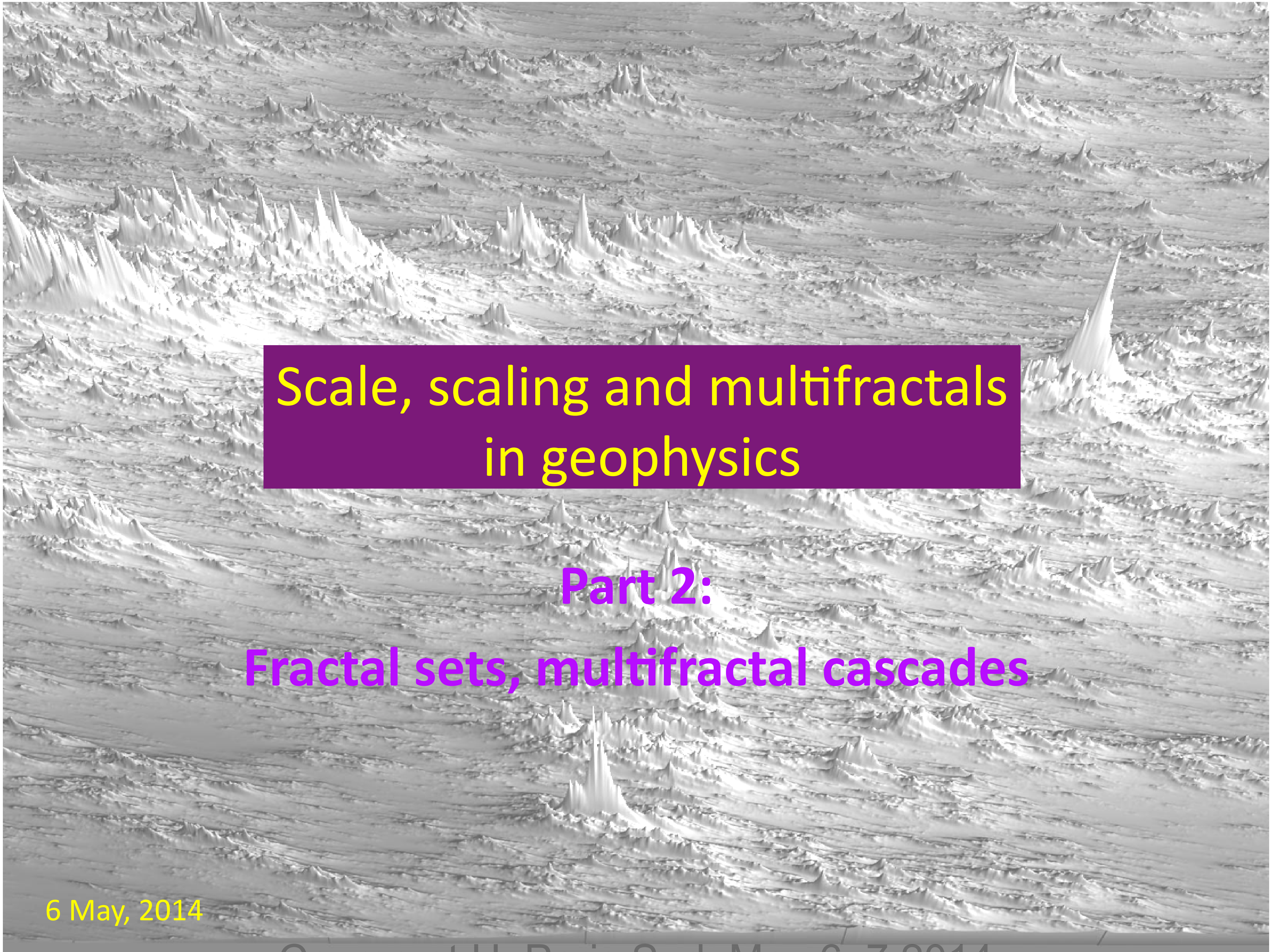


Scale, scaling and multifractals in geophysics

Part 2:

Fractal sets, multifractal cascades

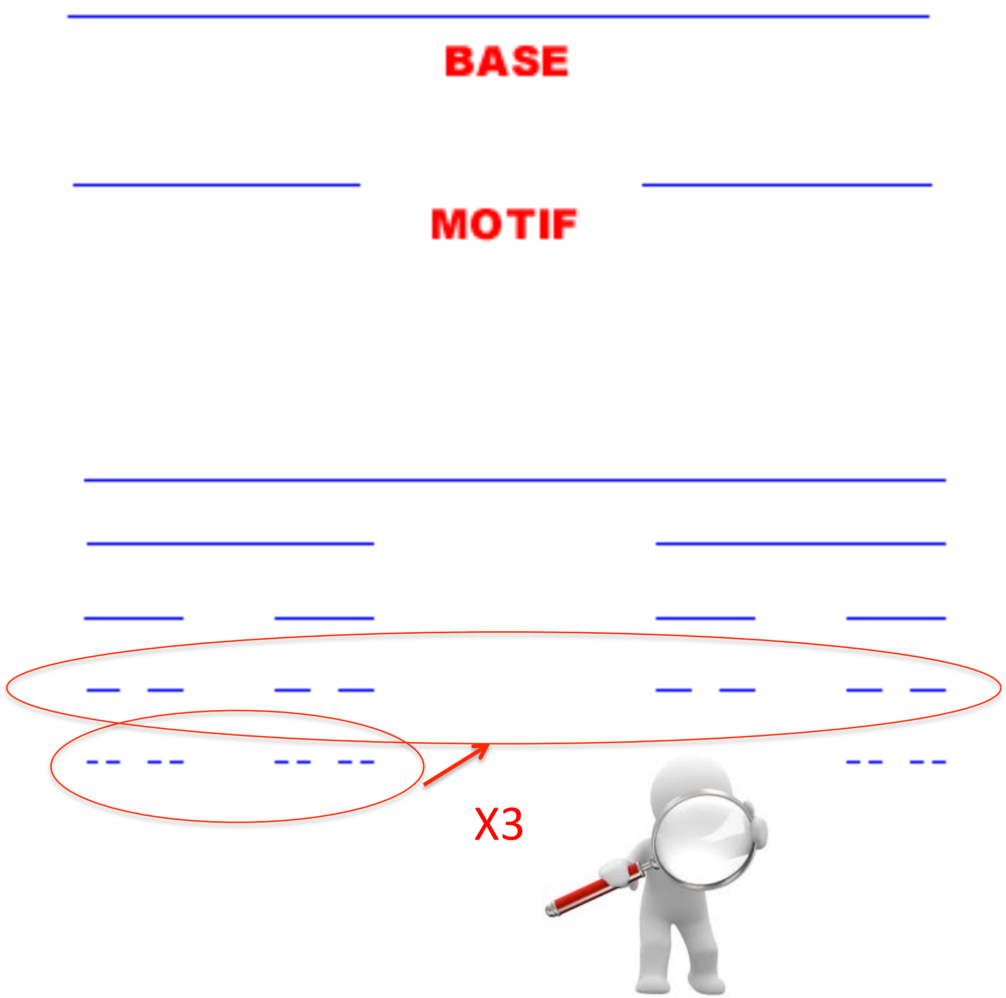
6 May, 2014

Scale Invariance sets and fields

Scale invariant geometric sets: Fractals

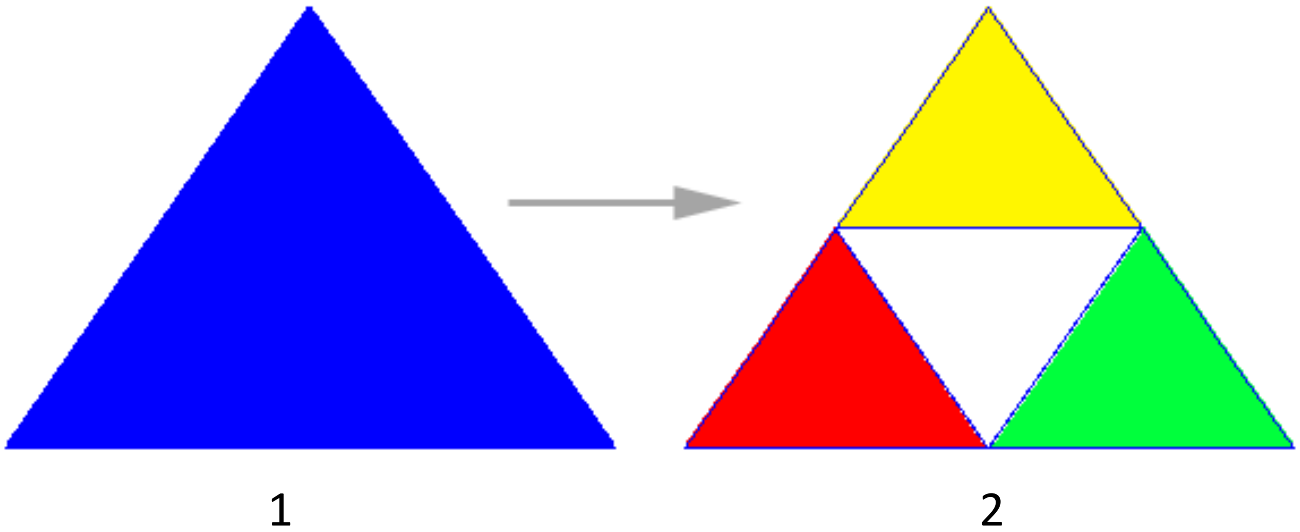
The simplest fractal, the Cantor set (1871)

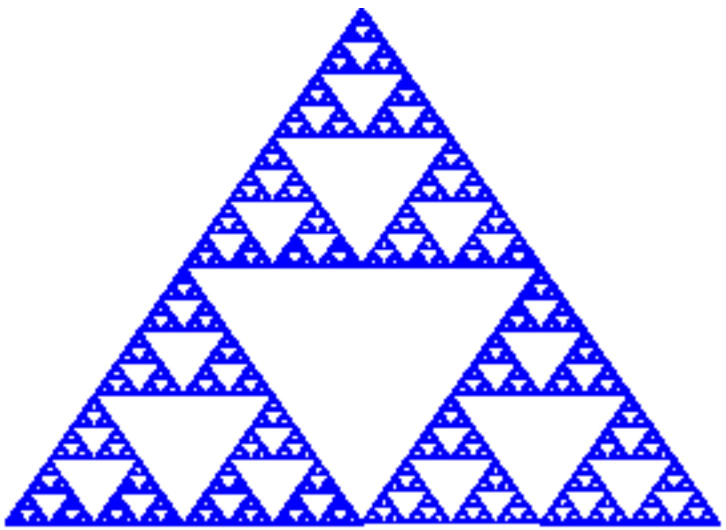
- Start with:



iterate:

Sierpinski Triangle

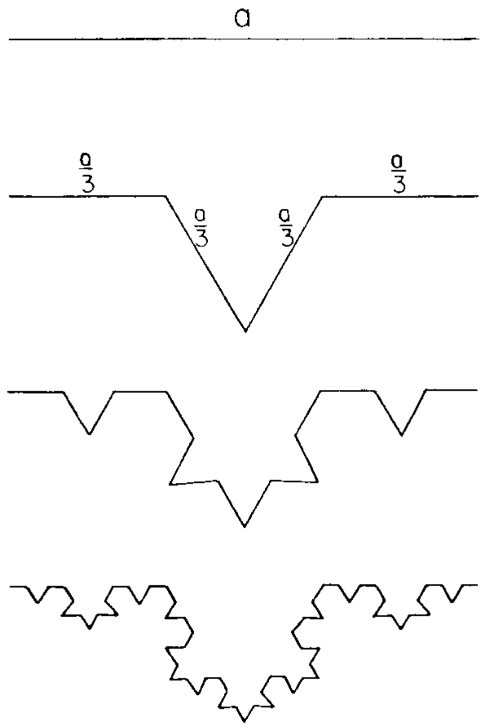




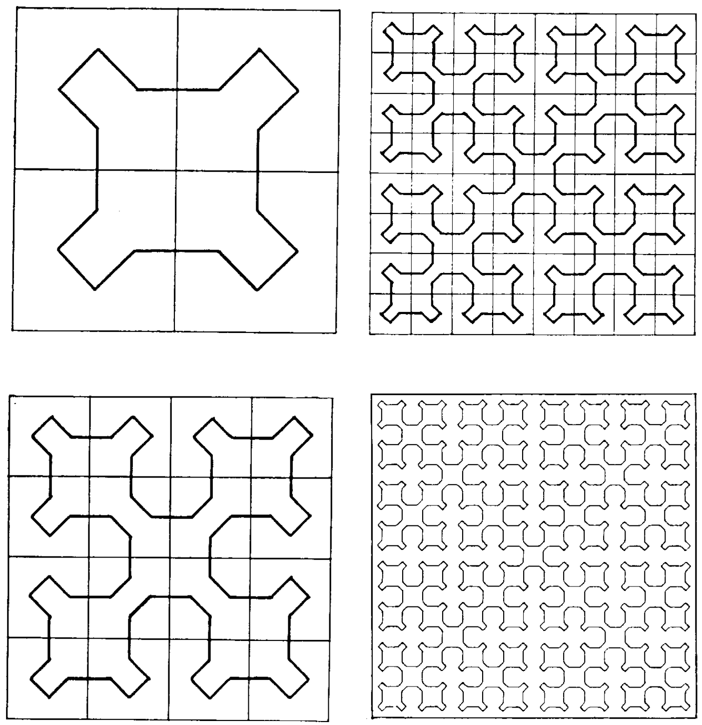
10 iterations

Early Geophysical applications of Fractal sets

Set: Black / white, single fractal dimension



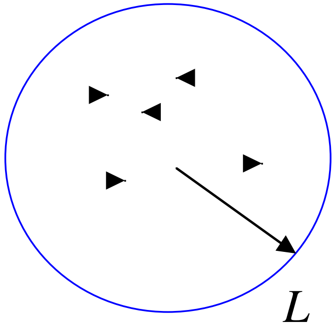
A fractal Koch curve ([*Koch*, 1904]), reproduced from [*Welander*, 1955] to illustrate the mixing of a two dimensional fluid.



A fractal Peano curve, reproduced from [*Steinhaus*, 1960] showing how a line (dimension 1) can literally fill the plane (dimension 2), illustrating how streams can fill a surface.

Isotropic Scale Invariance and fractal sets

Fractal Dimension:



$$n(L) \propto L^D$$

Number of points

$$\rho(L) = \frac{n(L)}{L^d} = \propto L^{D-d} = L^{-C}$$

Density of points

d=dimension of space
D= fractal dimension of set
C=d-D= fractal codimension

Scale invariance:

$$n(\lambda L) = \lambda^D n(L)$$

D=scale invariant

Same form after zoom by factor λ.

Meteorological measuring network

Fractal set: each point is a station

9962 stations (WMO, 1986)

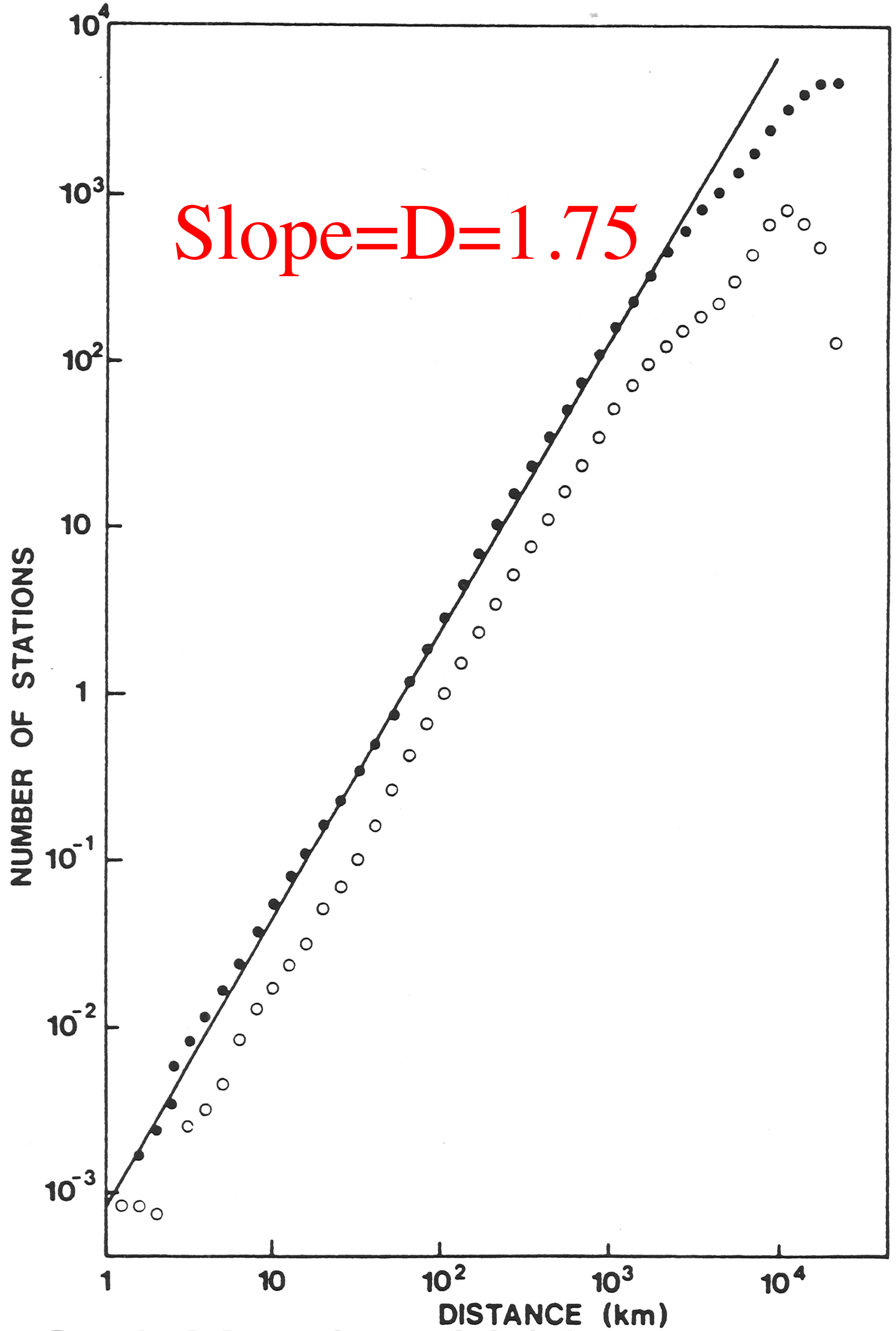
Number

$$n(L) \propto L^D$$

Density

$$\rho(L) = n(L)L^{-2} \propto L^{-C}; \quad C = d - D; \quad d = 2$$

The fractal dimension of the network= 1.75



(1) Fractal Codimensions: Geometric

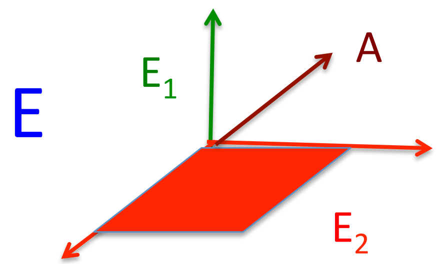
The notion of fractal codimension C_F can be defined both statistically and geometrically. The former is more useful and general since it applies not only to deterministic but also to stochastic processes.

Let $A \subset E$ (the embedding space) with $\dim(E) = D$ and $\dim(A) = D_F(A)$. Then the codimension $C_F(A)$ is defined as:

$$C_F(A) = D - D_F(A)$$

This definition corresponds merely to an extension of the (integer) codimension definition for vector sub-spaces, i.e., E_1 and E_2 being in direct sum (i.e., $E_1 \cap E_2 = \varnothing$):

$$E = E_1 \oplus E_2 \quad \Rightarrow \quad \mathrm{codim}(E_1) = \dim(E_2)$$



Example: E_1= line, E_2= plane, E= 3-D space
a line ($D_F(A)$=1) in three dimensional space: dim(E)=D=3, hence: $C_F(A)$=3-1=2

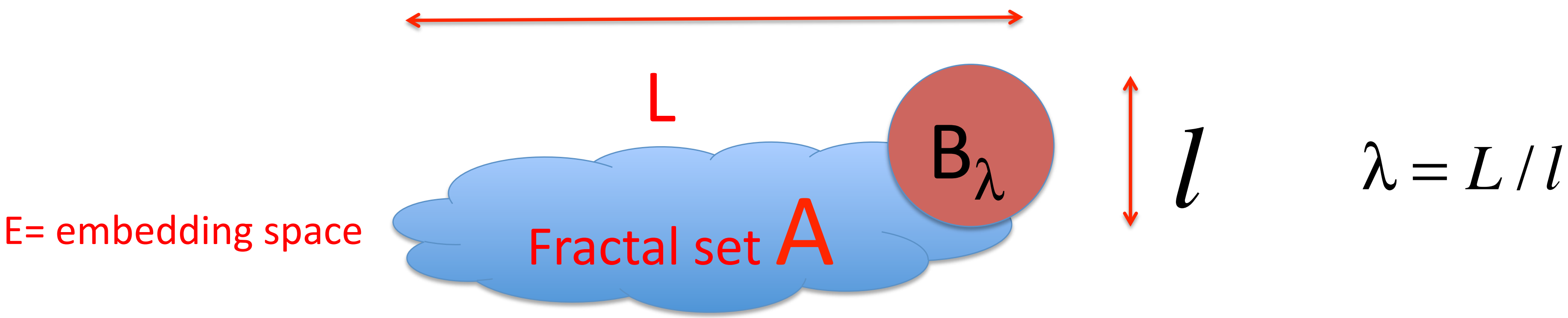
(2) Fractal Codimensions: Probabilistic

The codimension C_F can be introduced directly.

Consider the (scaling) behaviour the probability ("*Pr*") that a ball B_λ (of size $\ell = L/\lambda$) intersects the set A is:

$$\Pr(B_\lambda \cap A) \sim \lambda^{-C_F(A)}$$

where B_λ = ball of size and $\ell = L/\lambda$ and C_F is thus directly defined as an exponent measure of the fraction of the space occupied by the fractal set A (size L) in an embedding space E which can even be an infinite dimensional space.



Geometric versus probabilistic

Relating the two definitions

Since the probability of the event $(B_\lambda \cap A)$ is defined as:

$$\Pr\left(B_\lambda \cap A\right) \sim \frac{N\left(B_\lambda \cap A\right)}{N\left(B_\lambda \cap E\right)} \sim \frac{\lambda^{D_F(A)}}{\lambda^{D(E)}}$$

Number of balls B_λ needed to cover A

Number of balls B_λ needed to cover E

where $N\left(B_\lambda \cap A\right)$ refers to for example the number of balls B_λ needed to cover the set A and $N\left(B_\lambda \cap E\right)$ is the corresponding number for the entire space. It is easy to check that when $C_F(A) < D = \dim(E) < \infty$ the two definitions are equivalent:

$$C_F(A) \leq D < \infty, \ \{\text{definition 1} \ \equiv \ \text{definition 2}\} \ \forall \ D_F \geq 0$$

However, when $C_F(A)>D$, then they no longer agree since it implies $D_F(A)<0$ which is impossible.

Multifractal fields: Cascades and Multifractals

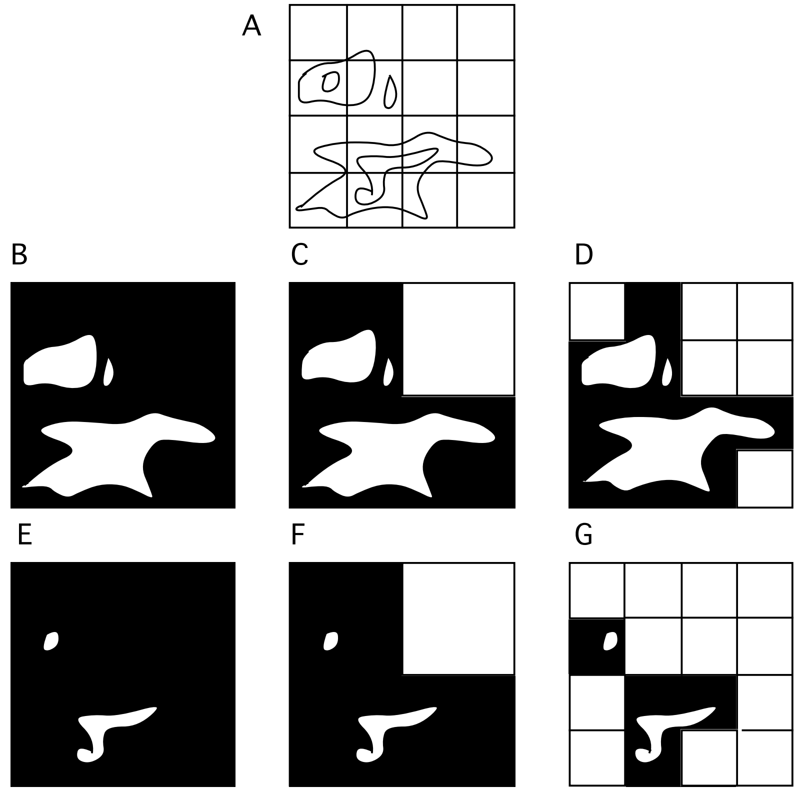
Course at U. Paris Sud, May 6, 7 2014

Multifractality and Functional Box Counting

$$N_T(L) \approx L^{-D(T)}$$

Box counting low threshold (large D)

Box counting high threshold (low D)

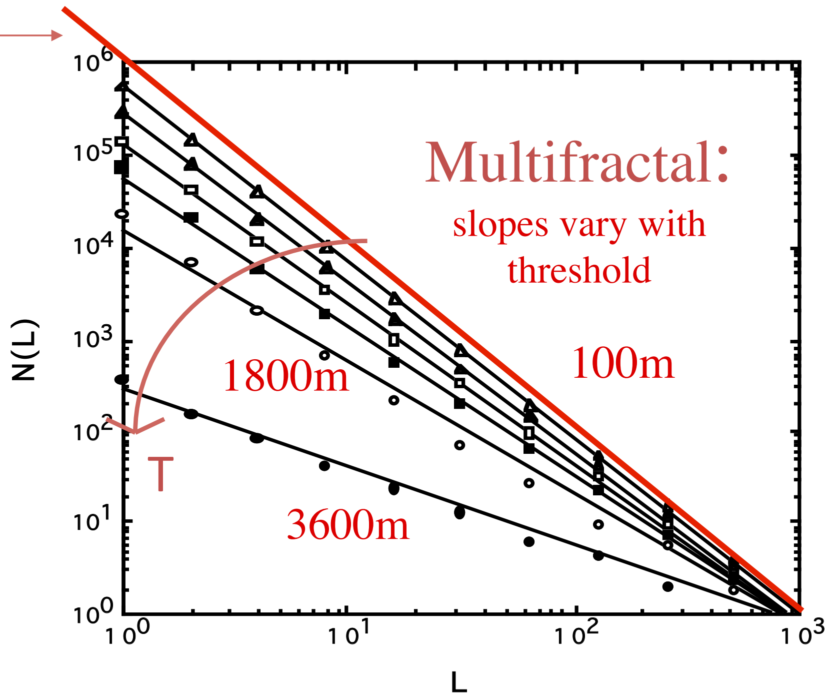


-Monofractal: D(T) <2, constant

-Multifractal: D(T)<2, decreasing

Functional box counting on French topography: 1 -1000km

Slope =2 (required for classical geostatistics - regularity of Lebesgue measures)



$N_T(L) \approx L^{-D(T)}$

Systematic resolution dependence

km

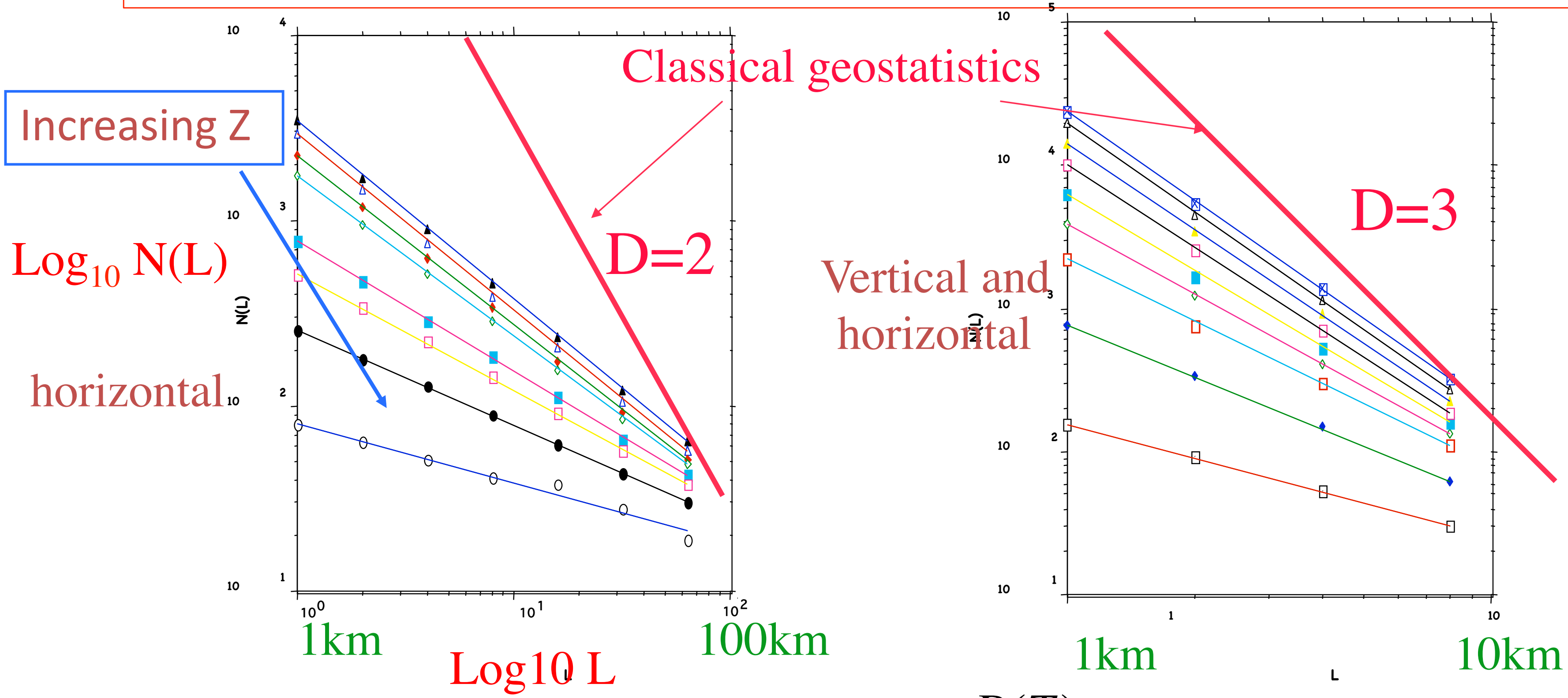
Implications for geostatistics:

The areas A_T exceeding a given threshold decrease as the resolution becomes finer (decreasing *L*):

$A_T = L^d N_T = L^{C(T)}$; $C(T) = d - D(T)$

Unless $C(T) = 0$, the areas depend on the subjective resolution *L*; the reference lines indicate that for the topography, all the regions defined by the thresholds have $C(T) = d-D(T) > 0$ so that they have systematic resolution dependencies.

Functional Box counting on 3D radar rain scans

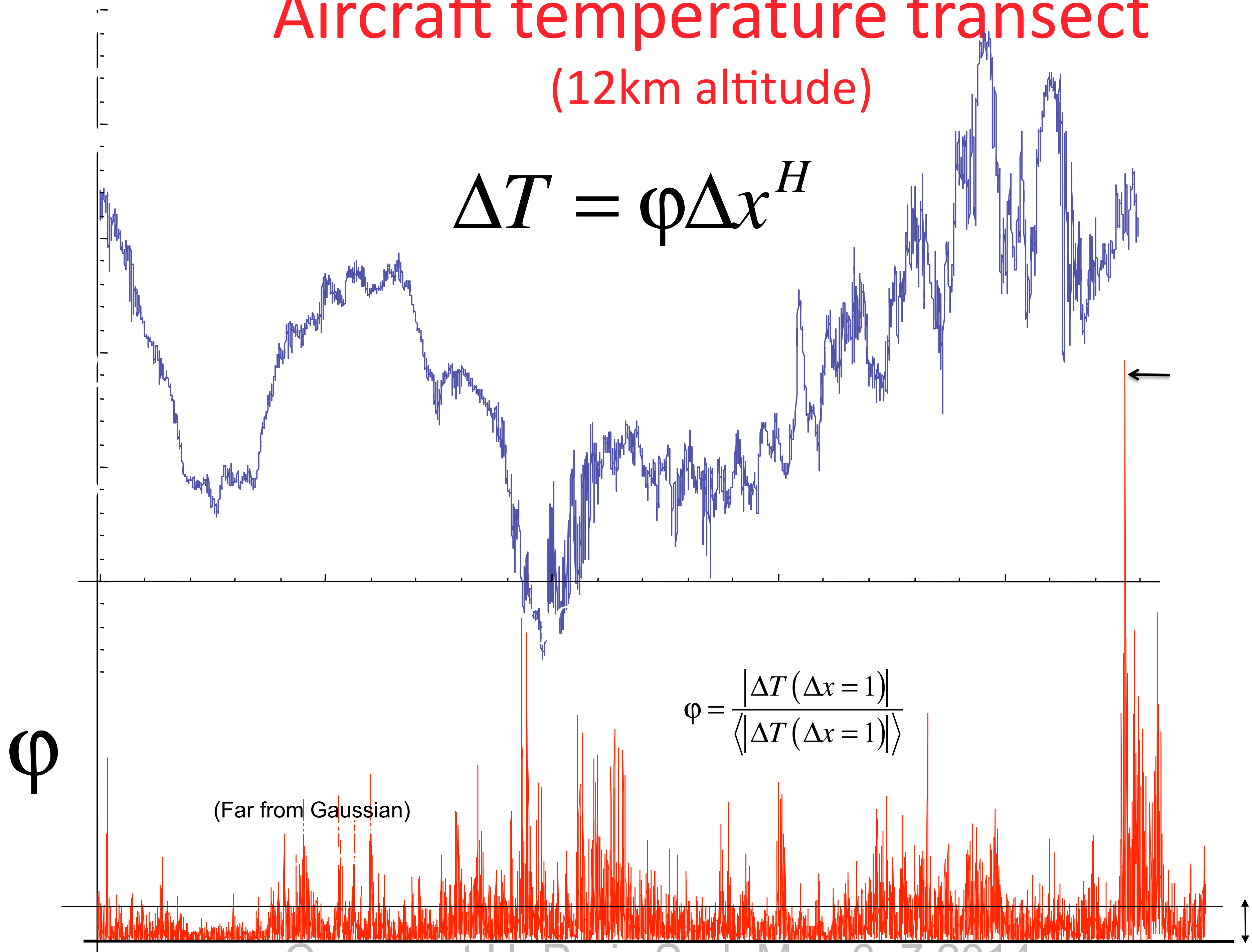


$$N_T(L) \approx L^{-D(T)}$$

Radar reflectivity thresholds increasing (top to bottom) by factors of 2.5 (data from Montreal).

Aircraft temperature transect

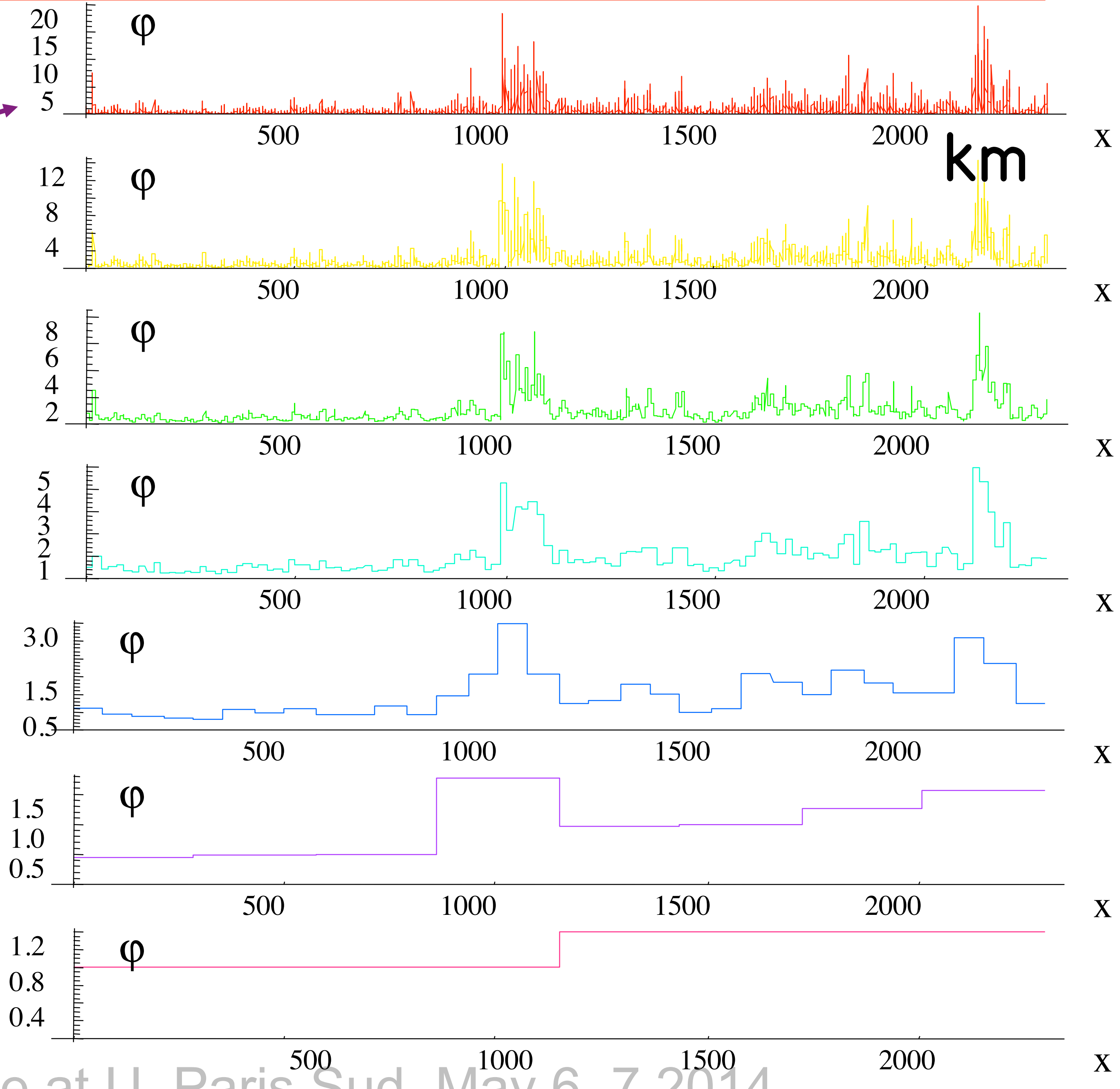
(12km altitude)



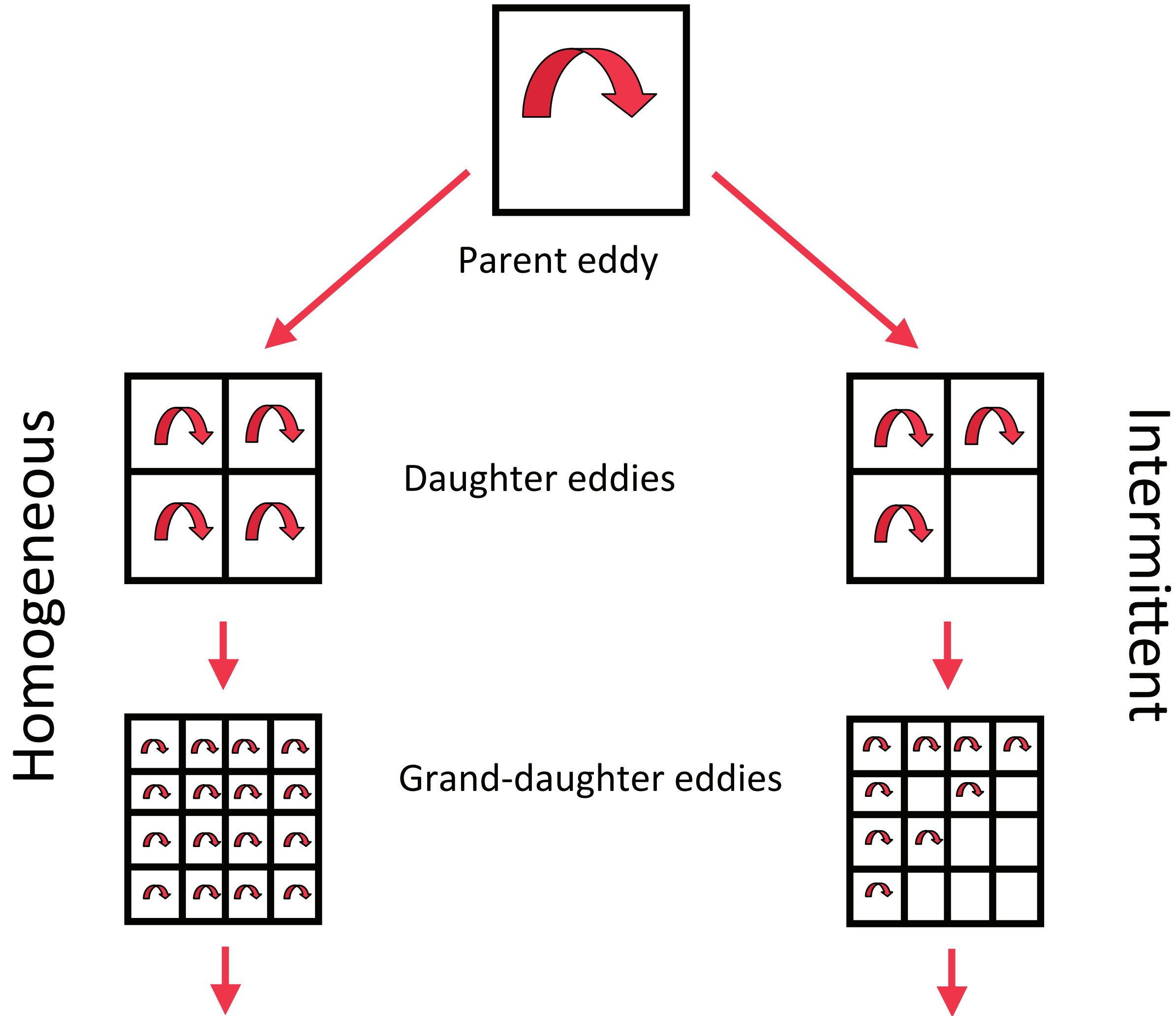
Degrading the resolution

Temperature turbulent flux φ at 280m resolution

High to low Resolution: degrading by factors of 4



Cascades



$$D = \frac{Log4}{Log2} = 2$$

$$D = \frac{Log3}{Log2} = 1.58...$$

Beta model

An initial attempt to handle intermittency reduces it to the simple notion of "on/off" intermittency, i.e. a cascade with the simple alternative alive/dead of the offspring.

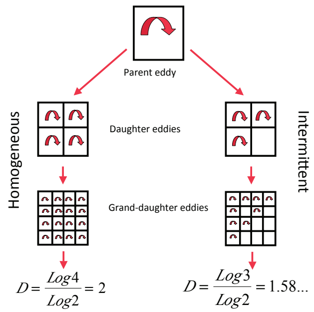
This leads to a confinement of the turbulence to a tiny support; a very small subregion of the flow. The right hand side of the figure shows the result of such a stochastic cascade obtained by randomly multiplying the energy flux of a "mother" eddy to obtain that of the "daughter" eddies either by 0 (dead sub-eddy) or by a positive value λ_0^c

(corresponding to an active sub-eddy, with fixed probability . λ_0^{-c}
In this model, we divide the spatial scales by λ_0 (here λ_0 = 2) and then flip coins to determine the on or off state; more precisely:

Each step:

$$\Pr\left(\mu\varepsilon = \lambda_0^c\right) = \lambda_0^{-c}$$

$$\Pr\left(\mu\varepsilon = 0\right) = 1 - \lambda_0^{-c}$$

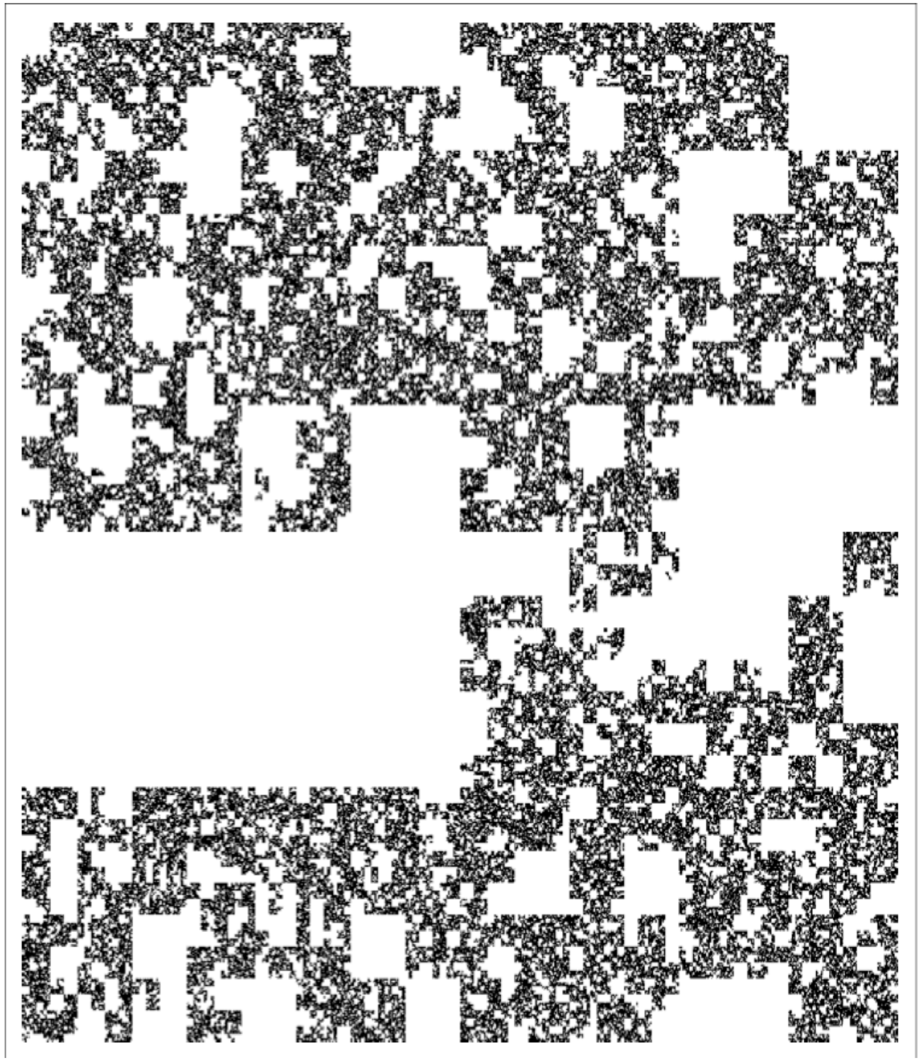


After n steps: $$\varepsilon_n = \prod_{j=1}^{n} \mu\varepsilon_j$$

("*Pr*" indicates "probability"). The nonzero value is taken as $\mu\varepsilon = \lambda_0^c$ so that the mean $\langle\mu\varepsilon\rangle = 1$; this implies a scale by scale conservation of the flux ε.

After n steps: $\lambda = \lambda_0^n$ $\Pr(alive) = \left(\lambda_0^{-c}\right)^n = \lambda^{-c}$ Relation to dimension: $N_{alive} = N_{tot}\Pr = \lambda^d \lambda^{-c} = \lambda^D; \quad D = d - c$

Beta model



In this example, the probability that an eddy will remain alive is λ_0^{-C} =0.87 (using the scale ratio at each step λ_0 = 4 here and the codimension C = 0.2).

Alpha model

The α model is a two state (binomial) process with $\mu\varepsilon$ = either $\lambda_0^{\gamma+}$ or $\lambda_0^{\gamma-}$ where $\gamma_+>0$ corresponds to a boost ($\mu\varepsilon>1$) and γ_- to a decrease ($\mu\varepsilon<1$) .

$$\Pr\left(\mu\varepsilon = \lambda_0^{\gamma_+}\right) = \lambda_0^{-c}$$

$$\Pr\left(\mu\varepsilon = \lambda_0^{\gamma_-}\right) = 1 - \lambda_0^{-c}$$

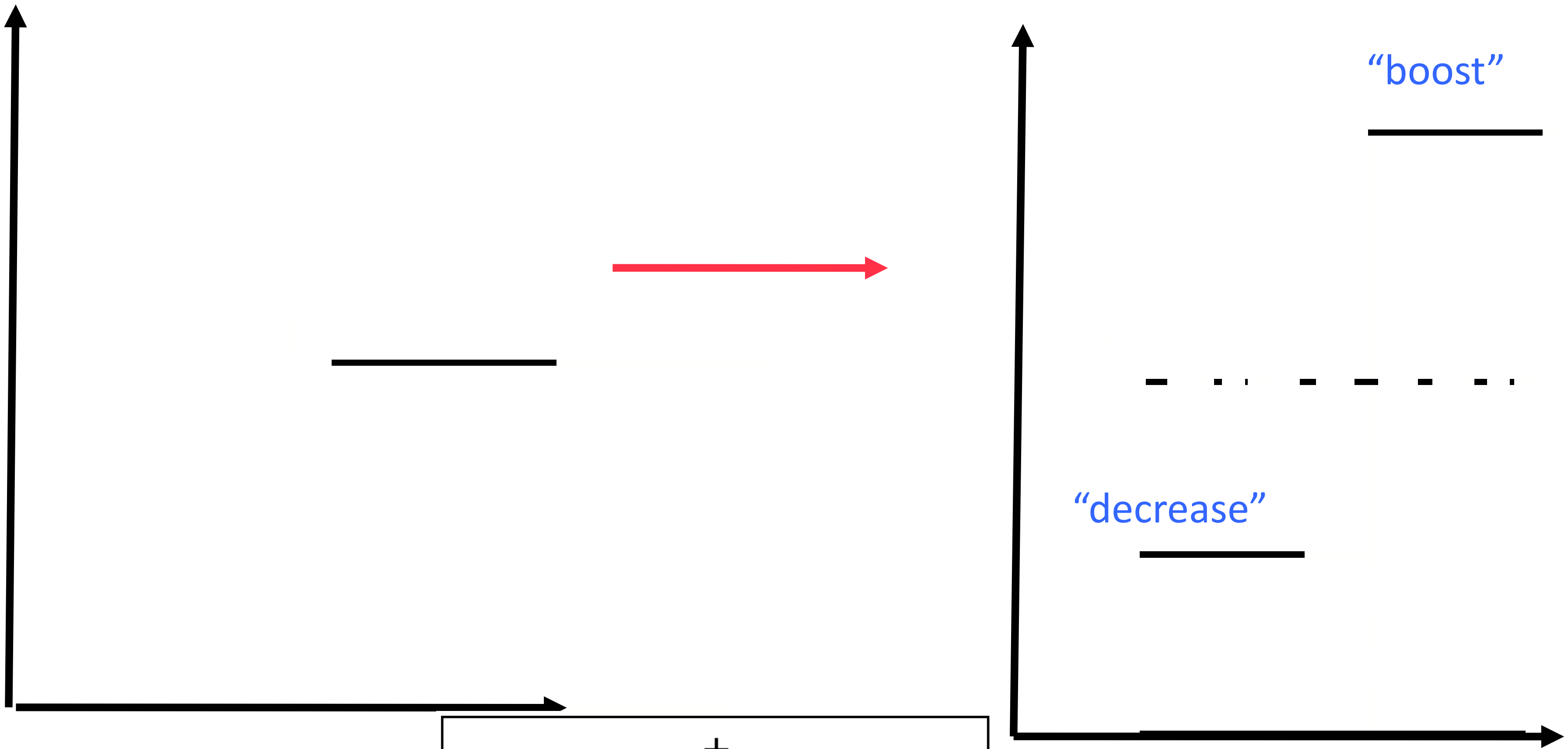
Although the α model apparently involves three parameters (γ_+, γ_-, c), due to the conservation constraint:

$$\langle \mu\varepsilon \rangle = \lambda_0^{-c}\lambda_0^{\gamma_+} + \left(1 - \lambda_0^{-c}\right)\lambda_0^{\gamma_-} = 1$$

We can see that the β model is recovered in the limit $\gamma_- \to -\infty$
which is the same as $\gamma_+ \to c$

The α model

Simulations: **multiplicative** introduction of small scale details (low resolution to high)

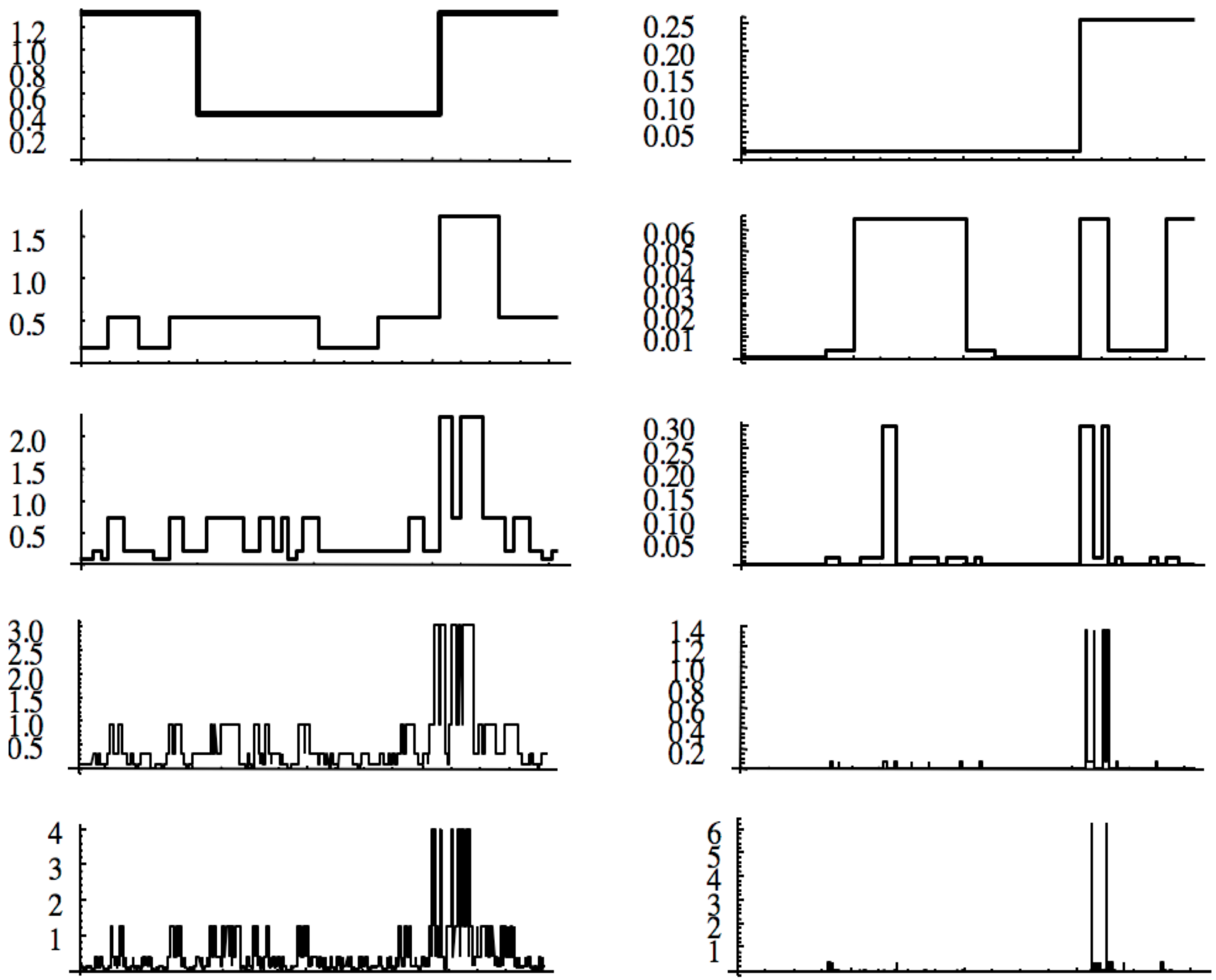


+							
+				−			
+		−		−		−	
−	+	+	+	−	+	+	−

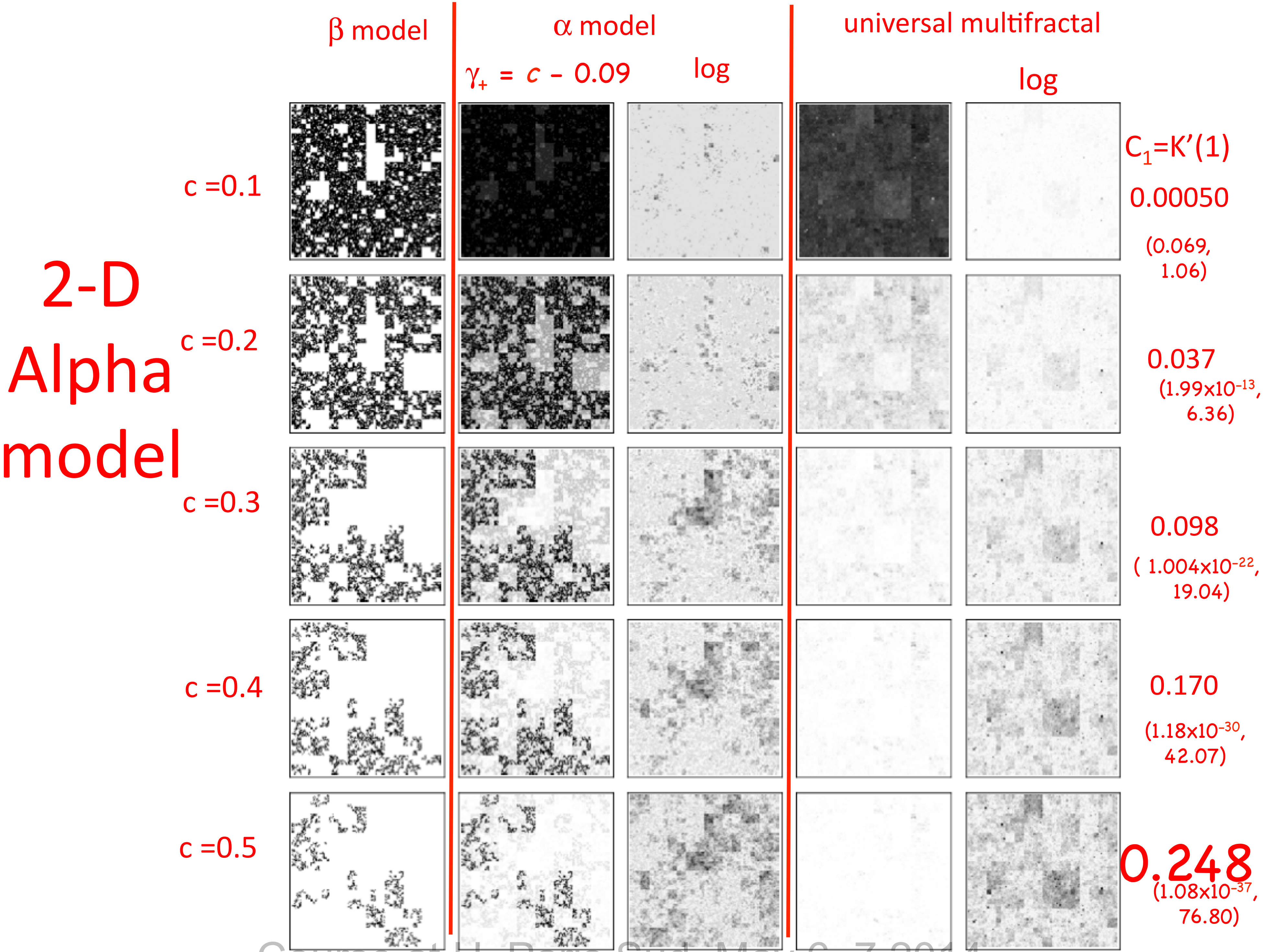
S+L 1983

Alpha model

$\gamma_+ = 0.2$, $c = 0.3$ ($C_1 = 0.087$) $\gamma_+ = 1.1$, $c = 1.2$ ($C_1 = 0.82$)



From top to bottom every second cascade step is shown (a factor of λ_0^2) is shown, 10 steps in all, the total range of scales is $2^{10} = 1024$). Notice the changing vertical scales


2-D
Alpha
model
β model
α model
$\gamma_+ = c - 0.09$
log
universal multifractal
log
c =0.1
c =0.2
c =0.3
c =0.4
c =0.5
$C_1=K'(1)$
0.00050
(0.069, 1.06)
0.037
$(1.99\times10^{-13}$, 6.36)
0.098
(1.004×10^{-22}, 19.04)
0.170
$(1.18\times10^{-30}$, 42.07)
0.248
$(1.08\times10^{-37}$, 76.80)

General cascade statistics

Characterize the statistics of με by K(q): $\langle \mu\varepsilon^{q} \rangle = \lambda_0^{K(q)}$ ← Scale ratio of each cascade step

The notation "μ" indicating "multiplicative increment"; it is analogous to the use of the "Δ" to denote an additive increment.

$$\langle \varepsilon_n^q \rangle = \left\langle \prod_{j=1}^{n} \mu\varepsilon_j^q \right\rangle = \prod_{j=1}^{n} \langle \mu\varepsilon_j^q \rangle = \langle \mu\varepsilon^q \rangle^n = \lambda_0^{nK(q)}$$

We can now write the general expression for the statistical properties after a total scale range λ:

$$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)}$$

Overall scale ratio since the cascade started:

$$\lambda = \lambda_0^n$$

This is the basic formula for cascade statistics. The specification of the statistics of με, and hence of ε_λ via statistical moments is equivalent to their specification by probabilities.

The cascade generator: Γ

The overall characterization of the statistical properties is conveniently through the "moment scaling exponent" $K(q)$:

$$K(q) = Log_{\lambda_0}\langle \mu\varepsilon^q \rangle = Log\langle \mu\varepsilon^q \rangle / Log\lambda_0$$

$$= Log\langle \mu\varepsilon^q \rangle^n / Log(\lambda_0)^n = Log\langle \varepsilon_\lambda^q \rangle / Log\lambda$$

Introducing the (random) cascade "generator" Γ, the logarithm of the multiplier:

$$\Gamma = Log\varepsilon_\lambda$$

$K(q)$ is is the (Laplace, base λ_0) second characteristic function ("cumulant generating function") of Γ:

$$K(q) = \log_\lambda \langle e^{q\Gamma} \rangle$$

Examples of second characteristic Functions

Ex.1 Gaussian

$$\langle e^{qx}\rangle = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{qx}\underbrace{e^{-x^2/(2\sigma^2)}}_{p(x)}\,dx = e^{q^2\sigma^2/2}$$

$$K(q) = \log\langle e^{qx}\rangle = \frac{q^2\sigma^2}{2}$$

Base e Laplace characteristic function

Ex.2 Exponential:

$$p(x) = \frac{1}{2}e^{-|x|}$$

$$K(q) = -\log(1-q) - \log(1+q); \quad -1 \le q \le 1$$

Properties of the Moment scaling exponent *K*(*q*)

1) In order to see the general shape of the *K*(*q*) function, we may first note that conservation from one scale to another requires *K*(1) = 0:

$$\langle \varepsilon_\lambda \rangle = 1 = \lambda^0 \qquad \text{hence} \qquad K(1) = Log_\lambda \langle \varepsilon_\lambda \rangle = Log_\lambda 1 = 0$$

2) In addition, because any positive number raised to the zero power is one, we have <1> = 1, hence *K*(0) = 0. $\langle \varepsilon_\lambda^0 \rangle = 1 = \lambda^0$

3) Finally, a basic property of second characteristic functions is that *K*(*q*) must be convex, i.e. $K''(q)>0$; this can be shown directly by doubly differentiating $K(q) = \log\langle e^{q\Gamma}\rangle/\log\lambda$.

The typical *K*(*q*) looks something like the next slide which shows the *K*(*q*) for the α model and the universal multifractal models in the fourth and fifth columns of the earlier example. The models are tangent to each other at *q* =1 because the derivatives at *q* = 1 were deliberately chosen to be equal to each other. This value:

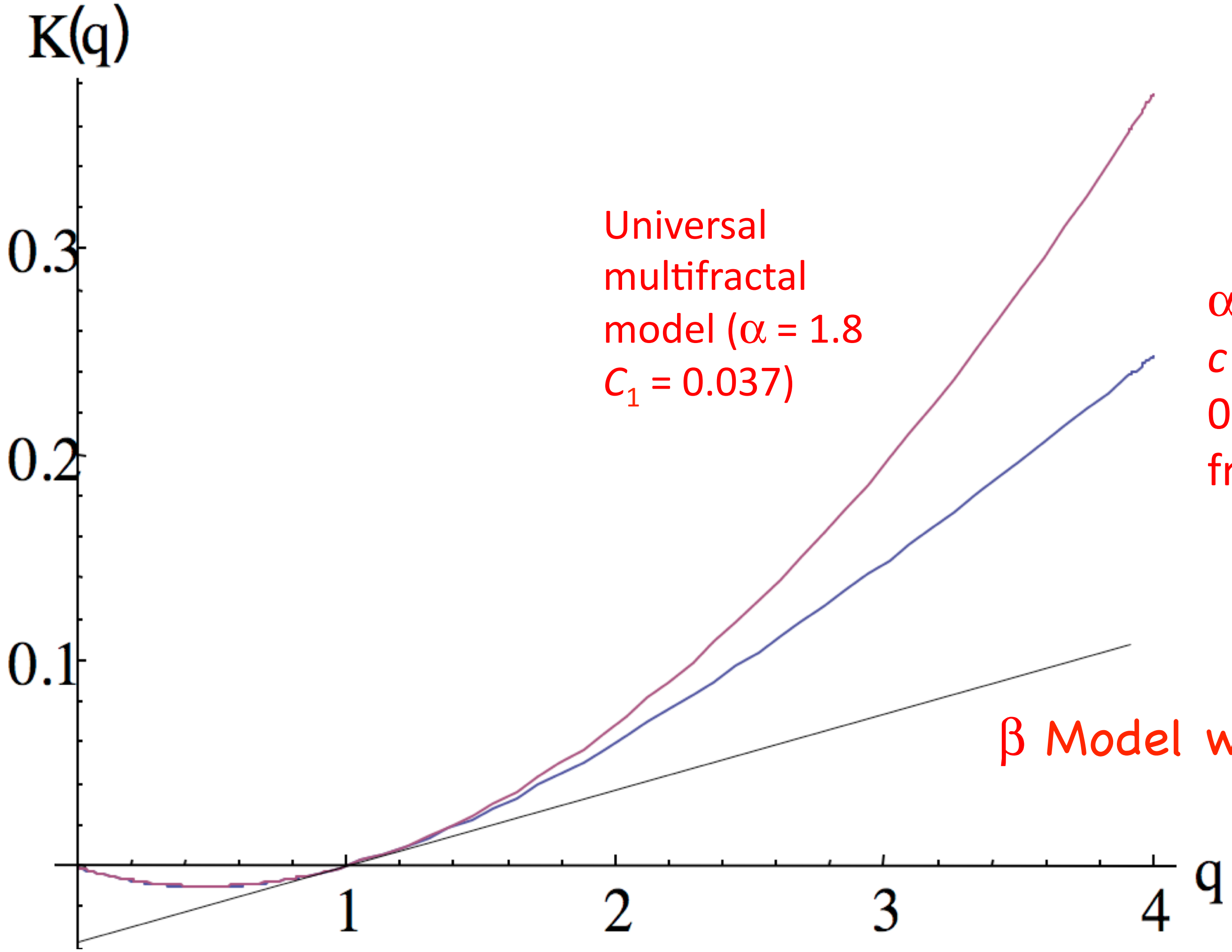
$C_1 = K'(1)$ tangent at the mean

C_1 = "the codimension of the mean"; a characterization of the variability near the mean

We can already use this idea to give a "local" (in *q* space) definition of the "degree of multifractality" α:

$$\alpha = K''(1)/K'(1)$$ Curvature near the mean

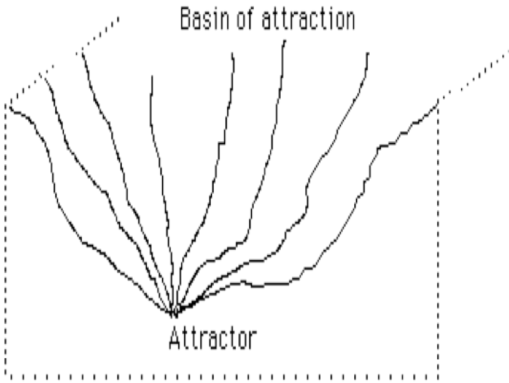
Comparison of the K(q) for examples



$C_1 = K'(1)$
$\alpha = K''(1)/K'(1)$

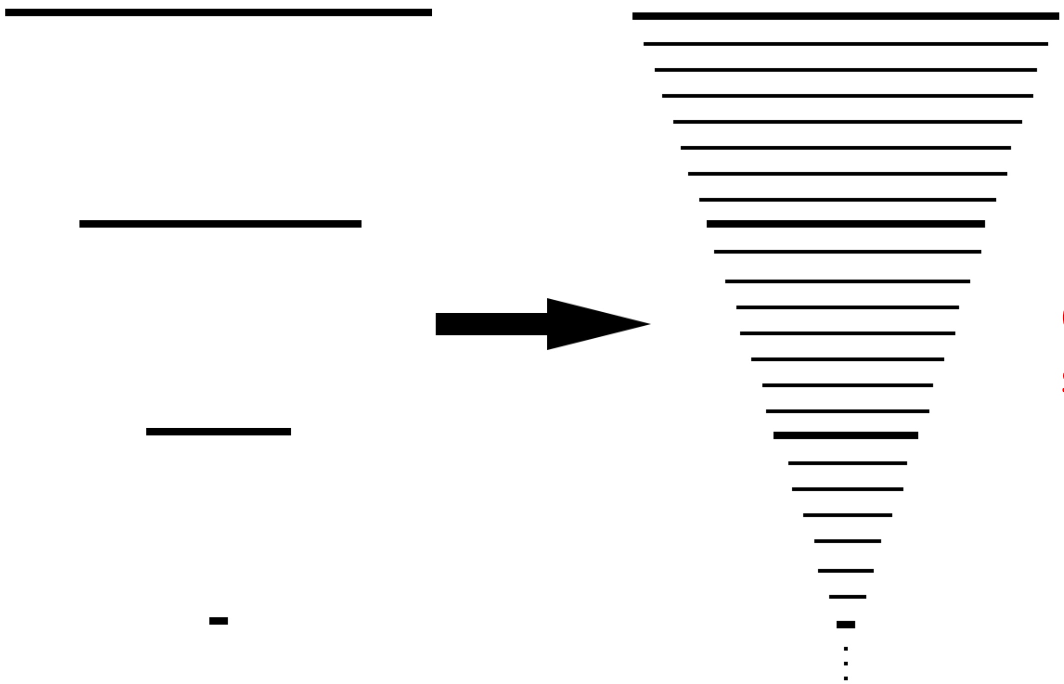
Universality: How many parameters for turbulence?

Answer	Date	References	Explanation	Parameters
1	1941	Kolmogorov (Homogeneous turbulence)	$\Delta v_\lambda \approx \overline{\varepsilon}^{1/3}\lambda^{-1/3}$	H=1/3
2	1962	Kolmogorov-Obukhov, (lognormal model)	$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)}$ $K(q) = \frac{\mu}{2}(q^2 - q)$	H, μ
2	1964	Novikov-Stewart, Mandelbrot, Frisch et al, β model	$K(q) = C_1(q-1)$	H, C_1
∞	1974	(Mandelbrot, 1974)	$K(q)$	Any $K(q)$ convex with $K(0)=K(1)=0$



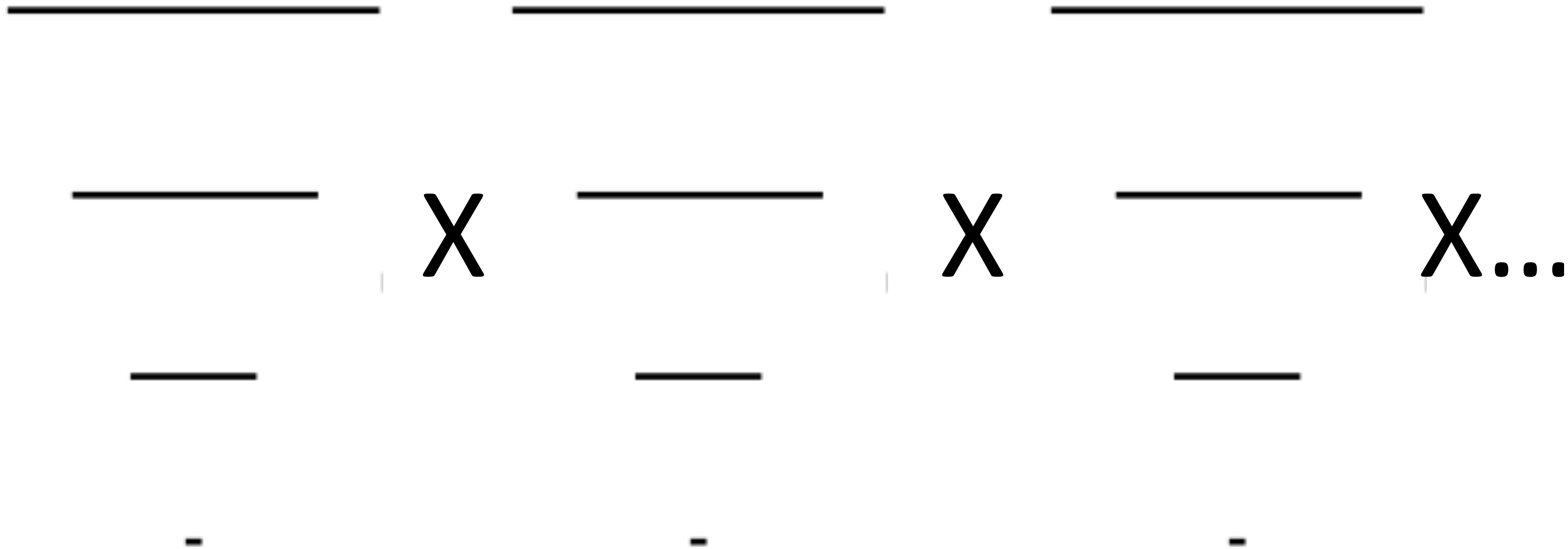
Routes to universality:
1) Densification of scales

Discrete in scale
(ex. β, α models)



Continuous in scale

Routes to universality: 2) "Mixing" of independent discrete cascades



ε_1 ε_2 ε_3

$$\varepsilon = \left(\prod_{i=1}^{N} \frac{\varepsilon_i}{a_N} \right)^{1/b_N}$$

N independent cascades, renormalized by a_N, b_N

For the generators $\Gamma = \log \varepsilon$ $\qquad \Gamma = \frac{1}{b_N} \sum_{i=1}^{N} \left(\Gamma_i - \log a_N \right)$

Normalized, centred sums

Universality in cascades: a "multiplicative central limit theorem"

Technical difficulty: the cascade requires a scale by scale conservation principle, otherwise there are no well defined small scale cascade limits, and it turns out that this normalization is in contradiction with the normalization required for central limit convergence.

Cascade convergence: $<\mu\varepsilon> = 1$ hence $<e^{\Delta\Gamma}> = 1$

Central limit convergence: $<\Delta\Gamma> = 0$, hence $<\log\mu\varepsilon> = 0$

Recall:
$\Delta\Gamma = \log\mu\varepsilon$

However, due to the convexity of the logarithm function, for any probability distribution of $\mu\varepsilon$ which is constrained such that $<\mu\varepsilon> = 1$, we have necessarily $<\Delta\Gamma> = <\log\mu\varepsilon> < 0$

Levy Generators (2)

The final normalization step needed for small scale convergence (analogous to the log-normal derivation: $K(q) \to K(q) - qK(1)$) leads to:

$$K'(1) = A_\alpha(\alpha - 1) = C_1$$
$$K''(1) = A_\alpha \alpha(\alpha - 1) = \alpha K'(1) = \alpha C_1$$

The local (near the mean) curvature characterization is satisfied:
$K'(1) = C_1$, $\alpha = K''(1)/K'(1)$
It is global.

Hence:

$$K(q) = \frac{C_1}{\alpha - 1}\left(q^\alpha - q\right); \quad 0 \le \alpha \le 2$$

(for $\alpha = 1$, using l'Hôpital's rule for the limit $\alpha \to 1$, we have $C_1 q \log q$).

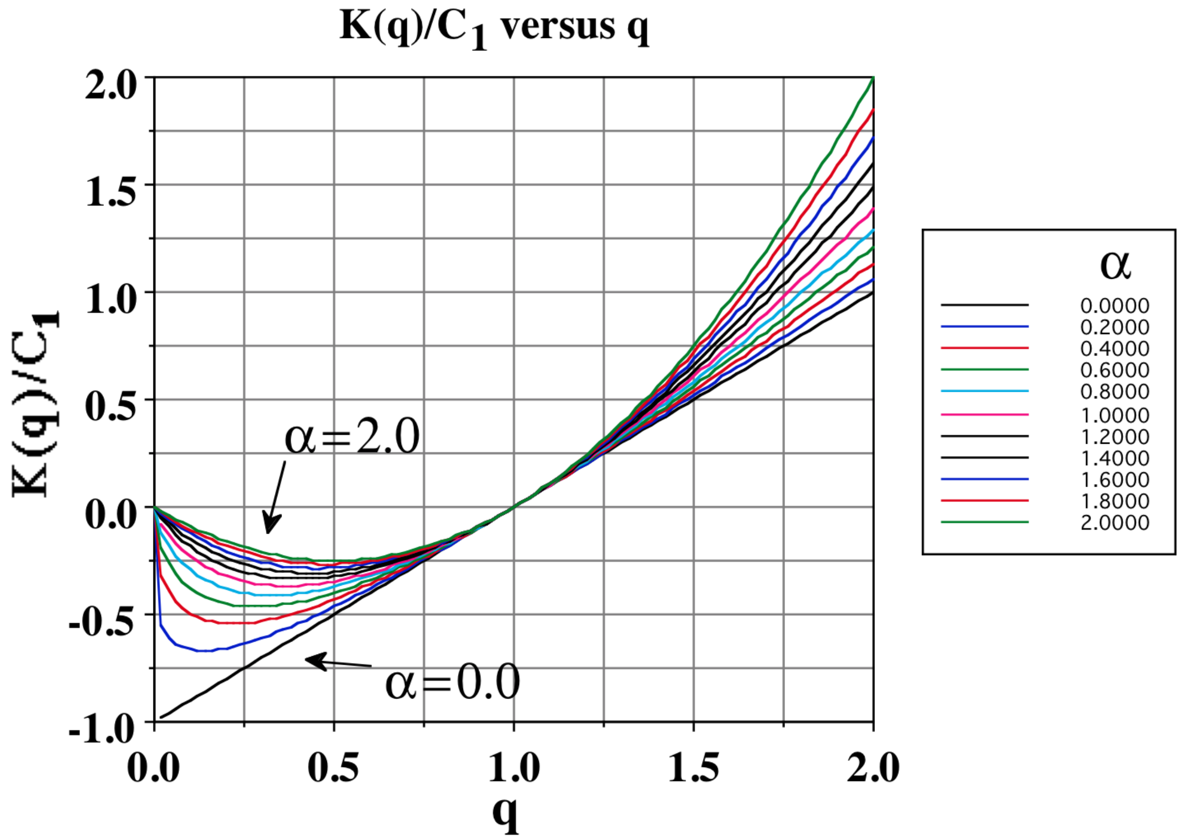
Note that when $\alpha<2$, and $q<0$, then ; this is a consequence of the extreme Lévy tail on the negative (but not positive) fluctuations of $\log\varepsilon$. The possibility (even likelihood) of:

$$\left\langle \varepsilon_\lambda^q \right\rangle \to \infty$$

for $q<0$ means that extreme caution should be used when analysing negative moments of empirical data.

K(q) for universal multifractals

$$K(q)/C_1 = \left(q^{\alpha} - q\right)/(\alpha - 1)$$



Universal $K(q)/C_1$ as a function of q, for different a values from 0 to 2 by increments of $\Delta\alpha = 0.2$.

Data Analysis

Fluctuation statistics and structure functions

The space-time variability of natural systems, can often be broken up into various "scaling ranges" over which the fluctuations vary in a power law manner with respect to scale. Over these ranges, the fluctuations follow

$$\Delta T = \varphi_{\Delta t} \Delta t^{H}$$

The flux at resolution Δt

Using Fluctuations:

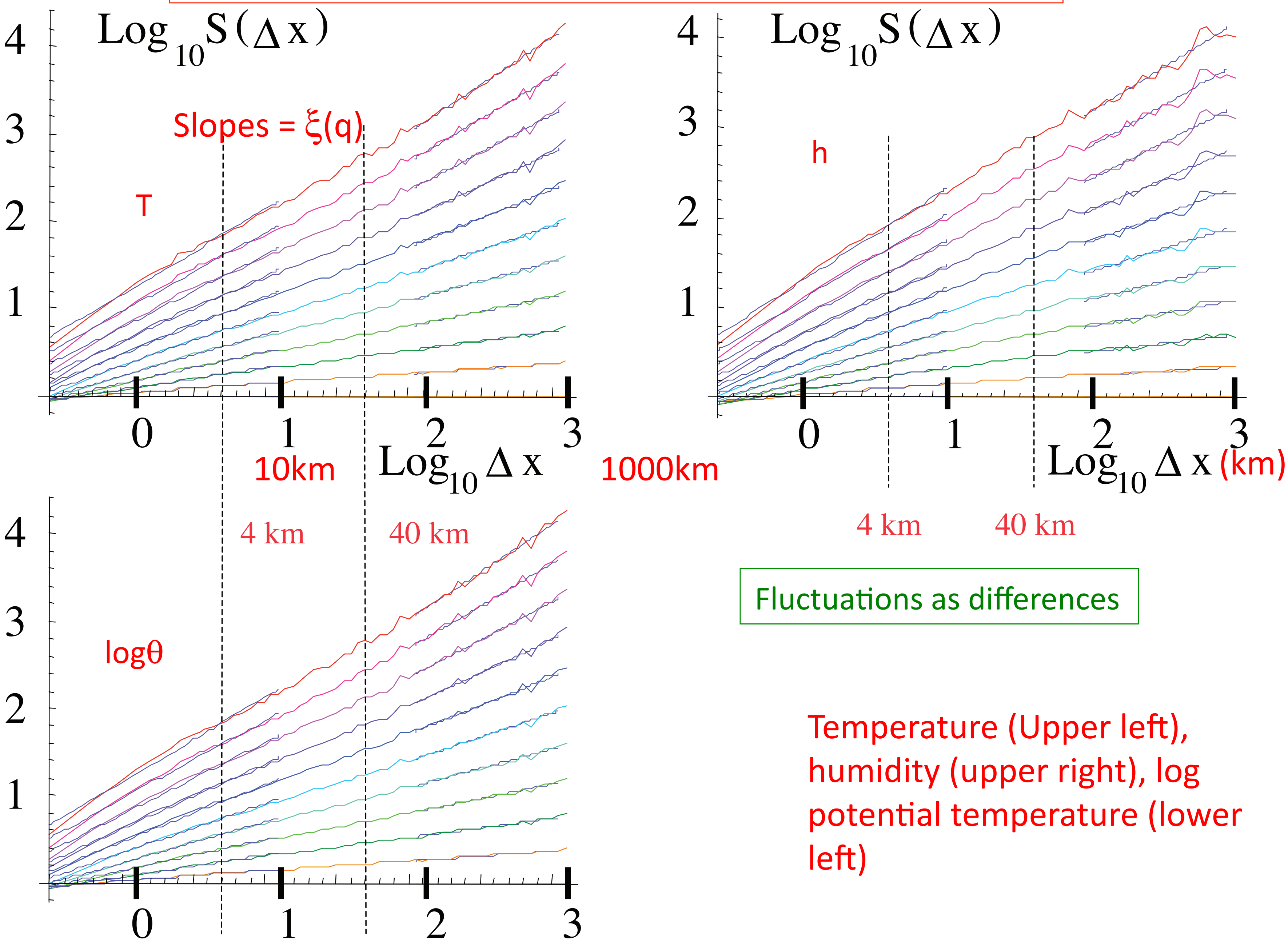
$$S_q(\Delta t) = \left\langle \Delta T(\Delta t)^q \right\rangle = \left\langle \varphi_{\Delta t}^q \right\rangle \Delta t^{qH} \approx \Delta t^{\xi(q)}; \qquad \left\langle \varphi_{\Delta t}^q \right\rangle = \left(\frac{\tau}{\Delta t} \right)^{K(q)}; \quad \xi(q) = qH - K(q)$$

(generalized, qth order) Structure function

Hence, we seek H, K(q)

With universality: $K(q) = \frac{C_1}{\alpha - 1}\left(q^{\alpha} - q\right)$ i.e. we seek H, C_1, α

Aircraft structure function estimates

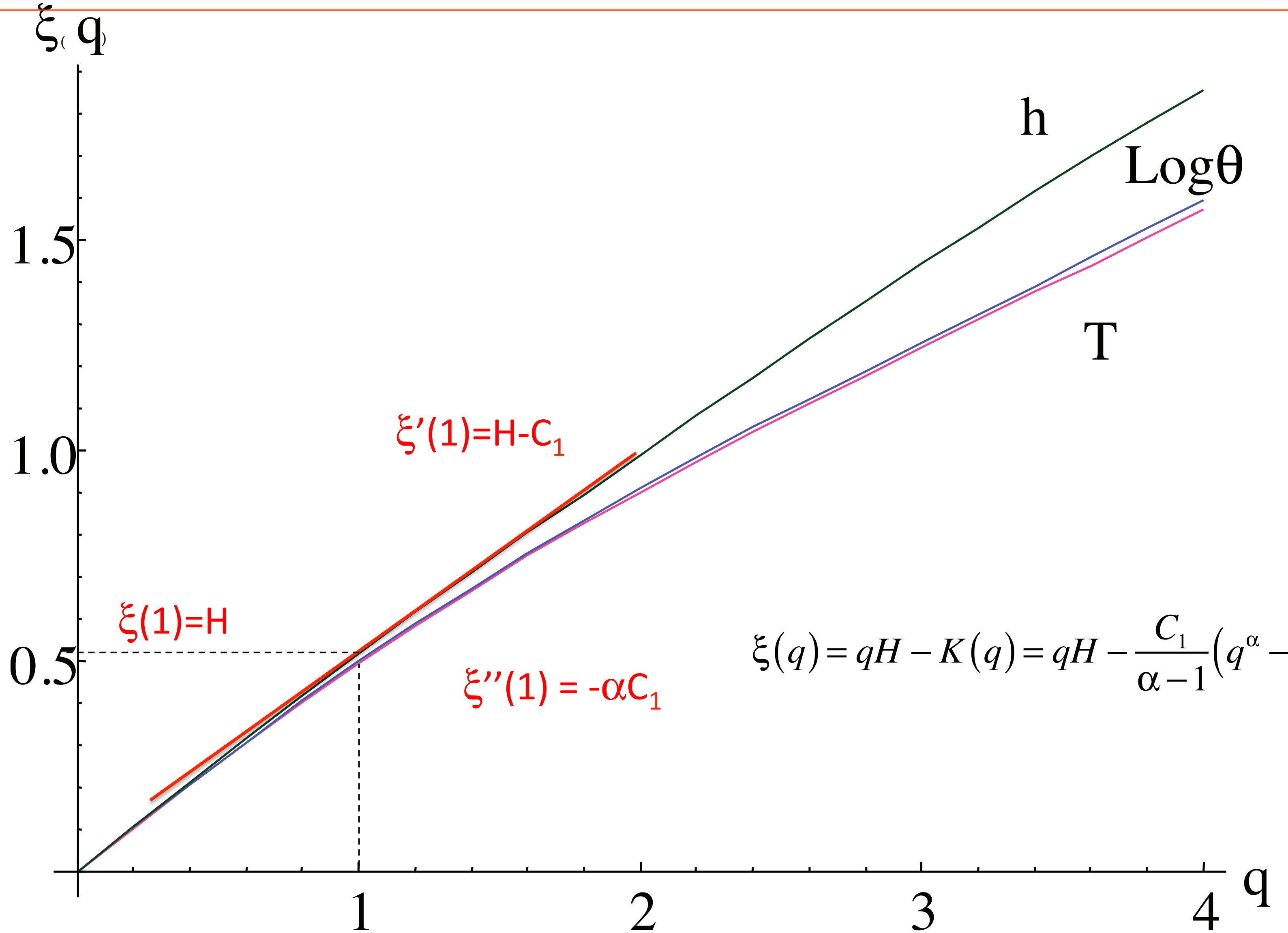


Fluctuations as differences

Temperature (Upper left), humidity (upper right), log potential temperature (lower left)

The structure functions of order q = 0.2, 0.4, …, 1.8, 2.0 are shown (from bottom to top). All have been nondimensionalized by dividing by the absolute mean first difference at the finest scale (280 *m*)

$\xi(q)$



$$\xi(q) = qH - K(q) = qH - \frac{C_1}{\alpha - 1}\left(q^{\alpha} - q\right)$$

The structure function exponents for *T*, logθ, *h* from the aircraft data analysed in the previous slide. The exponents were estimated by fitting the structure functions over the "optimal" range 4 – 40 *km*.

Difference, Tendency, Haar fluctuations

Differences: The difference in temperature between t and t+Δt

Tendency: The average of the temperature (with overall mean removed) between t and t+Δt

Haar: The difference between the average of the temperature from t and t+Δt/2 and from t+Δt/2 and t+Δt

Relations:
When 1 > H > 0: Haar ≈ difference
When 0 > H > -1: Haar ≈ tendency

Fluctuations and wavelets

$$\Delta T(\Delta t) = \frac{1}{\Delta t}\int T(t')\Psi\left(\frac{t-t'}{\Delta t}\right)dt'$$

mother wavelet

Difference

$$(\Delta T)_{diff} = T(t+\Delta t/2) - T(t-\Delta t/2) \qquad \Psi(t) = \delta(t-1/2) - \delta(t+1/2)$$

Tendency / Anomaly

$$(\Delta T)_{tend} = \frac{1}{\Delta t}\int_t^{t+\Delta t} T'(t')dt'; \quad T'(t) = T(t) - \overline{T(t)} \qquad \Psi(t) = I_{[-1/2,1/2]}(t) - \frac{I_{[-\tau/2,\tau/2]}(t)}{\tau}; \quad \tau >> 1 \qquad I_{[a,b]}(t) = \begin{matrix} 1 & a \le t \le b \\ 0 & otherwise \end{matrix}$$

I is the indicator function

Haar

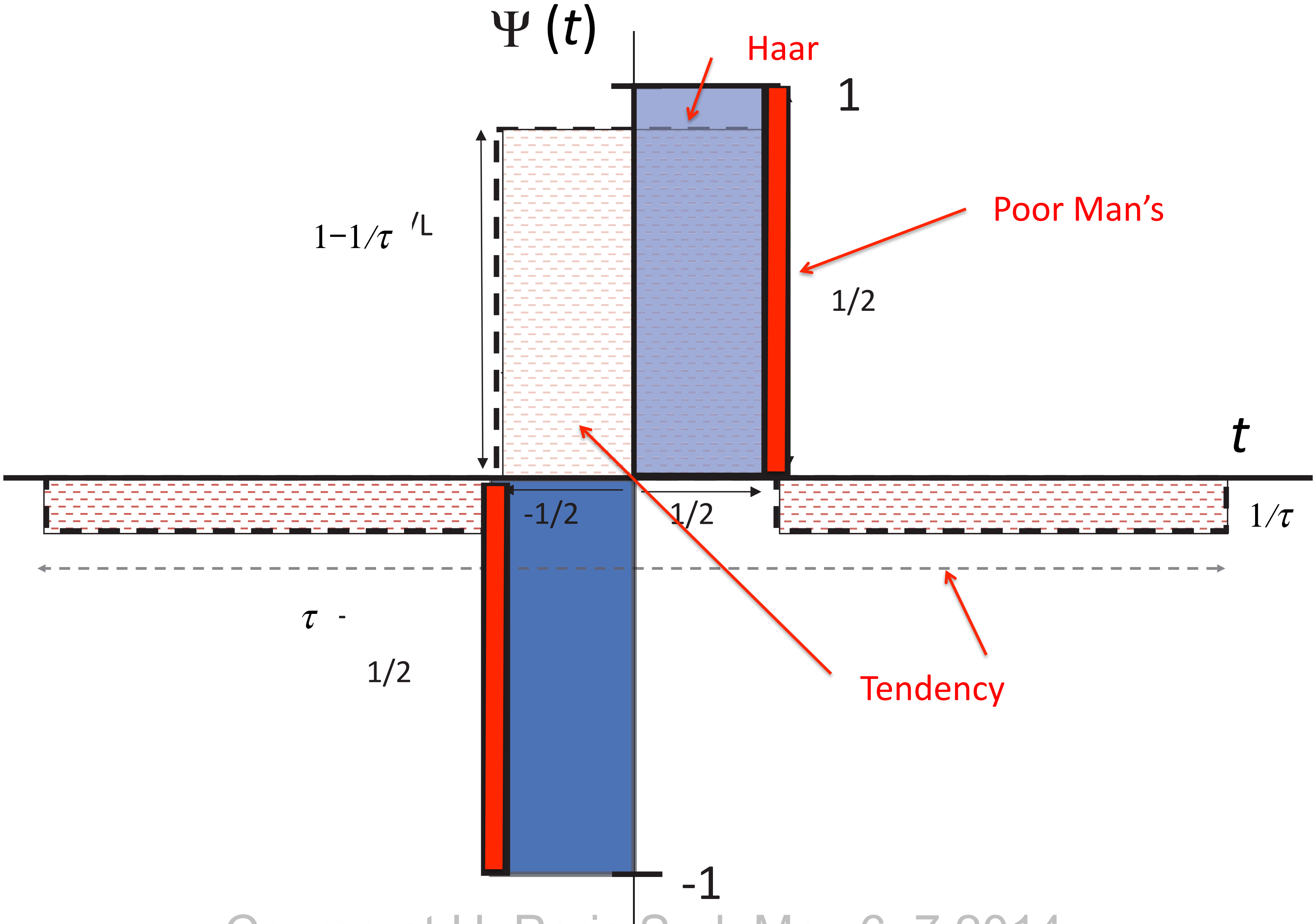
$$(\Delta T)_{Haar} = \frac{2}{\Delta t}\left[\int_t^{t+\Delta t/2} T(t')dt' - \int_{t+\Delta t/2}^{t+\Delta t} T(t')dt'\right] \qquad \Psi(t) = \begin{matrix} 1/2; & 0 \le t < 1/2 \\ -1/2; & -1/2 \le t < 0 \\ 0; & otherwise \end{matrix}$$

Relation between them:

$$(\Delta T)_{Haar} = \left(\Delta(\Delta T)_{tend}\right)_{diff}$$

Haar, tendency and poor man's wavelets

$$\Delta T(\Delta t) = \frac{1}{\Delta t}\int T(t')\Psi\left(\frac{t'-t}{\Delta t}\right)dt'$$



Spectrum of fluctuations

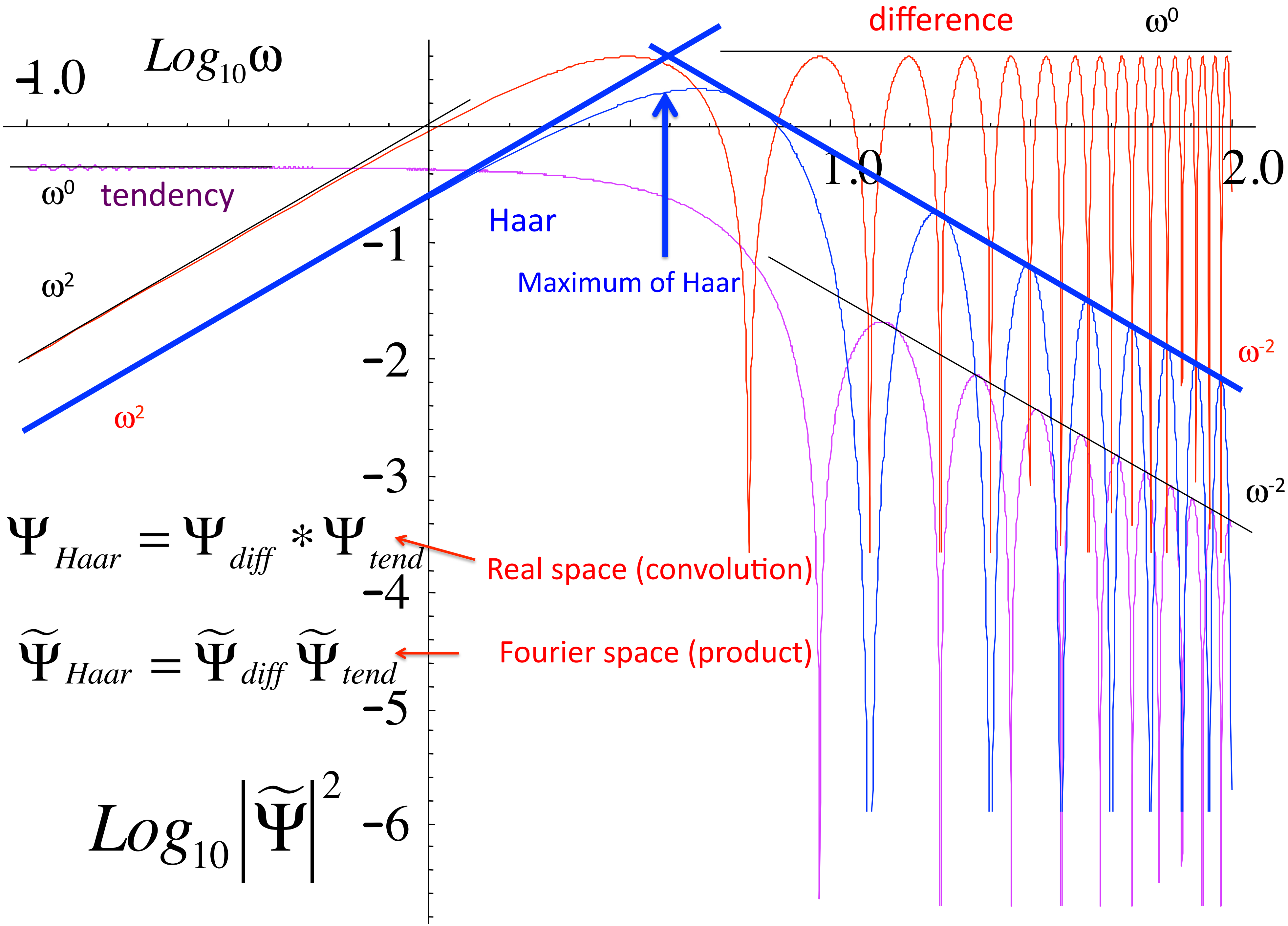
Fluctuations: $$\Delta T(\Delta t) = \frac{1}{\Delta t}\int T(t')\Psi\left(\frac{t'-t}{\Delta t}\right)dt'$$

Fourier transforms $$\widetilde{\Delta T}(\omega\Delta t) = \widetilde{T(\omega)}\widetilde{\Psi(\omega)}$$

Ensemble averaging of modulus squared: $$\left\langle\left|\widetilde{\Delta T}(\omega\Delta t)\right|^2\right\rangle = \left\langle\left|\tilde{T}(\omega)\right|^2\right\rangle\left|\widetilde{\Psi(\omega)}\right|^2$$

Spectra $$E_{\Delta T}(\omega\Delta t) = E_T(\omega)\left|\widetilde{\Psi(\omega)}\right|^2$$

If the maximum of $\left|\widetilde{\Psi(\omega)}\right|^2$ Occurs at ω_m, then the maximum in $E_{\Delta T}$ may be near $\omega_m\Delta t$



difference
ω0
-1.0
Log10ω
1.0
2.0
ω0
tendency
Haar
Maximum of Haar
ω2
ω2
ω-2
ω-2
-1
-2
-3
-4
-5
-6
Ψ Haar = Ψ diff * Ψ tend
Real space (convolution)
Ψ̃ Haar = Ψ̃ diff Ψ̃ tend
Fourier space (product)
Log10 |Ψ̃|2

Convergence of fluctuation variance

Spectra

$$E_{\Delta T}(\omega\Delta t) = E_T(\omega)\left|\widetilde{\Psi(\omega)}\right|^2$$

For scaling processes

$$E_T(\omega) \approx \omega^{-(1+2H')} \qquad \left|\widetilde{\Psi(\omega)}\right|^2 \approx \begin{matrix} \omega^{2H_{low}}; & \omega \to 0 \\ \omega^{2H_{high}}; & \omega \to \infty \end{matrix}$$

$$\left\langle \Delta T^2 \right\rangle = \int_{-\infty}^{\infty} E_{\Delta T}(\omega)\, d\omega$$

Parseval's theorem

Converges only if: $H_{low} > H' > H_{high}$

Various wavelets

Name	Wavelet	Frequency domain	low ω	high ω	H' range
Poor man's (first difference)	$\delta(t-1/2)-\delta(t+1/2)$	$2\sin(\omega/2)$	$\approx \omega$	≈ 0	$0 \le H' \le 1$
2^{nd} difference	$\frac{1}{2}(\delta(t+1/2)+\delta(t-1/2))-\delta(t)$	$\sin^2(\omega/4)$	$\approx \omega^2$	≈ 0	$0 \le H' \le 1$
Tendency	$I_{[-1/2,1/2]}(t)-\frac{I_{[-\tau/2,\tau/2]}(t)}{\tau}; \quad \tau >> 1$	$\frac{2}{\omega}\left(\sin\left(\frac{\omega}{2}\right)-\tau^{-1}\sin\left(\frac{\omega\tau}{2}\right)\right)$	$\frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega\tau}\approx 0; \quad \omega\tau >> 1$	$\approx \omega^{-1}$	$-1 \le H' \le 0$
Haar	$\psi(t)= \begin{matrix} 1/2; & 0 \le t < 1/2 \\ -1/2; & -1/2 \le t < 0 \\ 0; & otherwise \end{matrix}$	$2i\omega^{-1}\sin^2\left(\frac{\omega}{4}\right)$	$\approx \omega$	$\approx \omega^{-1}$	$-1 \le H' \le 1$
Quadratic Haar	$\psi(t)= \begin{matrix} -1/3 & 1/3 < t < 1 \\ 2/3; & -1/3 \le t \le 1/3 \\ -1/3; & -1 \le t < -1/3 \\ 0; & otherwise \end{matrix}$	$\frac{2}{3\omega}\left(3\sin\frac{\omega}{3}-\sin\omega\right)$	$\approx \omega^2$	$\approx \omega^{-1}$	$-1 \le H' \le 2$
First derivative Gaussian	$\Psi(t) \propto \frac{d}{dt}e^{-t^2/2}$	$\omega e^{-\omega^2/2}$	$\approx \omega$	$e^{-\omega^2/2}$	$-\infty \le H' \le 1$
Mexican Hat	$\Psi(t) \propto \frac{d^2}{dt^2}e^{-t^2/2}$	$\omega^2 e^{-\omega^2/2}$	$\approx \omega^2$	$e^{-\omega^2/2}$	$-\infty \le H' \le 2$

Range of exponents over which average fluctuations at scale Δt corresponds to frequency 1/Δt

Fluctuation → $\langle \Delta I \rangle = \langle \varphi \rangle \Delta t^{H}$; $\langle \varphi \rangle$ = constant

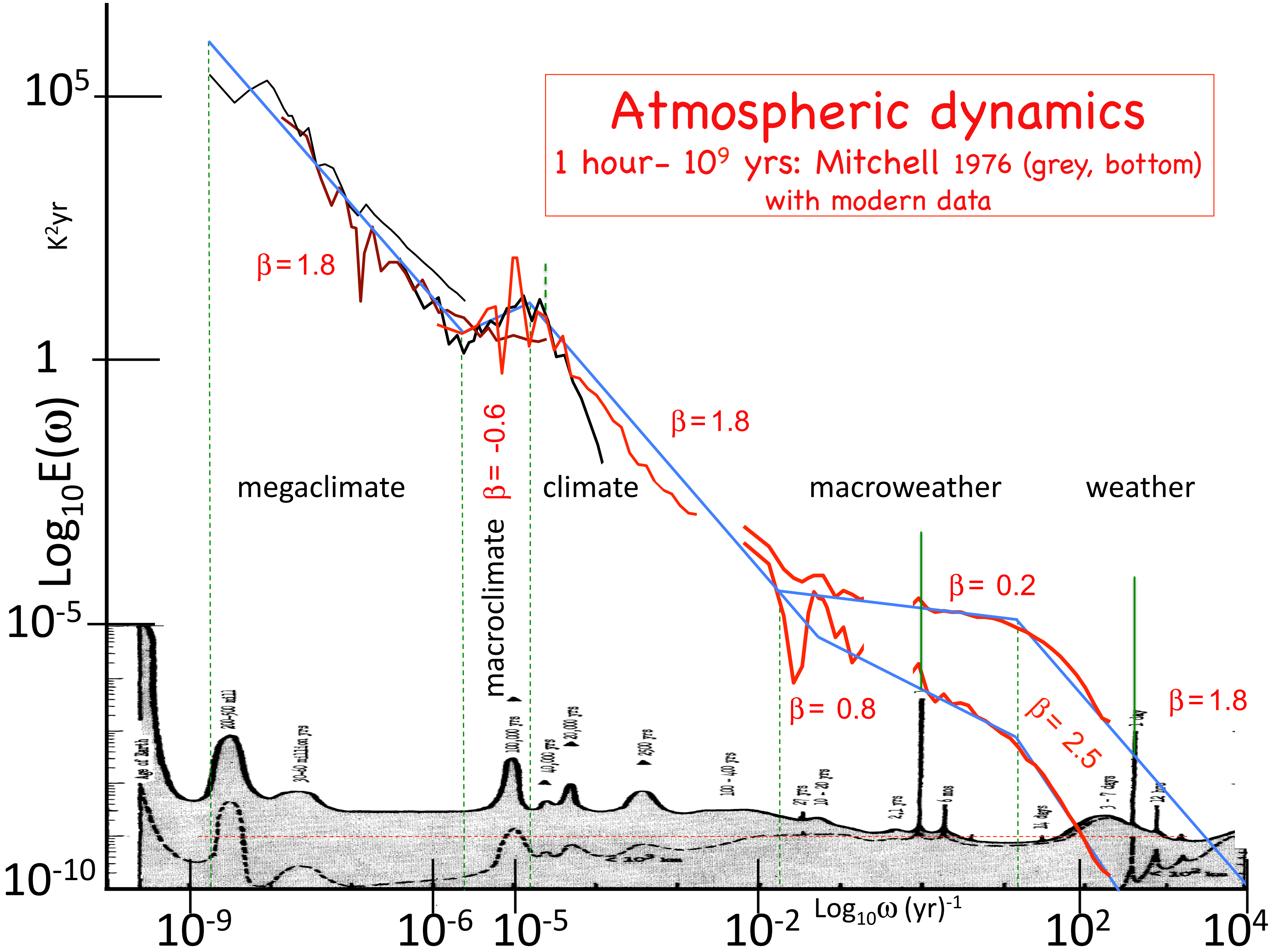
$E(\omega) = \left\langle \left| \tilde{I}(\omega) \right|^2 \right\rangle = \omega^{-\beta}$

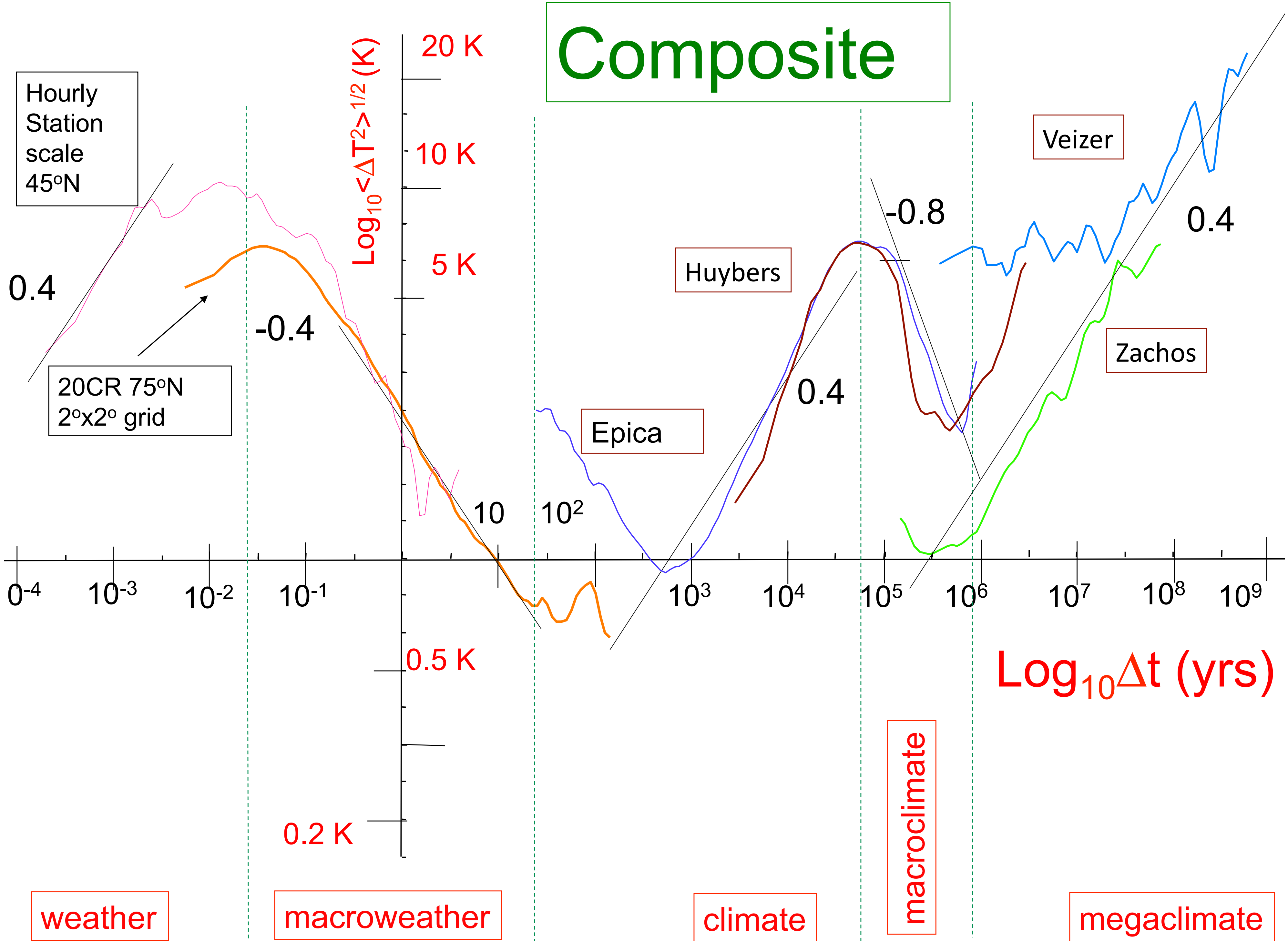
$\beta = 1 + 2H - K(2)$ ← Multifractal "correction"

$H' = H - K(2)/2$

Statistic	Range of H	Range of β	Comment
Spectrum	$-\infty < H < \infty$	$-\infty < \beta < \infty$	$E(\omega) \approx \omega^{-\beta}$
Difference	0<H<1	1<β+K(2)<3	"Poor man's wavelet"
Tendency Fluctuation	-1<H<0	-1<β+K(2)<1	Average with overall mean removed (standard deviation= "Climactogram", also called the "Aggregated Standard Deviation")
Haar	-1<H<1	-1<β+K(2)<3	Difference of means of first and second halves of interval
Detrended Fluctuation Analysis (DFA, polynomial order n	-1<H<(n+1)	-1<β+K(2)<3+2n	Also multifractal extension (MFDFA), usually linear: n=1, Not a wavelet
Mexican Hat Wavelet	$-\infty < H < 2$	$-\infty < \beta + K(2) < 5$	2nd Derivative of a Gaussian
Generalized Haar	-m<H<n	1-2m<β+K(2)<3+2n	Interpretation not simple

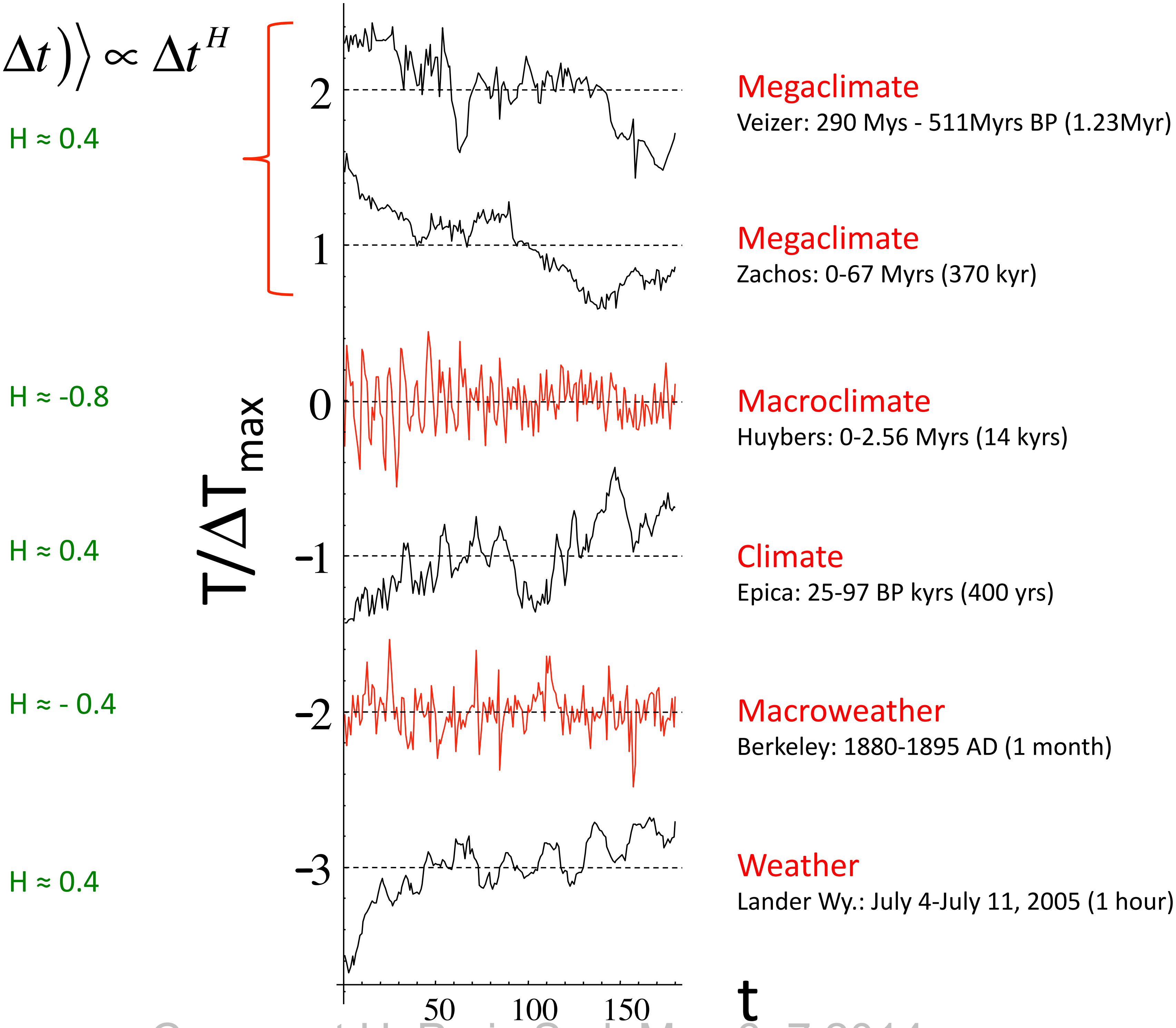
Simple interpretation (Difference, Tendency Fluctuation, Haar)


Atmospheric dynamics
1 hour- 10^9 yrs: Mitchell 1976 (grey, bottom)
with modern data
K²yr
Log10E(ω)
10^5
1
10^-5
10^-10
β=1.8
β= -0.6
β=1.8
β= 0.2
β= 0.8
β= 2.5
β=1.8
megaclimate
macroclimate
climate
macroweather
weather
10^-9
10^-6
10^-5
10^-2
Log10ω (yr)^-1
10^2
10^4



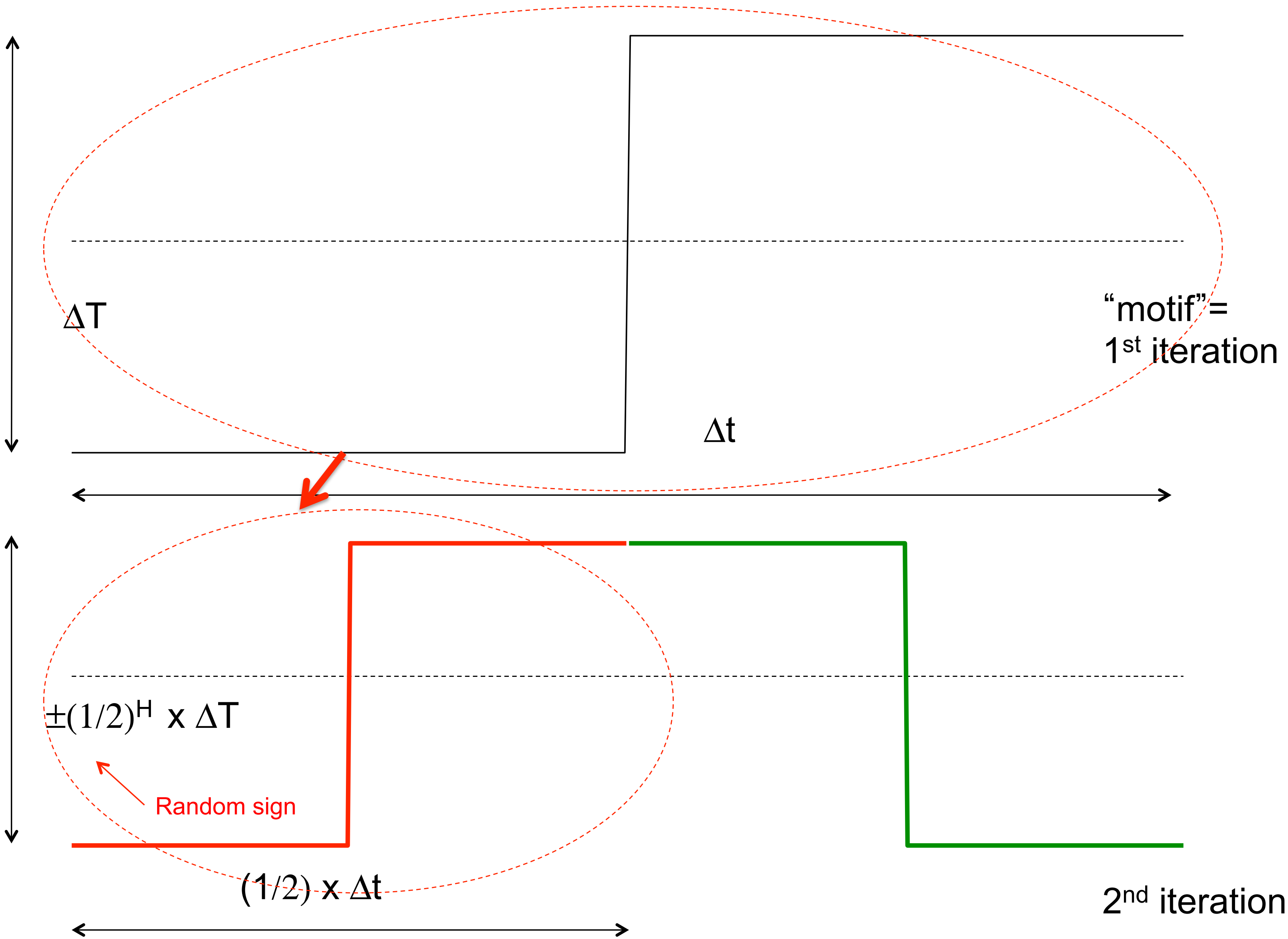
Composite
Hourly
Station
scale
45°N
0.4
20CR 75°N
2°x2° grid
-0.4
Log10<ΔT2>1/2 (K)
20 K
10 K
5 K
0.5 K
0.2 K
10
10²
Epica
Huybers
0.4
-0.8
Veizer
0.4
Zachos
10-4
10-3
10-2
10-1
10³
10⁴
10⁵
10⁶
10⁷
10⁸
10⁹
Log10Δt (yrs)
weather
macroweather
climate
macroclimate
megaclimate

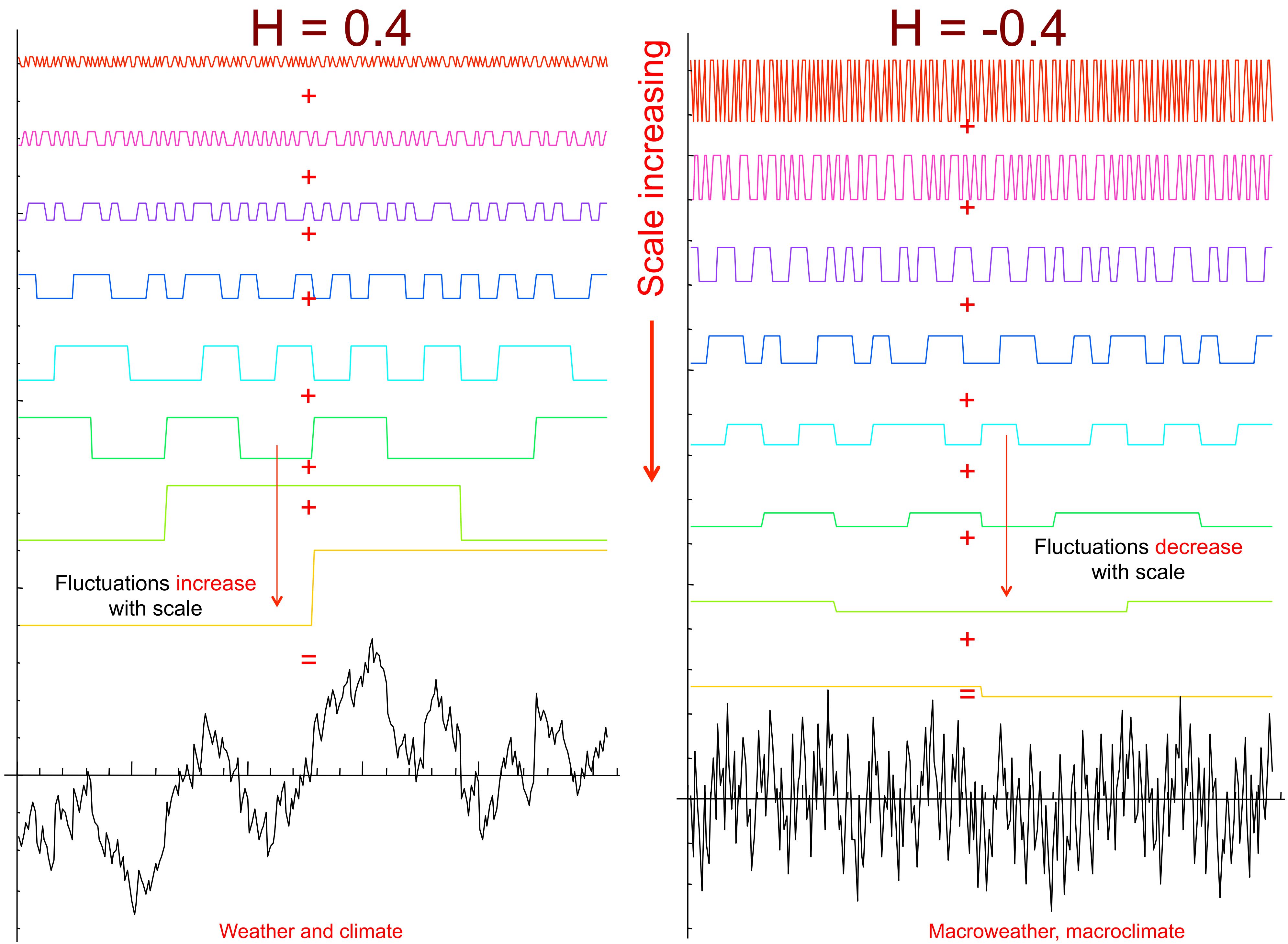
$$\langle \Delta T(\Delta t) \rangle \propto \Delta t^{H}$$



Understanding the Fluctuation exponent H

The fractal H model





H = 0.4
H = -0.4
Scale increasing
Fluctuations increase with scale
Fluctuations decrease with scale
Weather and climate
Macroweather, macroclimate