption and consider complex transient ("cross-over") regimes of models which are only monoscaling in the limit. However, let us emphasize that this corresponds to come back to a non-scaling framework, and by consequence to face many theoretical and practical difficulties.

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