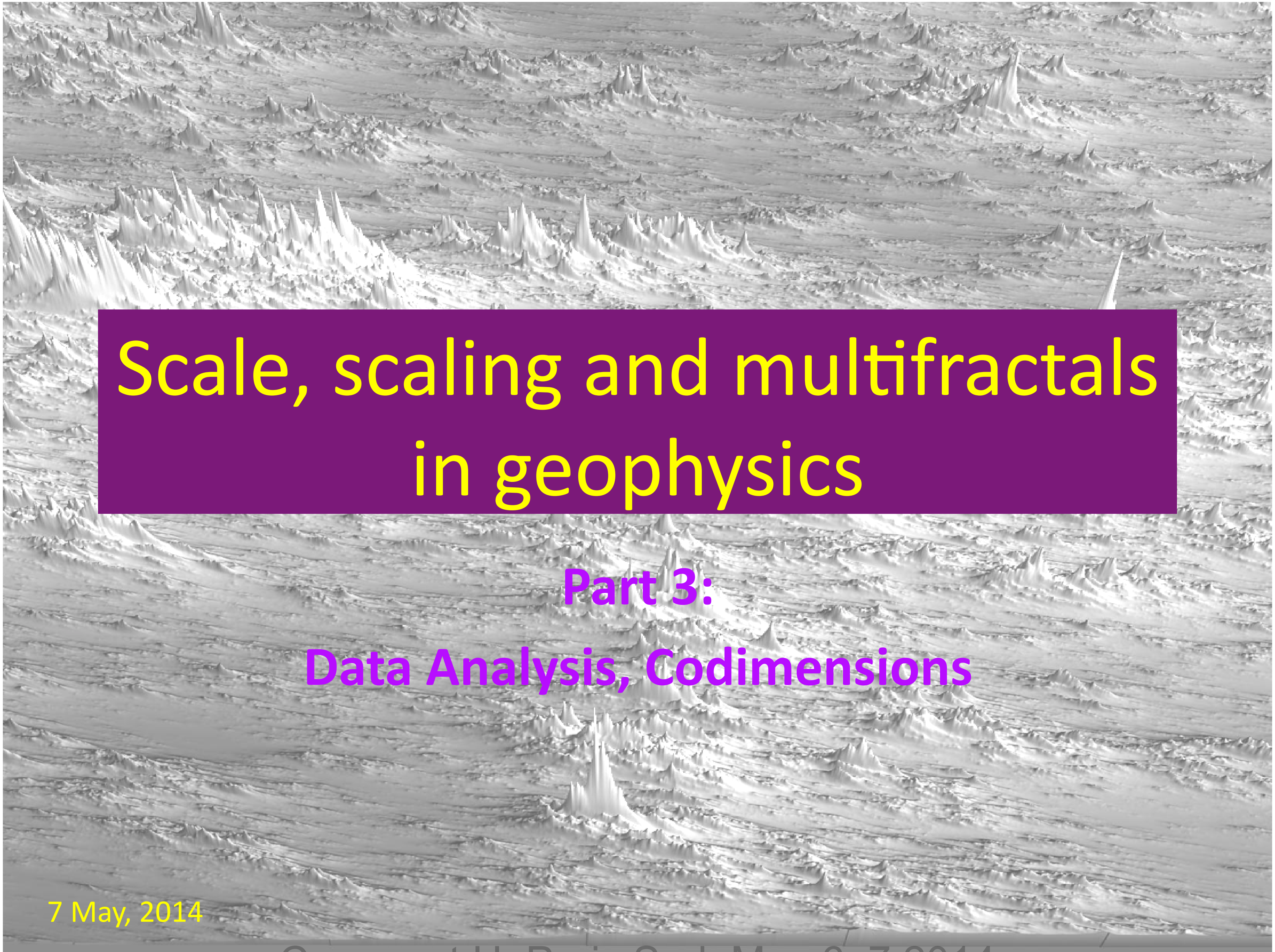


Scale, scaling and multifractals in geophysics

Part 3:

Data Analysis, Codimensions

7 May, 2014

Spectral analysis

$$R(\Delta t) = \langle T(t)T(t-\Delta t)\rangle$$ autocorrelation

$$S(\Delta t) = \left\langle \left(T(t) - T(t-\Delta t)\right)^2 \right\rangle = \left\langle \Delta T(\Delta t)^2 \right\rangle$$ Classical (q=2) structure function

$$S(\Delta t) = 2\left(R(0) - R(\Delta t)\right)$$ Relation between them

Spectral analysis (spectrum E(ω))

$$\left\langle \widetilde{T}(\omega)\widetilde{T}^*(\omega')\right\rangle = \delta(\omega+\omega')E(\omega); \quad \widetilde{T(\omega)} = \int_{-\infty}^{\infty} T(t)e^{-i\omega t}\,dt$$

Statistics independent of time (stationarity)

Note: for real T(t), we have: $\widetilde{T}(\omega) = \widetilde{T}^*(-\omega)$ hence $E(\omega) \propto \left\langle \left|\widetilde{T(\omega)}\right|^2 \right\rangle$

Wiener-Khintchine theorem

$$R(\Delta t) = \int_{-\infty}^{\infty} E(\omega)e^{i\omega\Delta t}\,d\omega$$

$$R(0) = \int_0^{\infty} E(\omega)\,d\omega$$

$$S(\Delta t) = 2\int_0^{\infty}\left(1 - e^{i\omega\Delta t}\right)E(\omega)\,d\omega$$ Real space-Fourier space relation

Tauberian theorem

If the spectrum is of power law form: $E(\omega) \approx \omega^{-\beta}$

Then: $E(\omega/\lambda) = \lambda^{\beta} E(\omega)$

Using: $\omega = \omega' / \lambda$

$$S(\Delta t) = 2\int_{-\infty}^{\infty} \left(1 - e^{i\omega'\Delta t/\lambda}\right) \lambda^{-\beta} E(\omega'/\lambda) \frac{d\omega'}{\lambda}$$

$$S(\lambda\Delta t) = 2\int_{-\infty}^{\infty} \left(1 - e^{i\omega'\Delta t}\right) E(\omega') \lambda^{-1+\beta} d\omega' = \lambda^{-1+\beta} S(\Delta t)$$

Hence:

$$S(\Delta t) \approx \Delta t^{2H'}; \quad H' = (\beta - 1)/2$$

We conclude:

POWER LAWS <->F.T. POWER LAWS

For convergence of the integral: $1<\beta<3$ ($0 \le H' \le 1$) for $S(\Delta t)$, $1>\beta>-1$; for $R(\Delta t)$ ($-1/2<H'<0$)

Empirical analysis: Estimating fluxes from the fluctuations

Multifractal cascade equation: $\left\langle \varphi_\lambda^q \right\rangle = \lambda^{K(q)}$

Fluctuations: $\Delta T = \varphi_{\Delta t} \Delta t^H$

Estimating the fluxes from the fluctuations

$$\varphi'_\lambda = \frac{\varphi_\lambda}{\left\langle \varphi_\lambda \right\rangle} \approx \frac{\Delta T(\Delta t)}{\left\langle \Delta T(\Delta t) \right\rangle}; \quad \lambda = \frac{\tau}{\Delta t}$$

outer cascade scale

Normalized flux at resolution λ

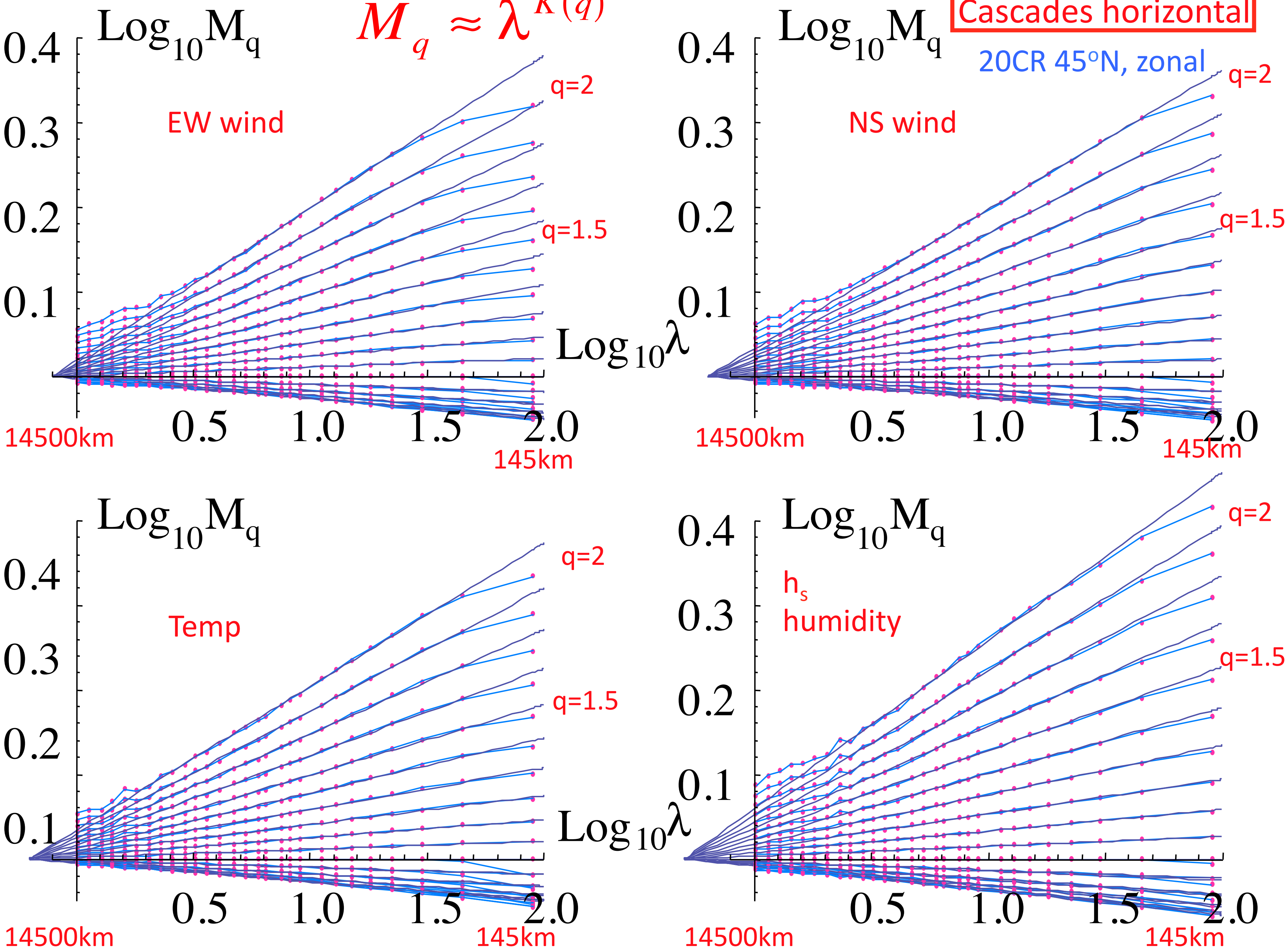
$$M_q = \left\langle \varphi'^q_\lambda \right\rangle$$

Estimate at finest resolution, then degrade to intermediate resolutions by averaging

"Trace moments"

Cascades horizontal

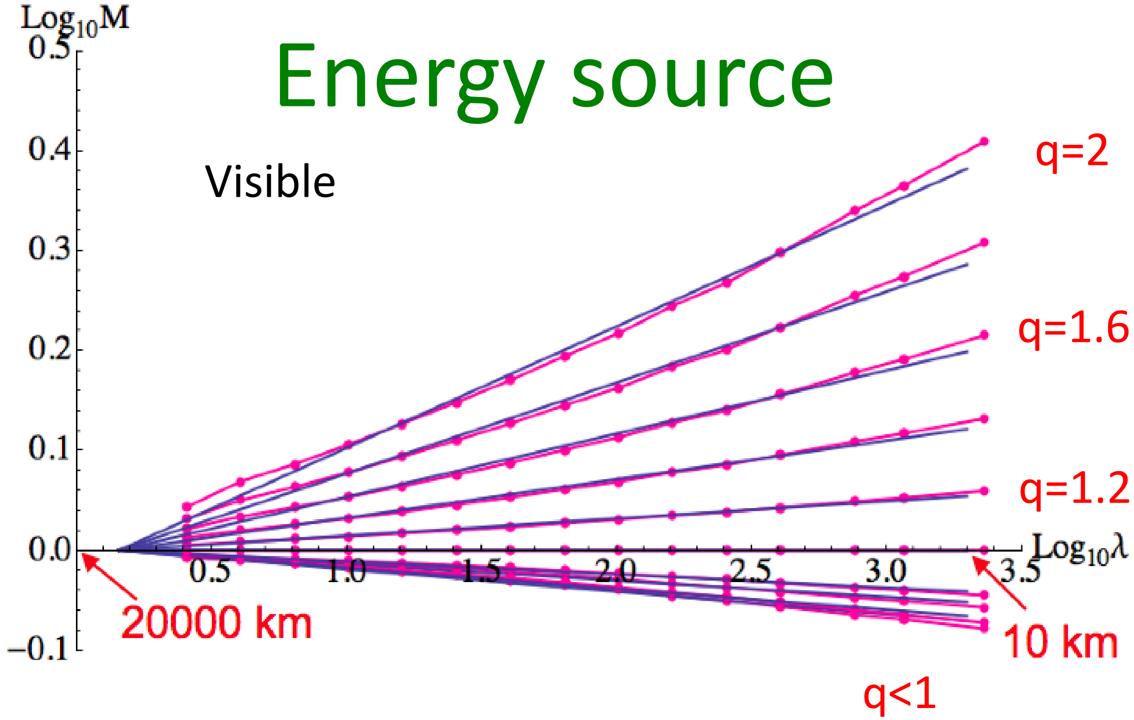
$$M_q \approx \lambda^{K(q)}$$

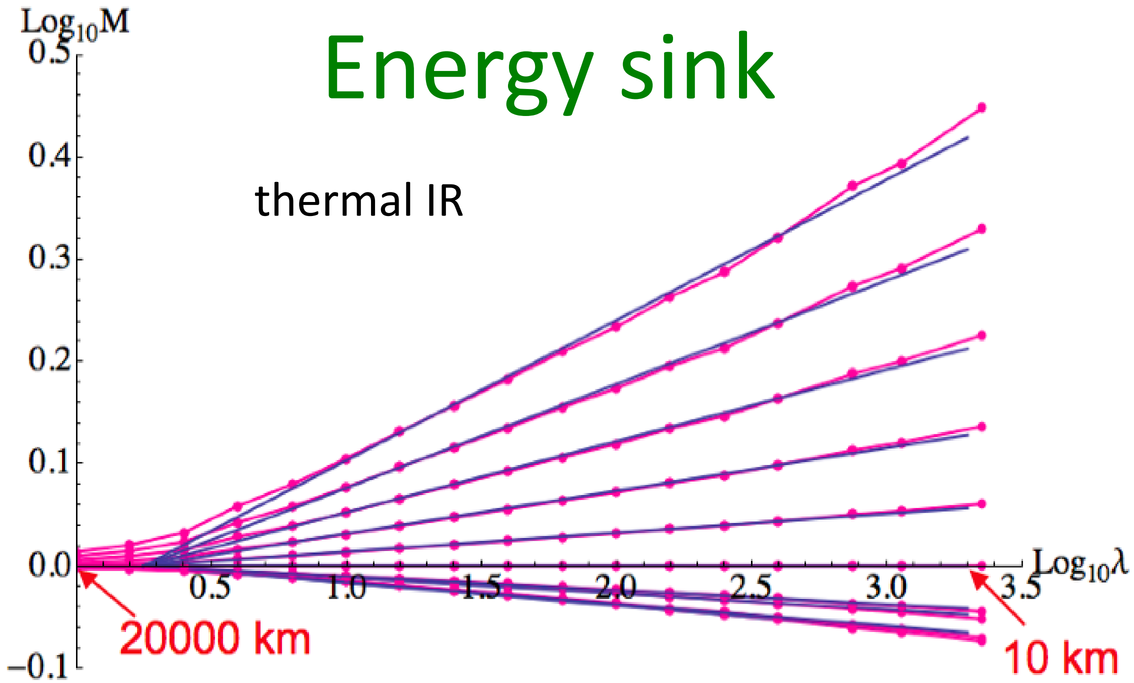


Course at U. Paris Sud, May 6, 7 2014

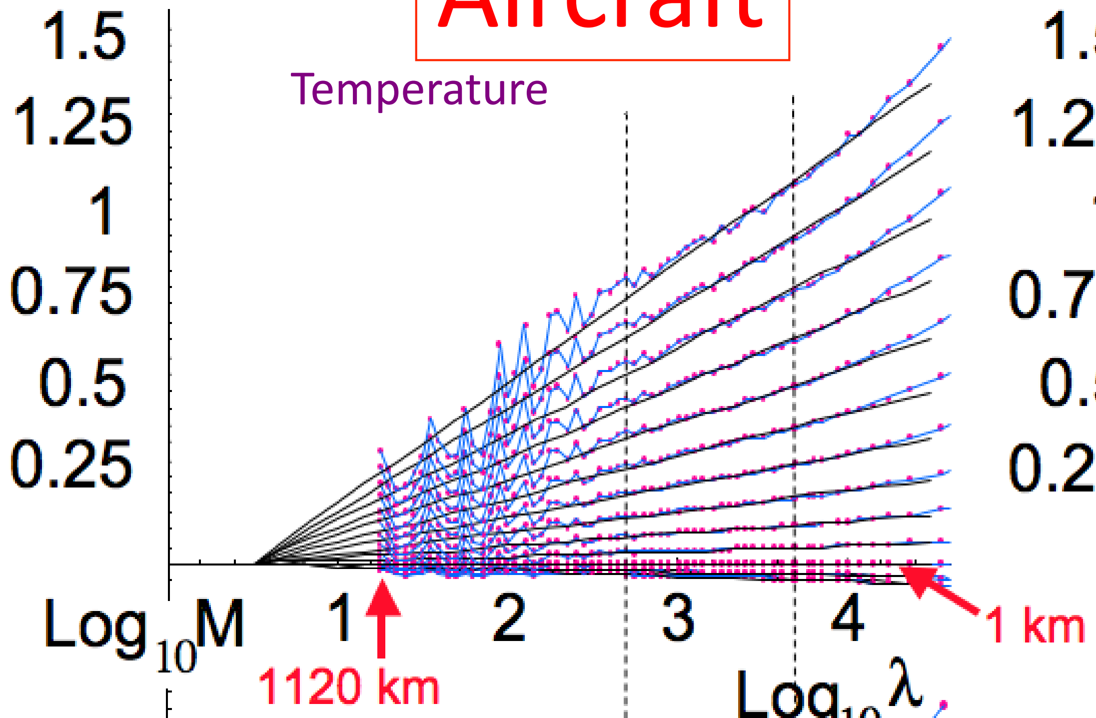
TRMM satellite data, ≈1000 orbits

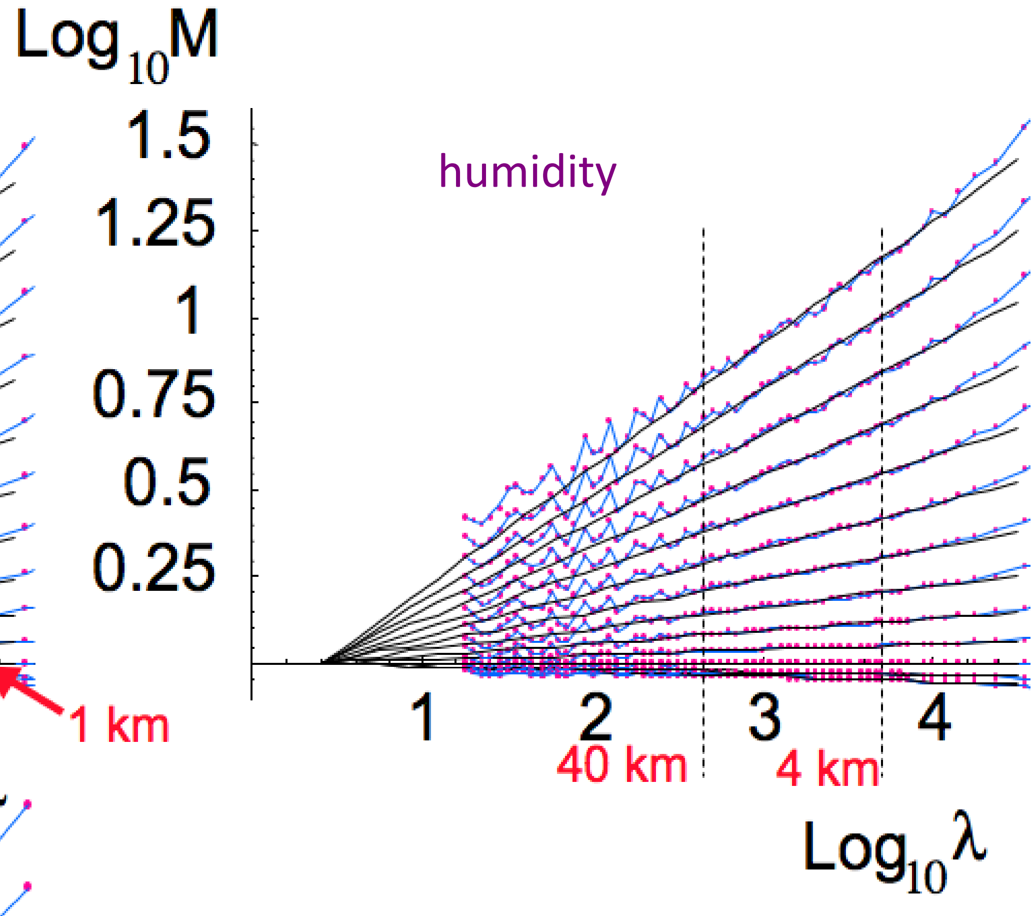
Energy budget

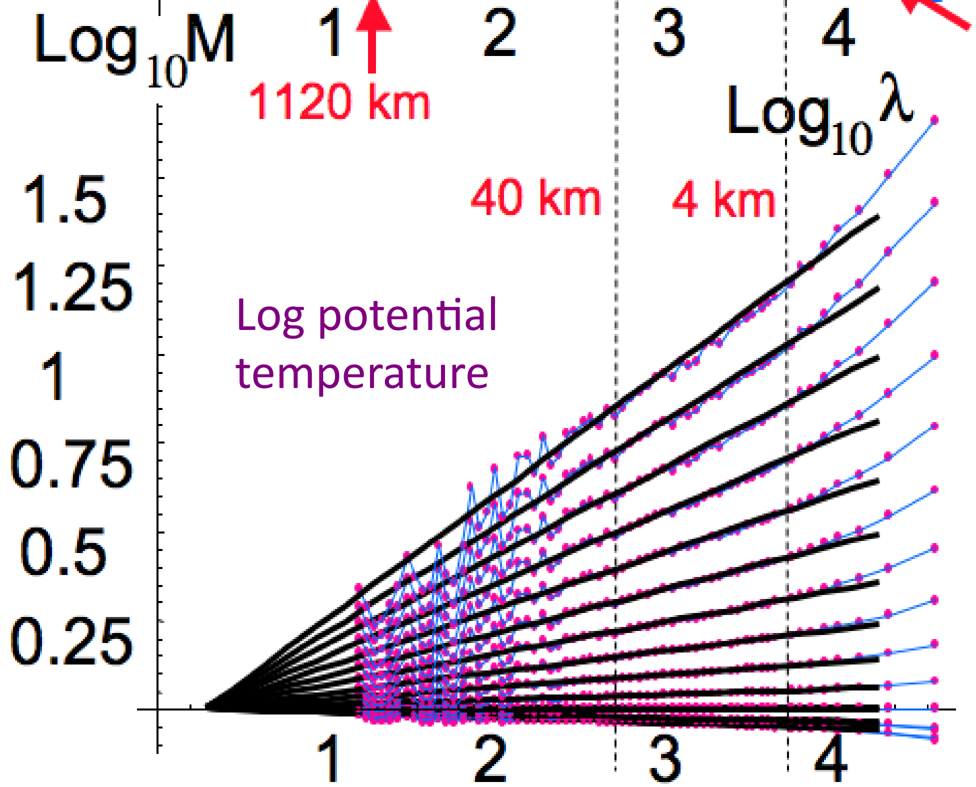




Aircraft





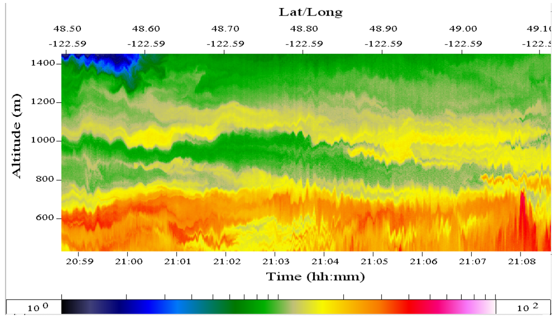


Horizontal cascades from 24 aircraft legs

(11-13km)

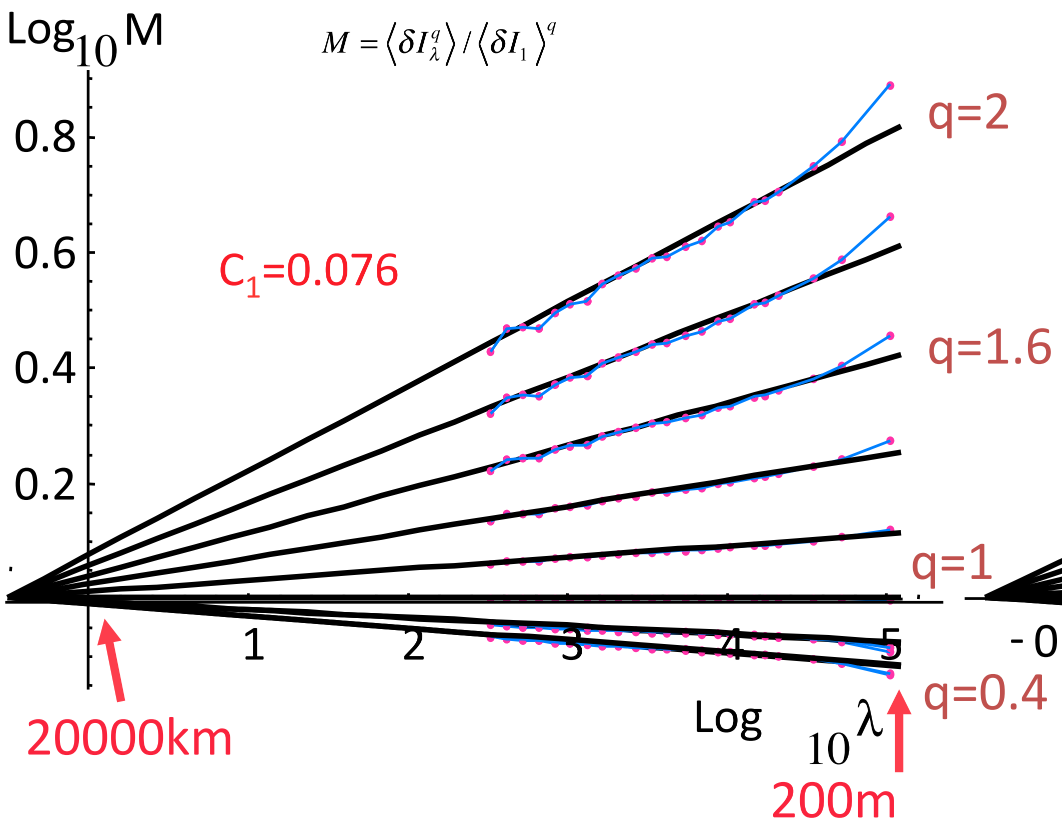
Fields that are relatively unaffected by the trajectories

Vertical cascades:
lidar backscatter

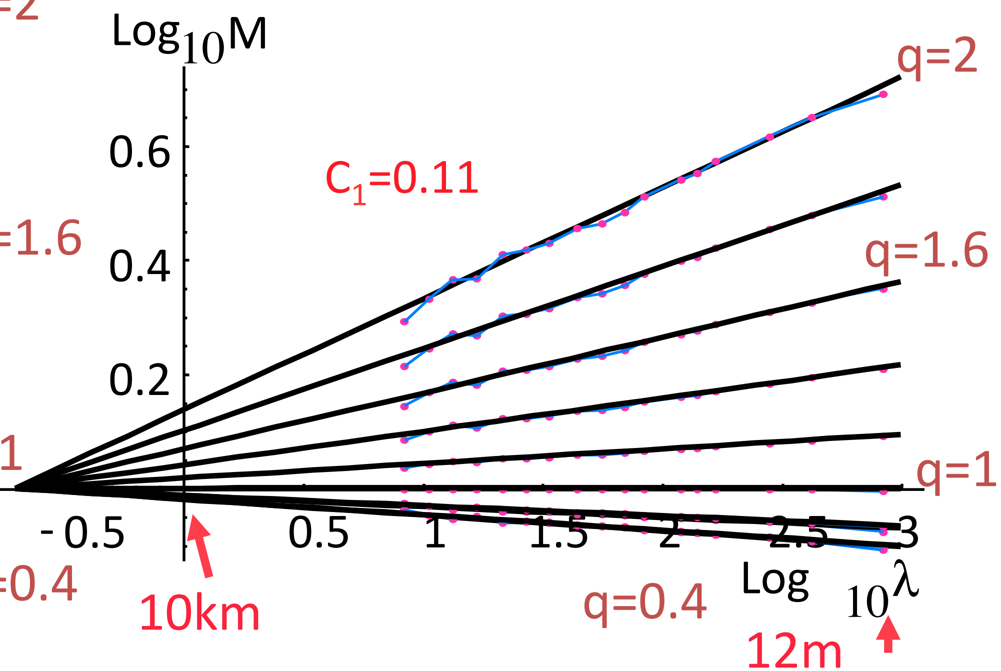


From 10 airborne lidar cross-sections near Vancouver B.C.

Horizontal cascade

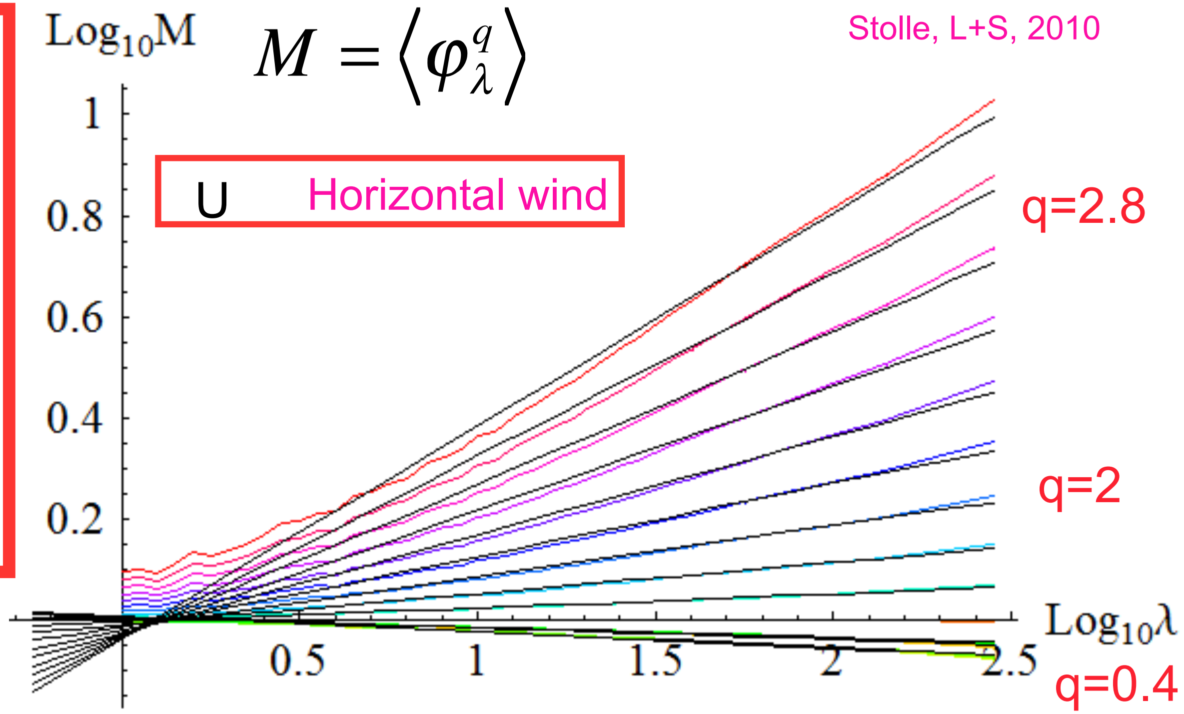


Vertical cascade

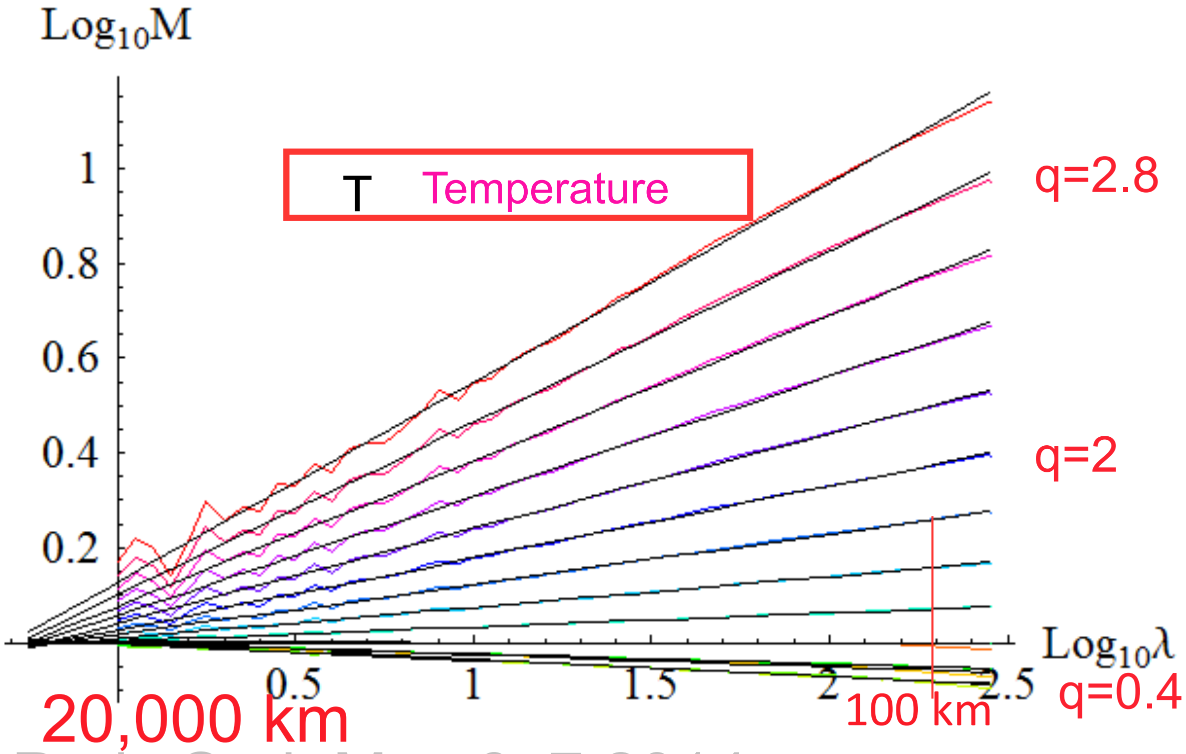


q=0, 0.2, 0.4..., 2

Global GEMS Model 00h

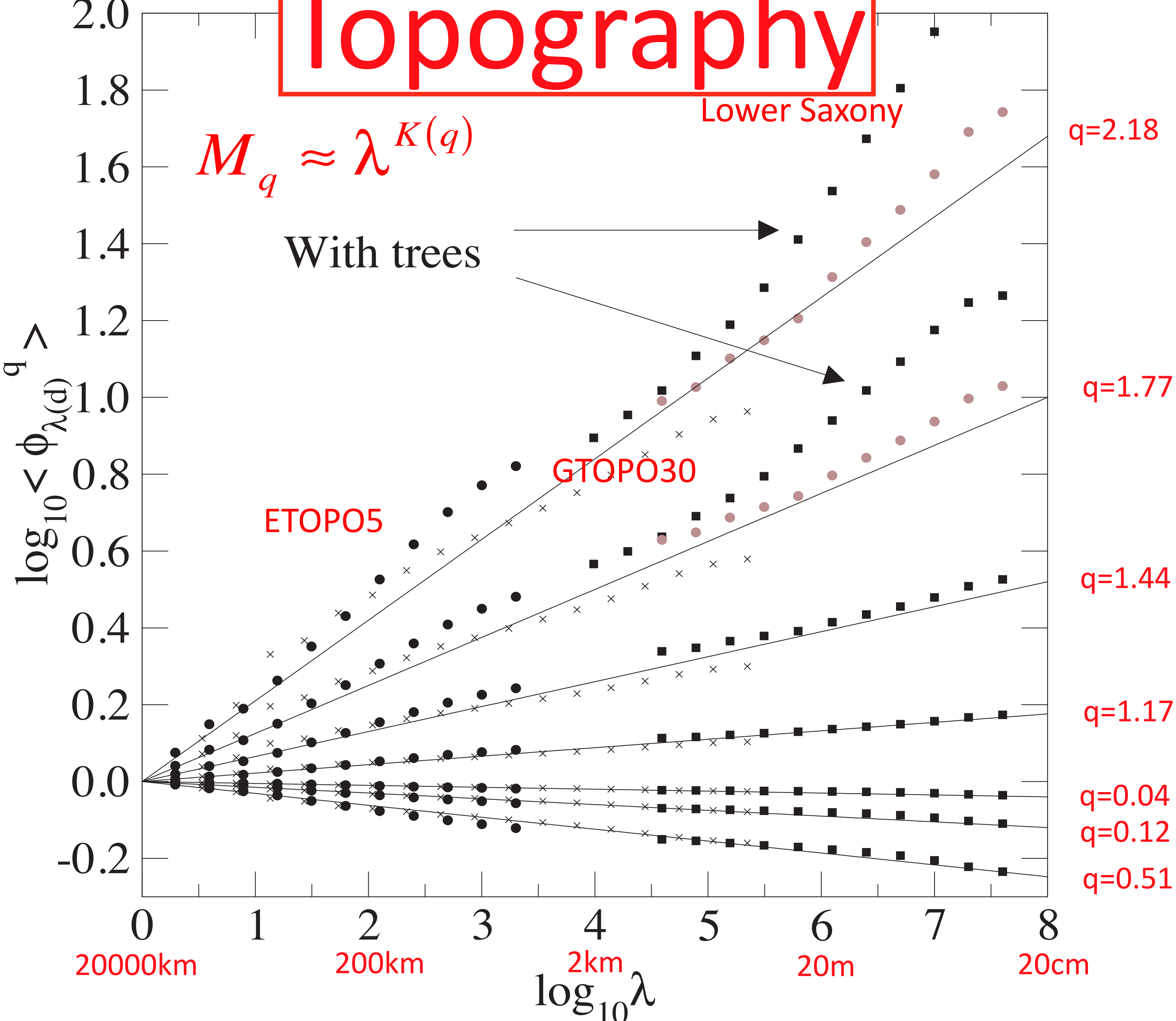


Analysis of four months
U,T at 1000 mb



(48 h forecasts are almost the same)

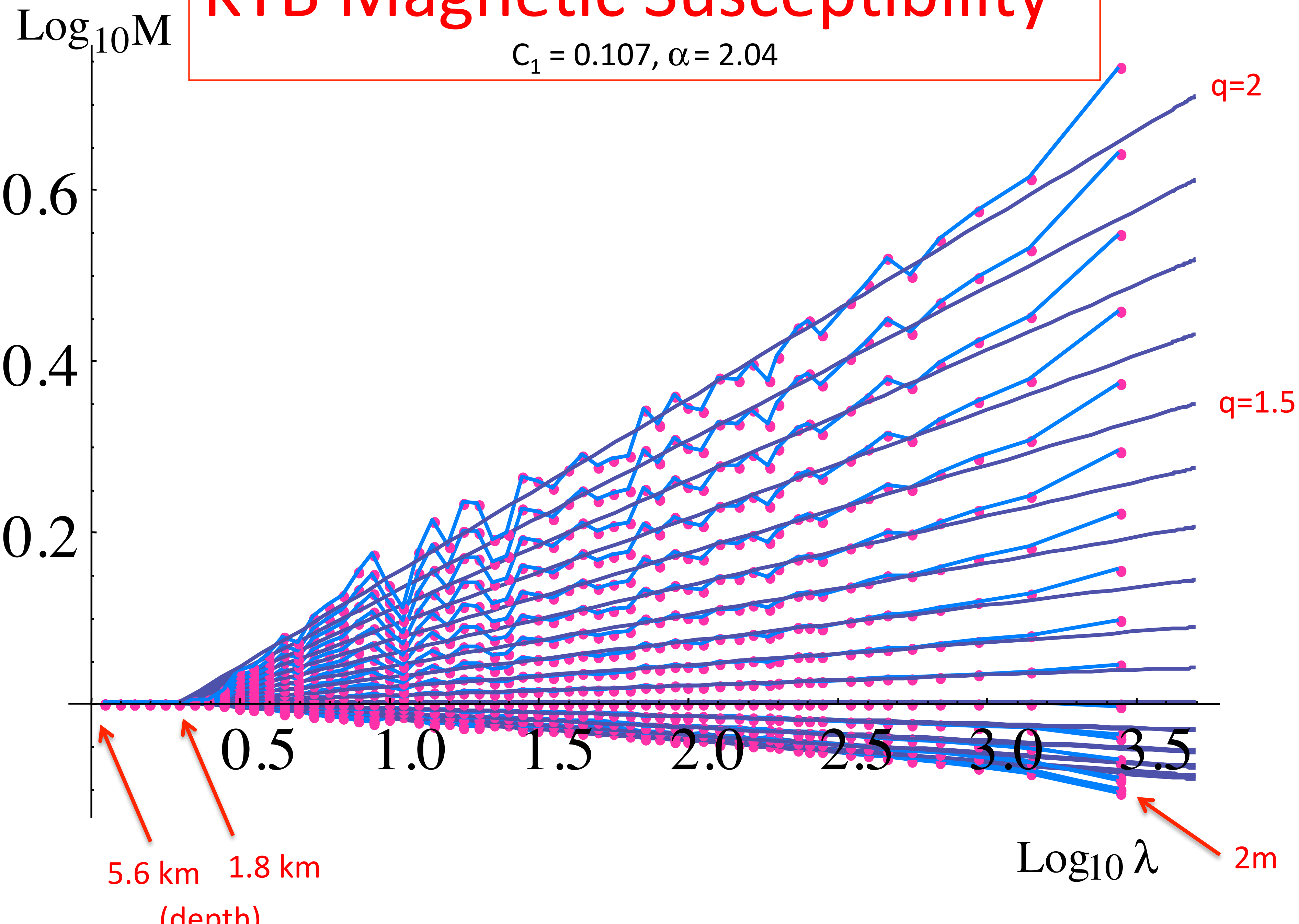
Topography



$M_q \approx \lambda^{K(q)}$
Lower Saxony
With trees
GTOPO30
ETOPO5
q=2.18
q=1.77
q=1.44
q=1.17
q=0.04
q=0.12
q=0.51
$\log_{10} < \phi_{\lambda(d)}^{q} >$
$\log_{10}\lambda$
20000km
200km
2km
20m
20cm

KTB Magnetic Susceptibility

$C_1 = 0.107, \alpha = 2.04$



Horizontal spatial Scaling exponents

		C_1	α	H	β	L_{eff}
State variables	u, v	0.09	1.9	1/3, (0.77)	1.6, (2.4)	(14 000)
	w	(0.12)	(1.9)	(−0.14)	(0.4)	(15 000)
	T	0.11, (0.08)	1.8	0.50, (0.77)	1.9, (2.4)	5000 (19 000)
	h	0.09	1.8	0.51	1.9	10 000
	z	(0.09)	(1.9)	(1.26)	(3.3)	(60 000)
Precipitation	R	0.4	1.5	0.00	0.2	32 000
Passive scalars	Aerosol concentration	0.08	1.8	0.33	1.6	25 000
Radiances	Infrared	0.08	1.5	0.3	1.5	15 000
	Visible	0.08	1.5	0.2	1.5	10 000
	Passive microwave	0.1–0.26	1.5	0.25–0.5	1.3–1.6	5000–15 000
Topography	Altitude	0.12	1.8	0.7	2.1	20 000
Sea surface temperature	SST (see Table 8.2)	0.12	1.9	0.50	1.8	16 000

$$\Delta I = \varphi \Delta x^{H} \qquad \langle \varphi_\lambda^q \rangle = \lambda^{K(q)} \qquad \lambda = L_{eff} / \Delta x \qquad K(q) = \frac{C_1}{\alpha - 1}(q^\alpha - q) \qquad E(k) \approx k^{-\beta}$$

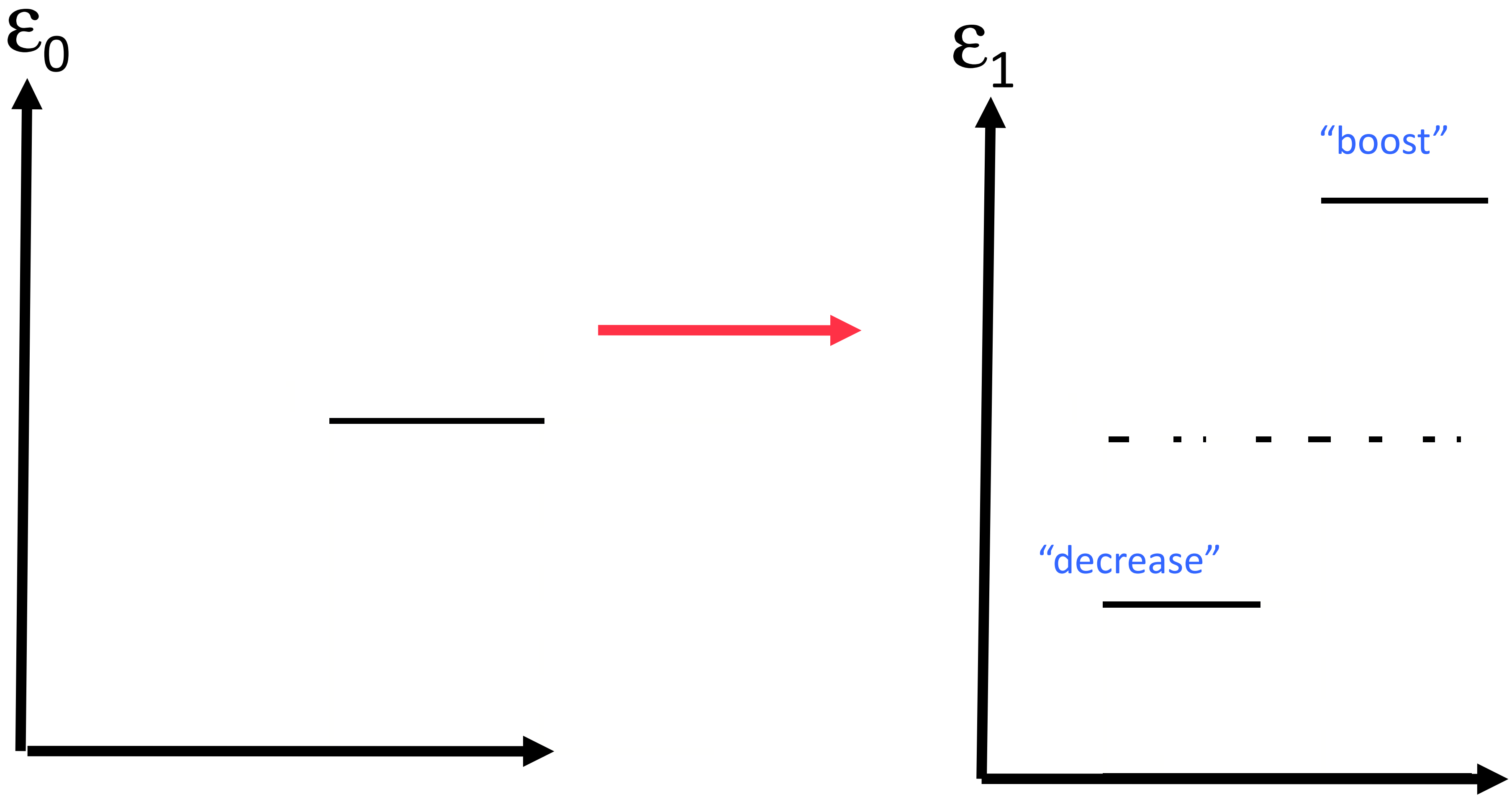
Surface, solid earth exponents

	C_1	α	H	β
Rock Density (vertical)	0.045	2.0	0.08	1.07
Magnetic susceptibility (vertical)	0.11	2.0	0.17	1.12
Topography	0.12	1.8	0.7	2.1
Vegetation index	0.064	2.0	0.16	1.19
Soil moisture index	0.053	2.0	0.14	1.17

$$\Delta I = \varphi \Delta x^{H} \quad \langle \varphi_\lambda^q \rangle = \lambda^{K(q)} \quad \lambda = L_{eff} / \Delta x \quad K(q) = \frac{C_1}{\alpha - 1}(q^\alpha - q) \quad E(k) \approx k^{-\beta}$$

Probabilities and codimensions

Revisiting the α Model



Revisiting the α Model

The α model is a binomial process:

$$\Pr(\mu\varepsilon = \lambda_0^{\gamma_+}) = \lambda_0^{-c} \qquad (>1 \;\Rightarrow\; \text{INCREASE})$$

$$\Pr(\mu\varepsilon = \lambda_0^{\gamma_-}) = 1 - \lambda_0^{-c} \qquad (<1 \;\Rightarrow\; \text{DECREASE})$$

where γ_+, γ_- correspond to boosts and decreases respectively, the β model being the special case where $\gamma_- = -\infty$ and $\gamma_+ = c$ (due to conservation $\langle\mu\varepsilon\rangle = 1$, there are only two free parameters):

$$\lambda_0^{\gamma_+ - c} + \lambda_0^{\gamma_-}\left(1 - \lambda_0^{-c}\right) = 1$$

Conservation constraint

Taking $\gamma_- > -\infty$, the pure orders of singularity γ_- and γ_+ lead to the appearance of mixed orders of singularity, of different orders γ ($\gamma_- \leq \gamma \leq \gamma_+$).

α Model after 2 steps

What is the behaviour as the number of cascade steps, $n \to \infty$? Consider two steps of the process, the various probabilities and random factors are:

Two steps: an equivalent 3 state model with $\lambda = \lambda_0^2$

$$\Pr(\mu\varepsilon = \lambda_0^{2\gamma_+}) = \lambda_0^{-2c} \qquad \text{(two boosts)}$$

$$\Pr(\mu\varepsilon = \lambda_0^{\gamma_+ + \gamma_-}) = 2\lambda_0^{-c}(1-\lambda_0^{-c}) \qquad \text{(one boost and one decrease)}$$

$$\Pr(\mu\varepsilon = \lambda_0^{2\gamma_-}) = (1-\lambda_0^{-c})^2 \qquad \text{(two decreases)}$$

Rewriting:

$$\Pr\left(\mu\varepsilon = \left(\lambda_0^2\right)^{\gamma_+}\right) = \left(\lambda_0^2\right)^{-c} \qquad \text{(one large)}$$

$$\Pr\left(\mu\varepsilon = \left(\lambda_0^2\right)^{(\gamma_+ + \gamma_-)/2}\right) = 2\left(\lambda_0^2\right)^{-c/2} - 2\left(\lambda_0^2\right)^{-c} \qquad \text{(intermediate)}$$

$$\Pr\left(\mu\varepsilon = \left(\lambda_0^2\right)^{\gamma_-}\right) = 1 - 2\left(\lambda_0^2\right)^{-c/2} + \left(\lambda_0^2\right)^{-c} \qquad \text{(large decrease)}$$

α Model after n steps

Iterating this procedure, after $n = n_+ + n_-$ steps we find:

$$\gamma_{n_+,n_-} = \frac{n_+\gamma_+ + n_-\gamma_-}{n_+ + n_-} \quad , \qquad n_+ = 1,\ldots,n$$

$$\Pr\left(\mu\varepsilon = \left(\lambda_0^n\right)^{\gamma_{n_+,n_-}}\right) = \begin{pmatrix} n \\ n_+ \end{pmatrix} \left(\lambda_0^n\right)^{-cn_+/n} \left(1-\left(\lambda_0^n\right)^{-c/n}\right)^{n_-}$$

where $\begin{pmatrix} n \\ n_+ \end{pmatrix}$ is the number of combinations of n objects taken n^+ at a time. This implies that we may write:

$$\Pr\left(\varepsilon_{\lambda_0^n} \geq \left(\lambda_0^n\right)^{\gamma_i}\right) = \sum_j p_{i,j} \left(\lambda_0^n\right)^{-c_{i,j}}$$

The p_{ij}'s are the "submultiplicities" (the prefactors in the above), c_{ij} are the corresponding exponents ("subcodimensions") and λ_0^n is the total ratio of scales from the outer scale to the smallest scale. Notice that the requirement that $\langle\mu\varepsilon\rangle = 1$ implies that some of the λ^{γ_i} are >1.

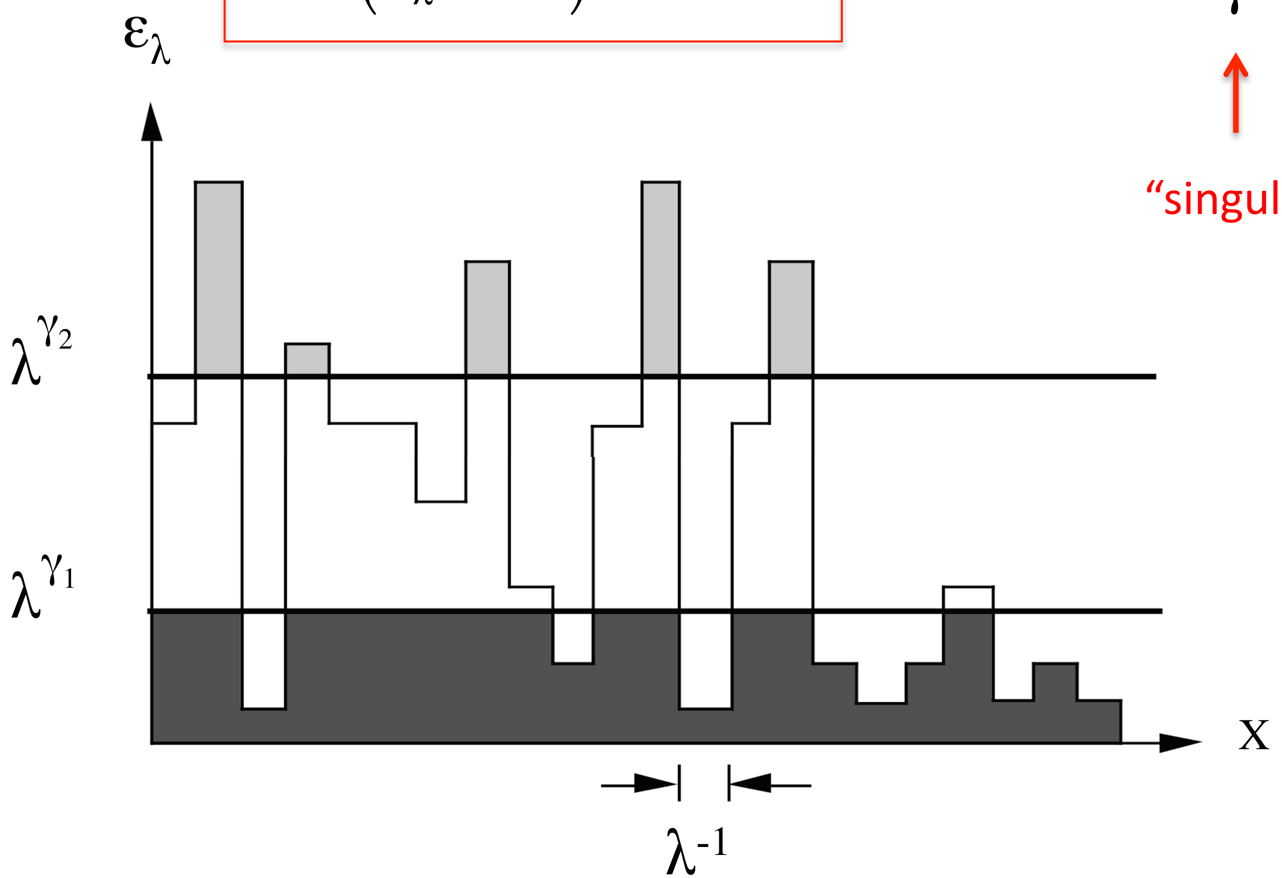
Values and singularities

General result:

$$\Pr\left(\varepsilon_\lambda > \lambda^\gamma\right) \approx \lambda^{-c(\gamma)}$$

$$\gamma = \frac{\log \varepsilon_\lambda}{\log \lambda}$$

"singularities"

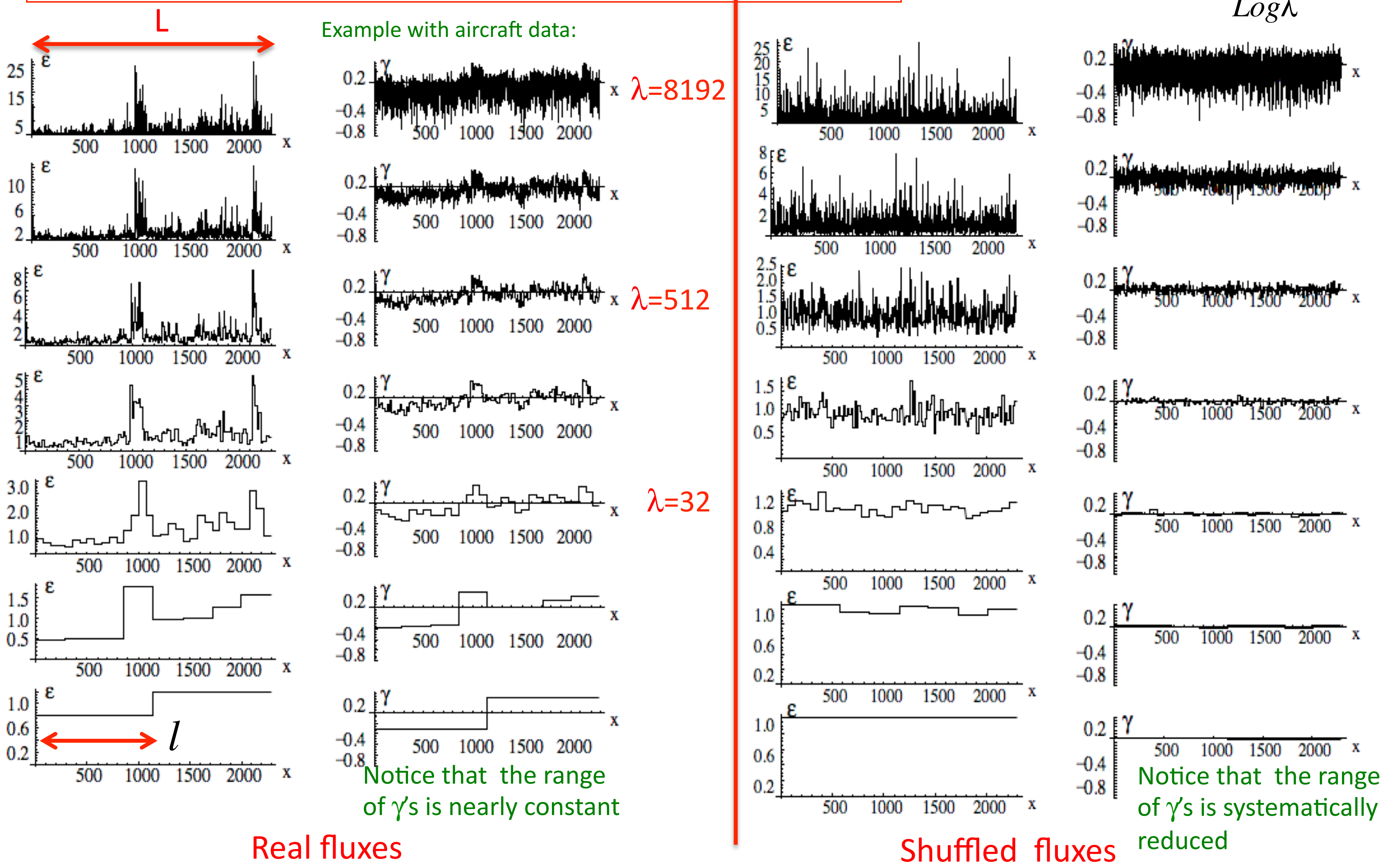


A schematic illustration of a multifractal field analyzed over a scale ratio λ, with two scaling thresholds λ^{γ_1} and λ^{γ_2}. , corresponding to two orders of singularity : $\gamma_2 > \gamma_1$.

Removing the sale dependency of the flux: γ

$$\varepsilon' = \lambda^{\gamma};\quad \varepsilon' = \frac{\varepsilon}{\langle\varepsilon\rangle};\quad \lambda = \frac{L}{l}$$

$$\gamma = \frac{Log\varepsilon'}{Log\lambda}$$



The Codimension Multifractal Formalism

Codimension of Singularities *c*(γ) and its relation to *K*(*q*)

We now derive the basic connection between $c(\gamma)$ and the moment scaling exponent $K(q)$. To relate the two; write the expression for the moments in terms of the probability density of the singularities:

$$p(\gamma) = \left|\frac{d\Pr}{d\gamma}\right| \sim c'(\gamma)(\log\lambda)\lambda^{-c(\gamma)} \sim \lambda^{-c(\gamma)}$$

Relation probability density and distribution:

$$\Pr(\gamma' > \gamma) = \int_{\gamma}^{\infty} p(\gamma')d\gamma'$$

(where we have absorbed the $c'(\gamma)\log\lambda$ factor into the "~" symbol since it is slowly varying, subexponential). This yields:

$$\langle \varepsilon_\lambda^q \rangle = \int d\Pr(\varepsilon_\lambda)\varepsilon_\lambda^q \approx \int d\gamma \lambda^{-c(\gamma)}\lambda^{q\gamma}$$

$$\Pr(\varepsilon_\lambda > \lambda^\gamma) = \int_{\lambda^\gamma}^{\infty} p(\varepsilon_\lambda)d\varepsilon_\lambda$$

where we have used $\varepsilon_\lambda = \lambda^\gamma$ (this is just a change of variables ε_λ for γ, λ is a fixed parameter). Hence:

$$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)} = e^{K(q)\log\lambda} \approx \int_{-\infty}^{\infty} d\gamma e^{\xi f(\gamma)}; \quad \xi = \log\lambda; \quad f(\gamma) = q\gamma - c(\gamma); \quad \lambda >> 1$$

Legendre transform

We need an asymptotic expansion of an integral with integrand of the form:

$\exp(\xi f(\gamma))$; $\xi = \log\lambda$ is a large parameter, $f(\gamma) = q\gamma - c(\gamma)$.

"Steepest descents" method shows that for $\xi = \log\lambda >> 1$, the integral is dominated by the γ which yields the maximum:

$$\int_{-\infty}^{\infty} e^{\xi f(\gamma)}\, d\gamma \approx e^{\xi \max_{\gamma}(f(\gamma))}$$

so that:

$$\langle \varepsilon_{\lambda}^{q} \rangle = e^{\xi K(q)} \approx e^{\xi \max_{\gamma}(q\gamma - c(\gamma))}; \quad \xi = \log\lambda$$

hence:

$$\boxed{K(q) = \max_{\gamma}\left(q\gamma - c(\gamma)\right)}$$

Legendre transform

This relation between $K(q)$ and $c(\gamma)$ is called a "Legendre transform" (Parisi and Frisch, 1985).

Inverse Legendre transform: $c(\gamma)$

We can also invert the relation to obtain $c(\gamma)$ from $K(q)$: aLegendre transform is equal to its inverse, hence we conclude:

$$c(\gamma) = \max_q \left(q\gamma - K(q)\right)$$

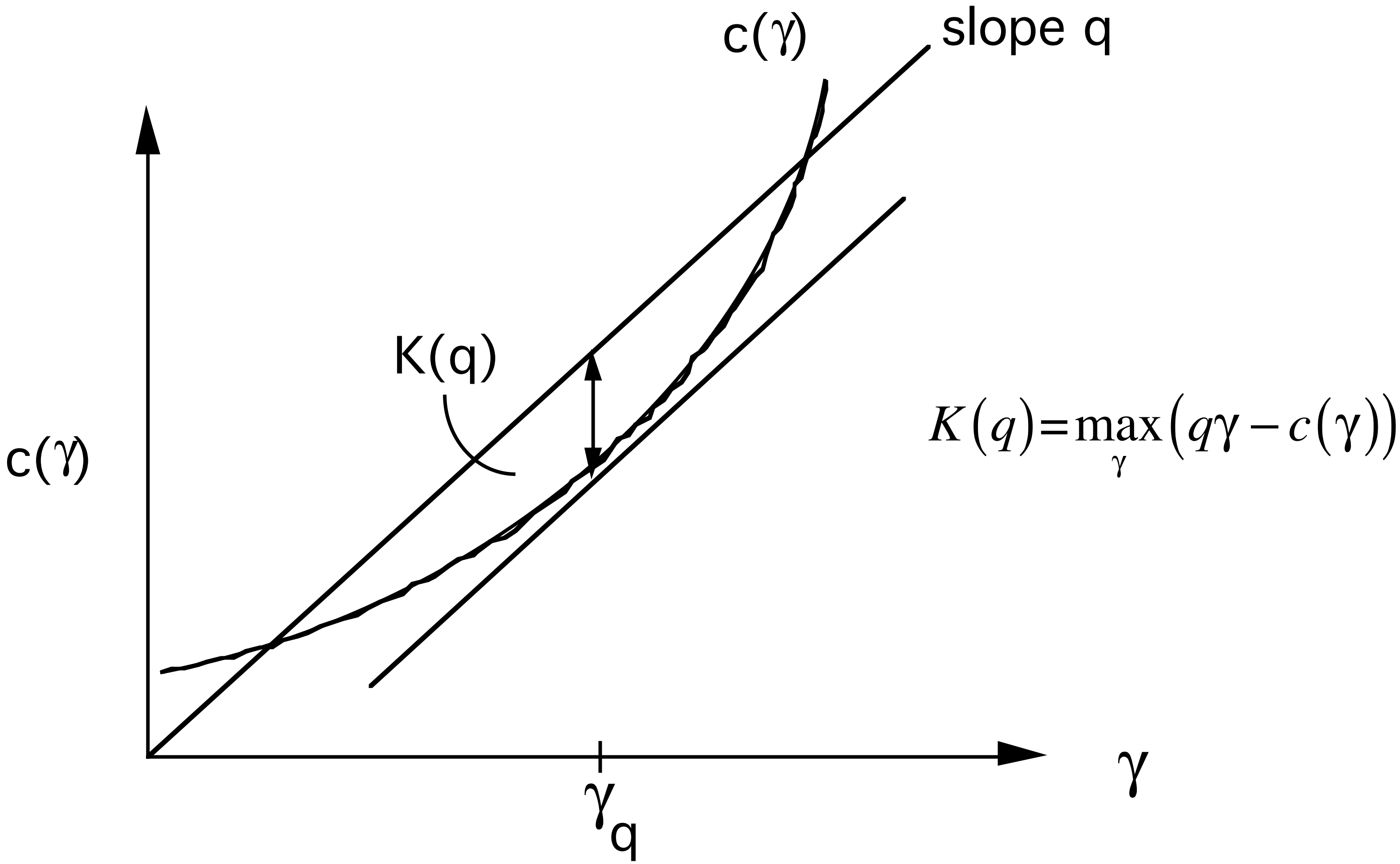
← Legendre transform

The γ which for a given q maximizes $q\gamma$ - $c(\gamma)$ is γ_q and is the solution of $c'(\gamma_q) = q$. Similarly, the value of q which for given γ maximizes $q\gamma$- $K(q)$ is q_γ so that:

$$q_\gamma = c'(\gamma)$$
$$\gamma_q = K'(q)$$

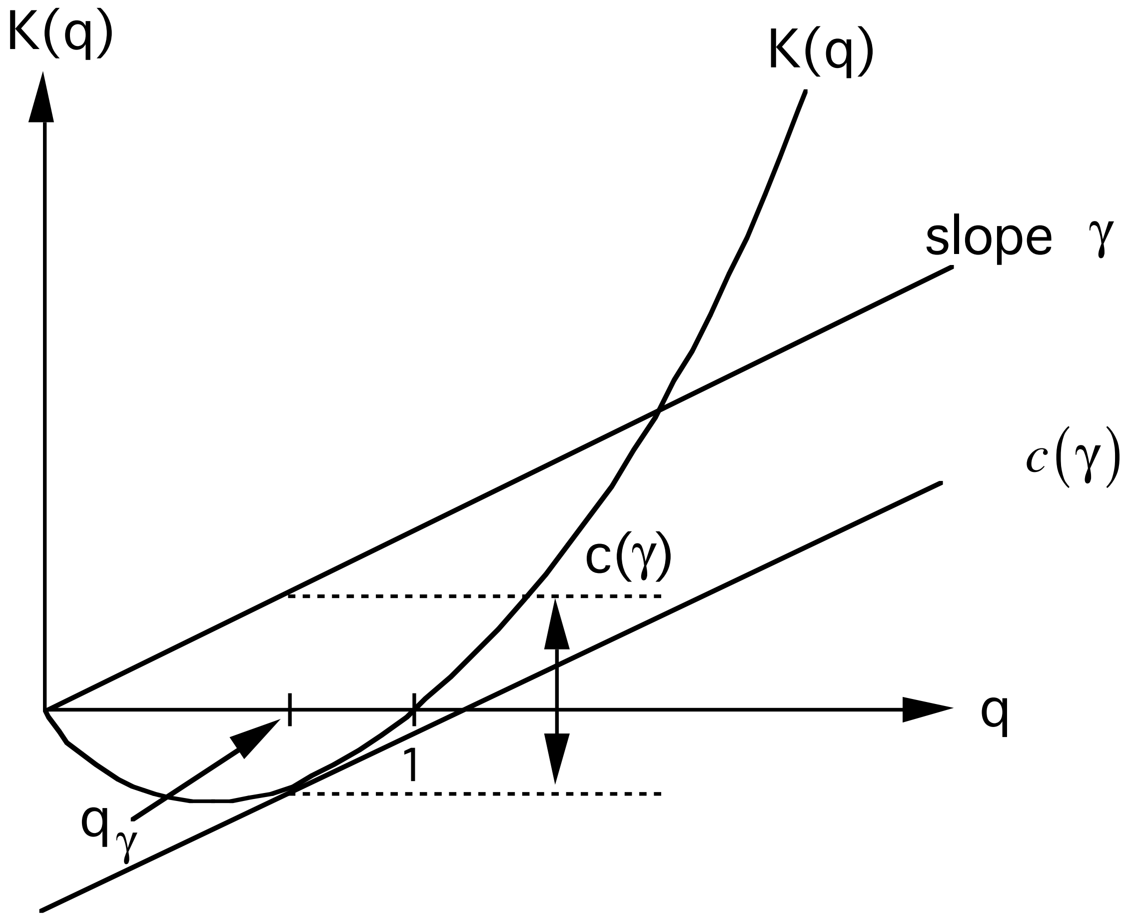
← One-to-one correspondence between moments and orders of singularities

Graphical Legendre transform



$c(\gamma)$ versus showing the tangent line $c'(\gamma_q) = q$ with the corresponding chord . Note that the equation is the same as $\gamma_q = K'(q)$.

Graphical Legendre transform



$$c(\gamma) = \max_q \left(q\gamma - K(q) \right)$$

K(*q*) versus *q* showing the tangent line $K'(q_\gamma) = \gamma$ with the corresponding chord .

Properties of codimension functions

$c(\gamma)$ is the statistical scaling exponent characterizing how its probability changes with scale.

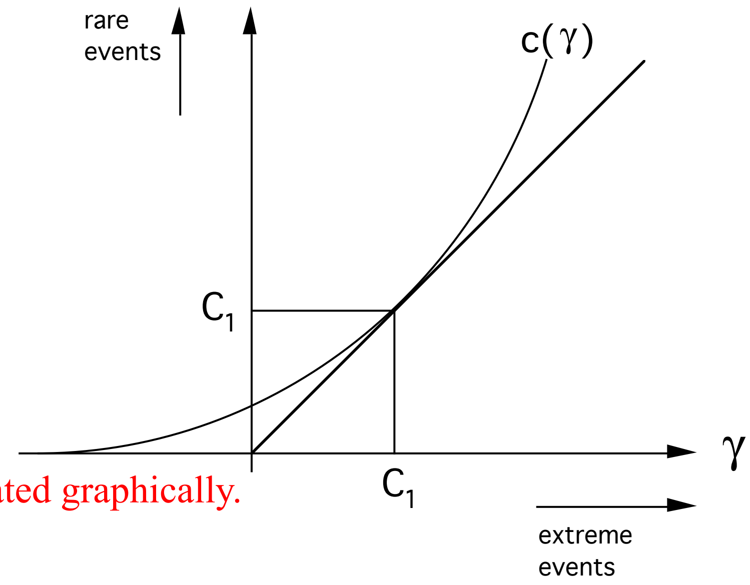
1) The first obvious property is that due to its very definition $c(\gamma)$ is an increasing function of γ: $c'(\gamma)>0$.

Reason: increasing γ must decrease the probability: $\Pr\left(\varepsilon_\lambda > \lambda^\gamma\right) \approx \lambda^{-c(\gamma)}$

1) Another fundamental property which follows directly from the Legendre relation with $K(q)$, is that $c(\gamma)$ must be convex: $c''(\gamma)>0$.

Reason: K(q) must be convex and the Legendre transform of a convex function is convex

The special properties of the singularity of the mean, C_1



Many properties of the codimension function can be illustrated graphically.
For example, consider the mean, q =1.

1) First, applying $K'(q) = \gamma$ we find $K'(1) = \gamma_1$ where is the singularity giving the dominant contribution to the mean (the q = 1 moment). We have already defined $C_1 = K'(1)$, so that this implies $C_1 = \gamma_1$; the Legendre relation thus justifies the name "codimension of the mean" for C_1.

2) Also at q = 1 we have $K(1) = 0$ (due to the scale by scale conservation of the flux) so that $C_1 = c(C_1)$ (this is a fixed point relation). C_1 is thus simultaneously the codimension of the mean of the process and the order of singularity giving the dominant contribution to the mean.

Reason: $$K(1) = \max_{\gamma}\left(\gamma - c(\gamma)\right)$$ Now, note that the γ that maximizes this is $\gamma_1 = C_1$ and $K(1)=0$

1) Finally, applying $c'(\gamma) = q$ we obtain $c'(C_1) = 1$ so that the curve $c(\gamma)$ is also tangent to the line $x = y$ (the bisectrix). If the process is observed on a space of dimension d, it must satisfy $d \geq C_1$, otherwise, following the above, the mean will be so sparse that the process will (almost surely) be zero everywhere; it will be "degenerate". We will see that when $C_1 > d$ that the ensemble mean of the spatial averages (the dressed mean) cannot converge.

Codimensions of Universal multifractals, cascades

$$K(q) = \frac{C_1}{\alpha - 1}\left(q^{\alpha} - q\right); \qquad \alpha \neq 1$$

Universal multifractal K(q)

$$K(q) = C_1 q Log(q); \quad \alpha = 1$$

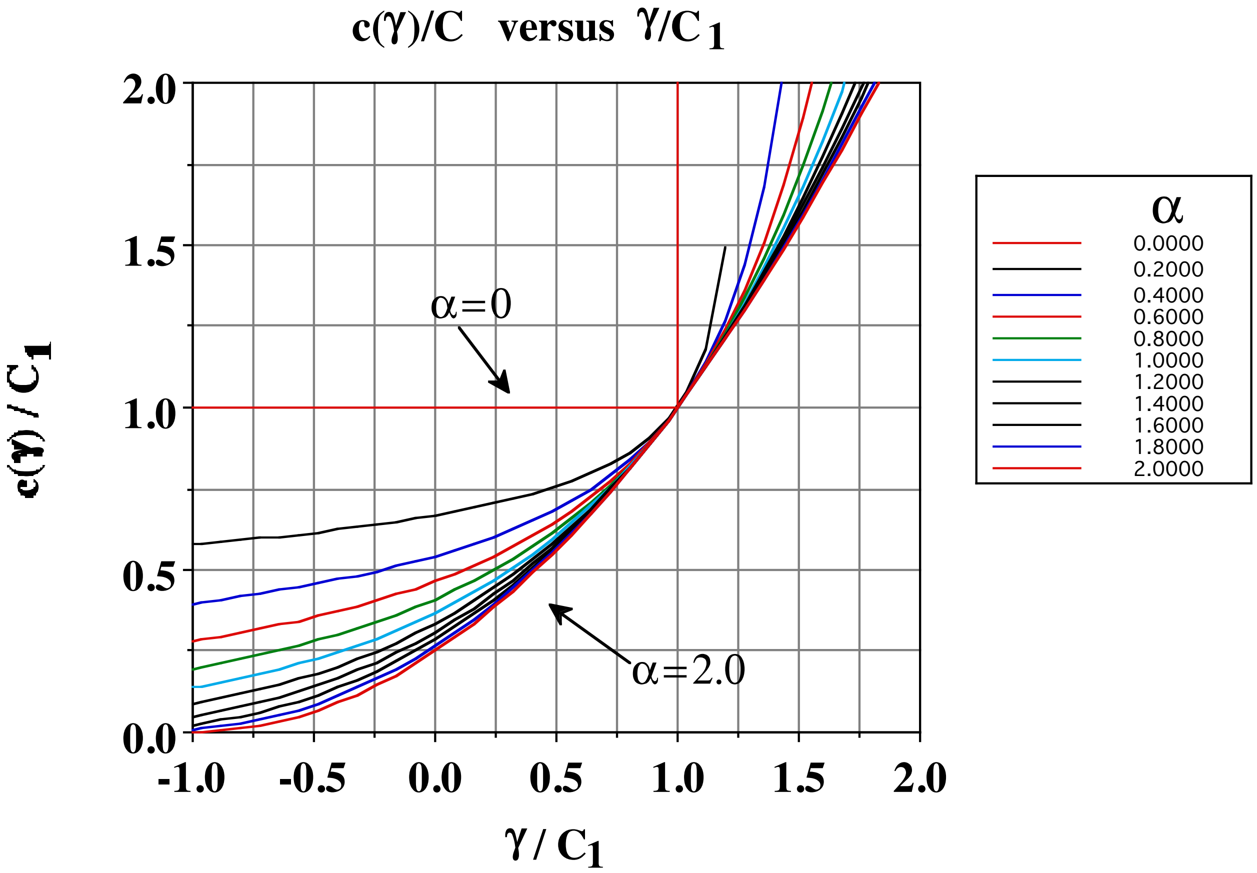
Valid for 0≤α≤2; however, *K* diverges for all *q* <0 except in the special ("log-normal") case α = 2 .
To obtain the corresponding *c*(γ), one can simply take the Legendre transformation to obtain Add: to obtain the α=1 case, just take limit as α->1.

$$c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'}; \quad \alpha \neq 1; \quad 1/\alpha' + 1/\alpha = 1$$

$$c(\gamma) = C_1 e^{\left(\frac{\gamma}{C_1} - 1\right)}; \quad \alpha = 1$$

Universal multifractal c(γ)

Universal c(γ)



Universal $c(\gamma)$ vs γ, for different $\alpha = 0$ to 2 by increment $\Delta\alpha = 0.2$.

Note that since α' changes sign at $\alpha = 1$, for $\alpha < 1$, there is a maximum order of singularity $\gamma_{max} = C_1/(1-\alpha)$ so that the cascade singularities are "bounded", whereas for $\alpha > 1$, there is on the contrary a minimum order $\gamma_{min} = -C_1/(\alpha-1)$ below which the prefactors dominate ($c(\gamma) = 0$ for $\gamma < \gamma_{min}$) but the singularities are unbounded.

α<1, α>1 cases: bounded, unbounded singularities

$2 \ge \alpha > 1; \quad \alpha' > 2$

$$c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'}; \quad \gamma > \frac{-C_1}{\alpha - 1}$$

$$c(\gamma) = 0; \quad \gamma \le \frac{-C_1}{\alpha - 1}$$

Frequent low values
"Levy holes"

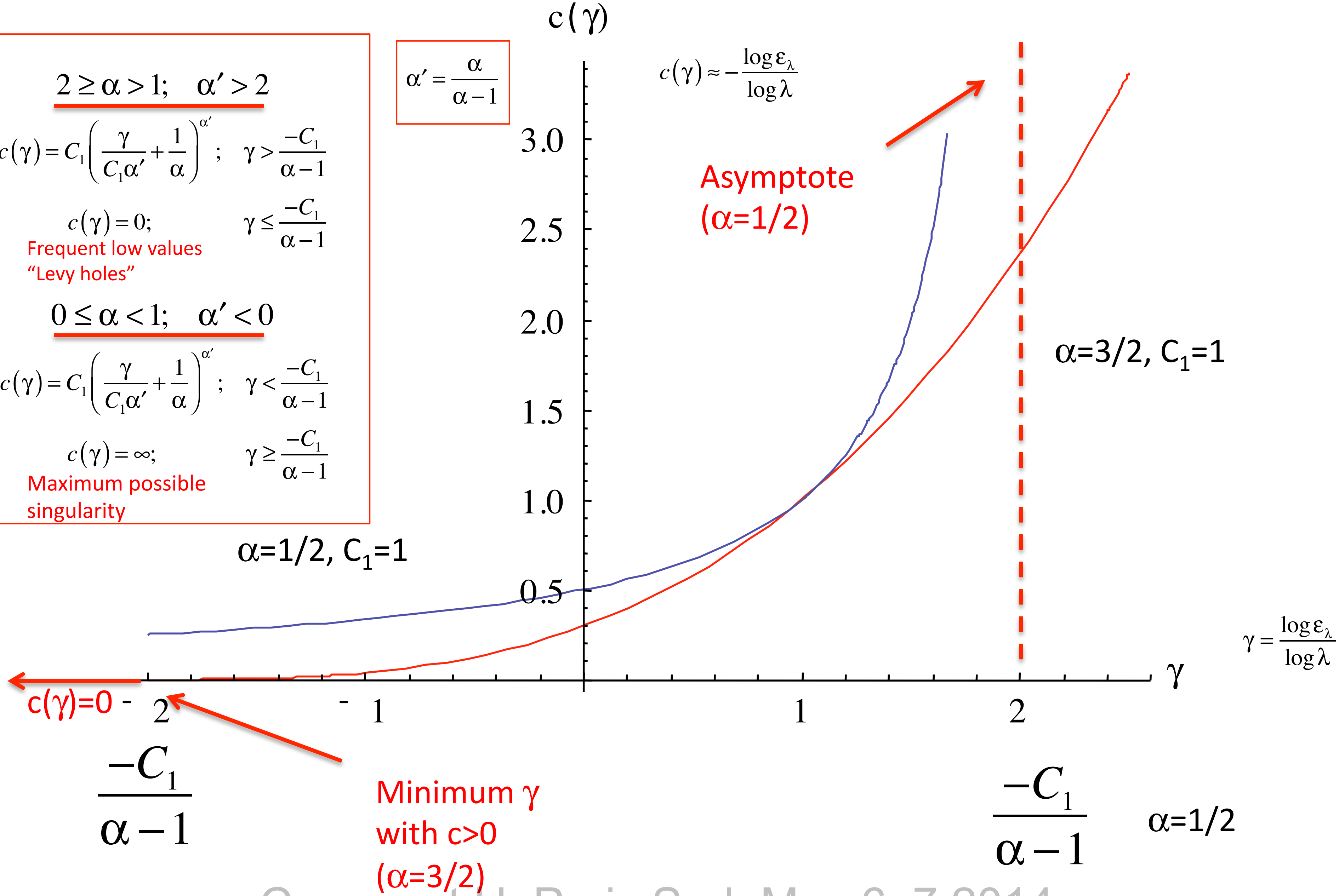
$0 \le \alpha < 1; \quad \alpha' < 0$

$$c(\gamma) = C_1\left(\frac{\gamma}{C_1\alpha'} + \frac{1}{\alpha}\right)^{\alpha'}; \quad \gamma < \frac{-C_1}{\alpha - 1}$$

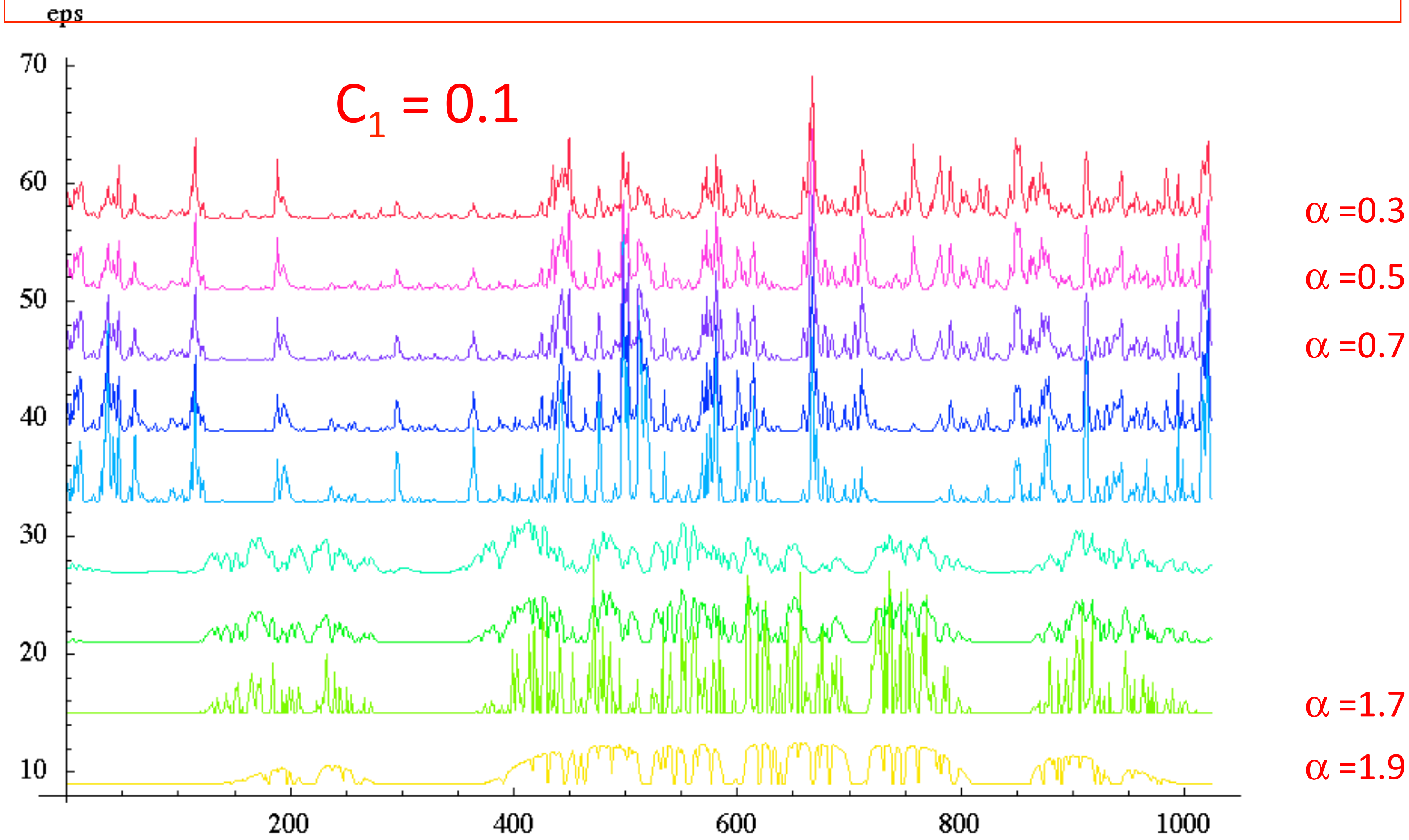
$$c(\gamma) = \infty; \quad \gamma \ge \frac{-C_1}{\alpha - 1}$$

Maximum possible
singularity

$$\alpha' = \frac{\alpha}{\alpha - 1}$$

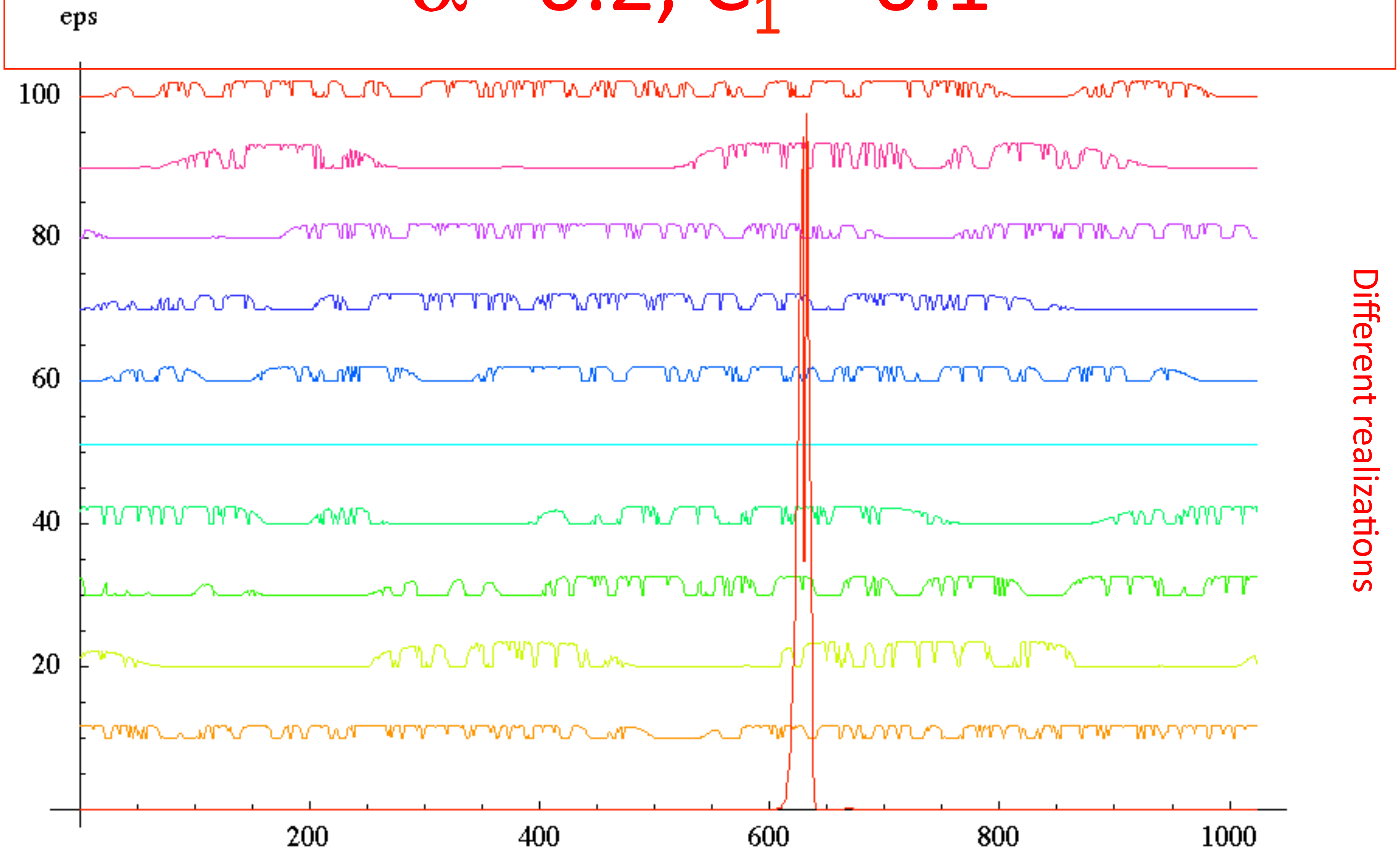


Examples



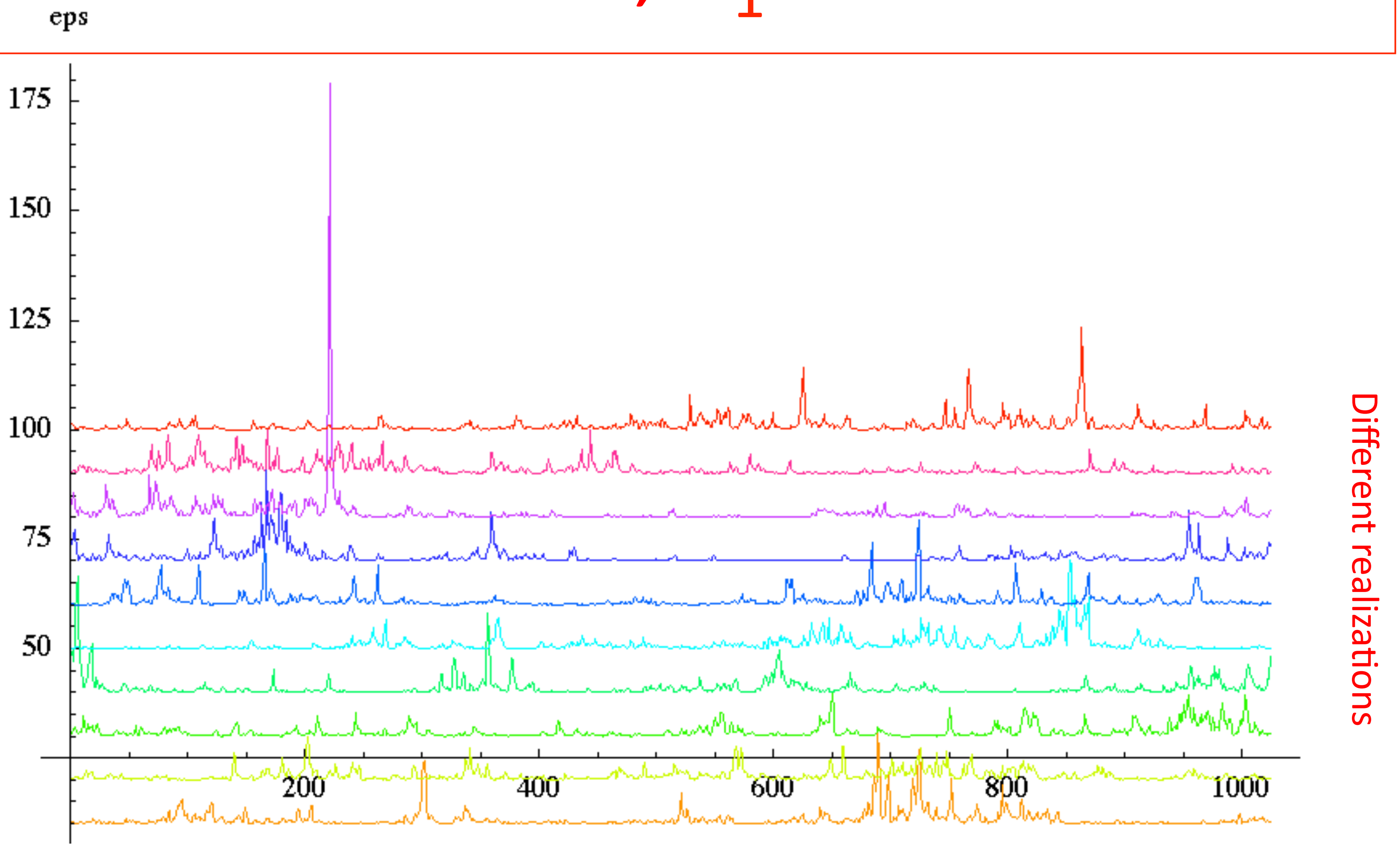
Multifractal simulations C_1=0.1 and α = 0.3, 0.5, …1.9 from bottom to top, offset for clarity (same random seed).

$\alpha = 0.2$, $C_1 = 0.1$

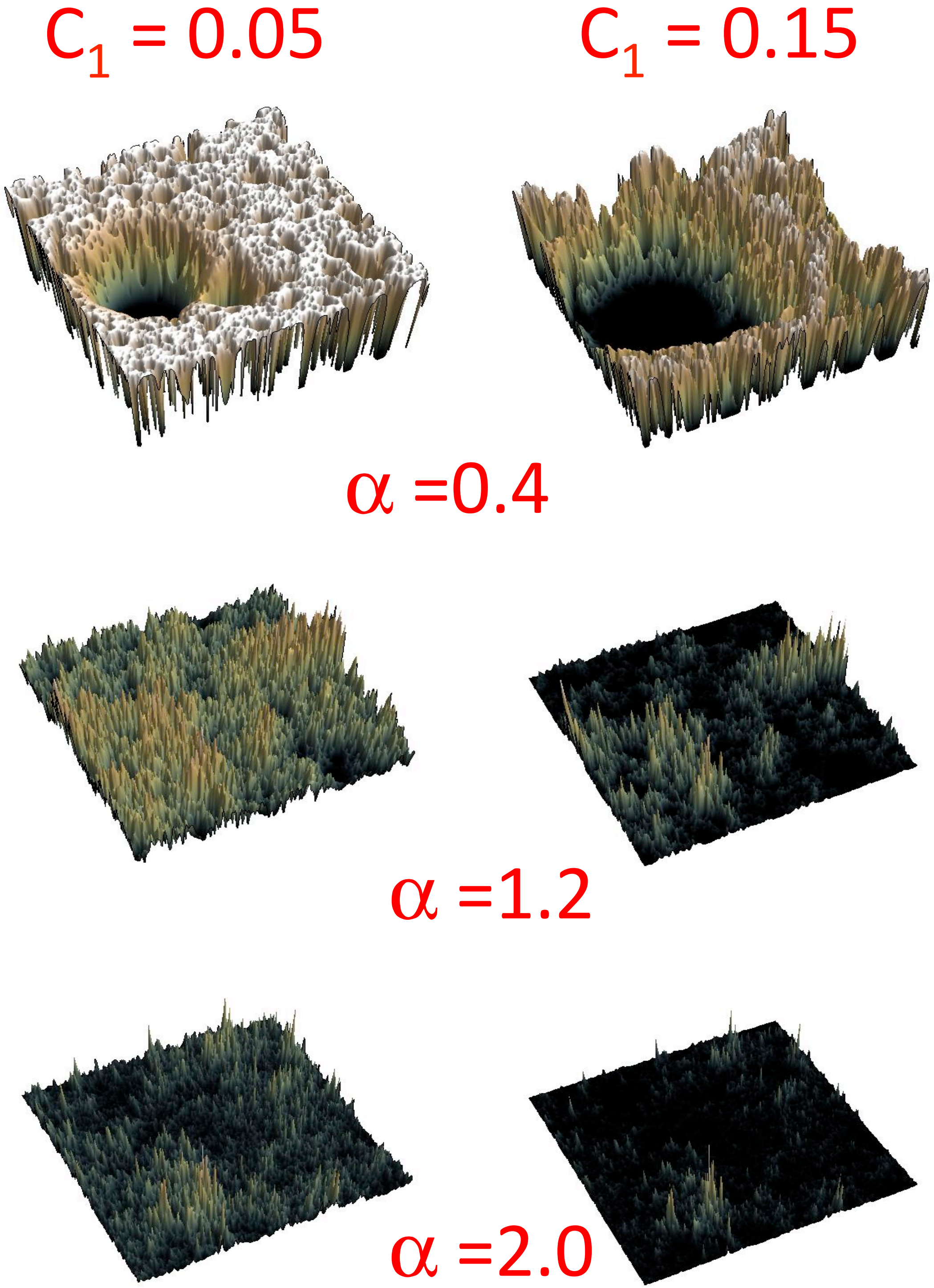


This shows 11 independent realizations of $\alpha = 0.2$, $C_1 = 0.1$ indicating the huge realization to realization variability : the bottom realization is not an outlier! no to so impressive .... with the only exception of a big spike !

α =1.9, C_1 = 0.1



Ten independent realizations of $\alpha = 1.9$, $C_1 = 0.1$, again notice the large realization to realization variability.



This shows isotropic realizations in two dimensions with α = 0.4, 1.2, 2, (top to bottom) and C_1 = 0.05, 0.15 (left to right). The random seed is the same so as to make clear the change in structures as the parameters are changed. The low α simulations are dominated by frequent very low values; the “Lévy holes”. The vertical scales are not the same. misleading, we need to find something else..

It’s too late to change the name… and if so, to what?

Direct empirical estimation of $c(\gamma)$: the probability distribution multiple scaling (PDMS) technique

$$\Pr(\varepsilon_\lambda > \lambda^\gamma) \approx \lambda^{-c(\gamma)}$$

Start from the fundamental defining equation, take logs of both sides and rewrite it as follows:

$$Log(\Pr(\varepsilon_\lambda > \lambda^\gamma) = -c(\gamma)Log(\lambda) + o(1/Log(\lambda))O(\gamma)$$

corresponds to the logarithm of slowly varying factors that are hidden in the “≈” sign.

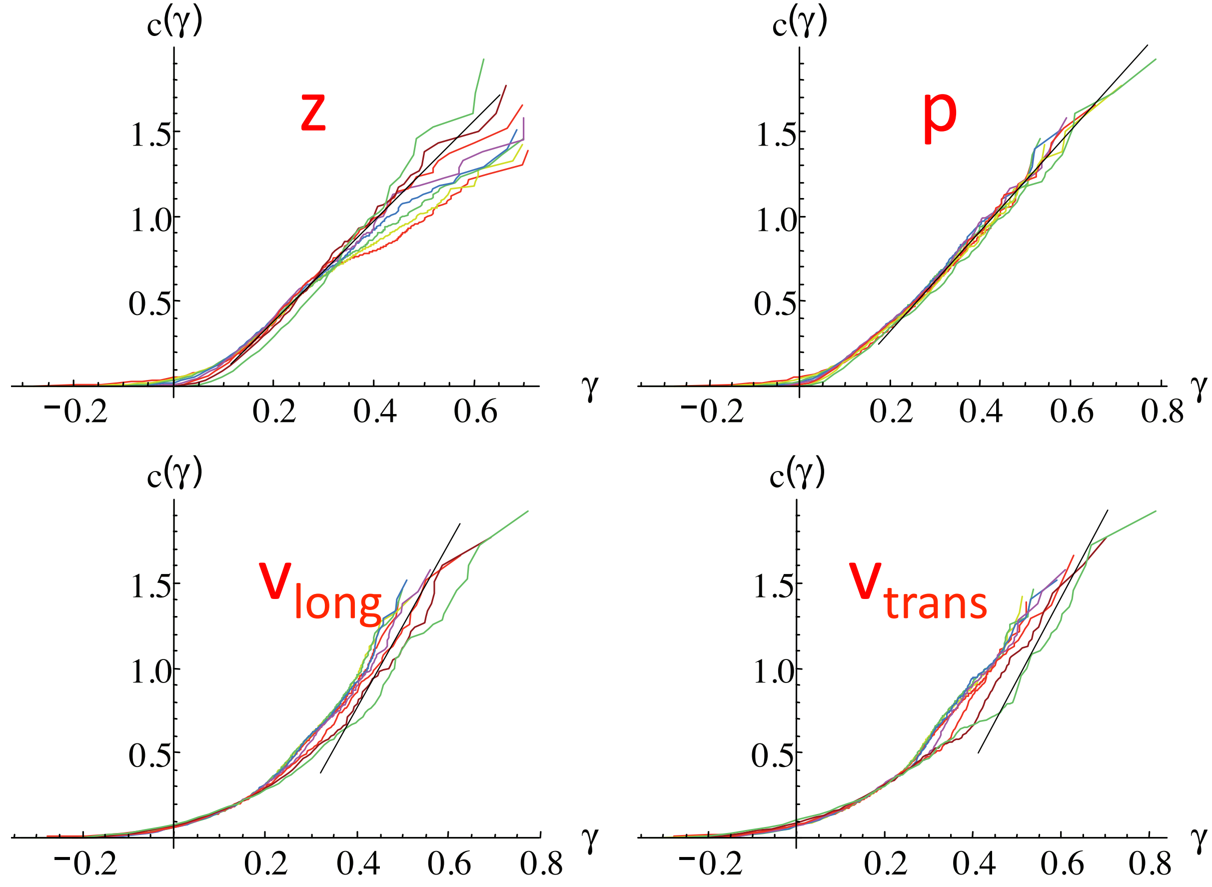
The singularity is estimated from the fluxes by:

$$\gamma = \frac{\log(\varepsilon_\lambda)}{\log \lambda}$$

Probability Distribution Multiple Scaling technique

PDMS examples

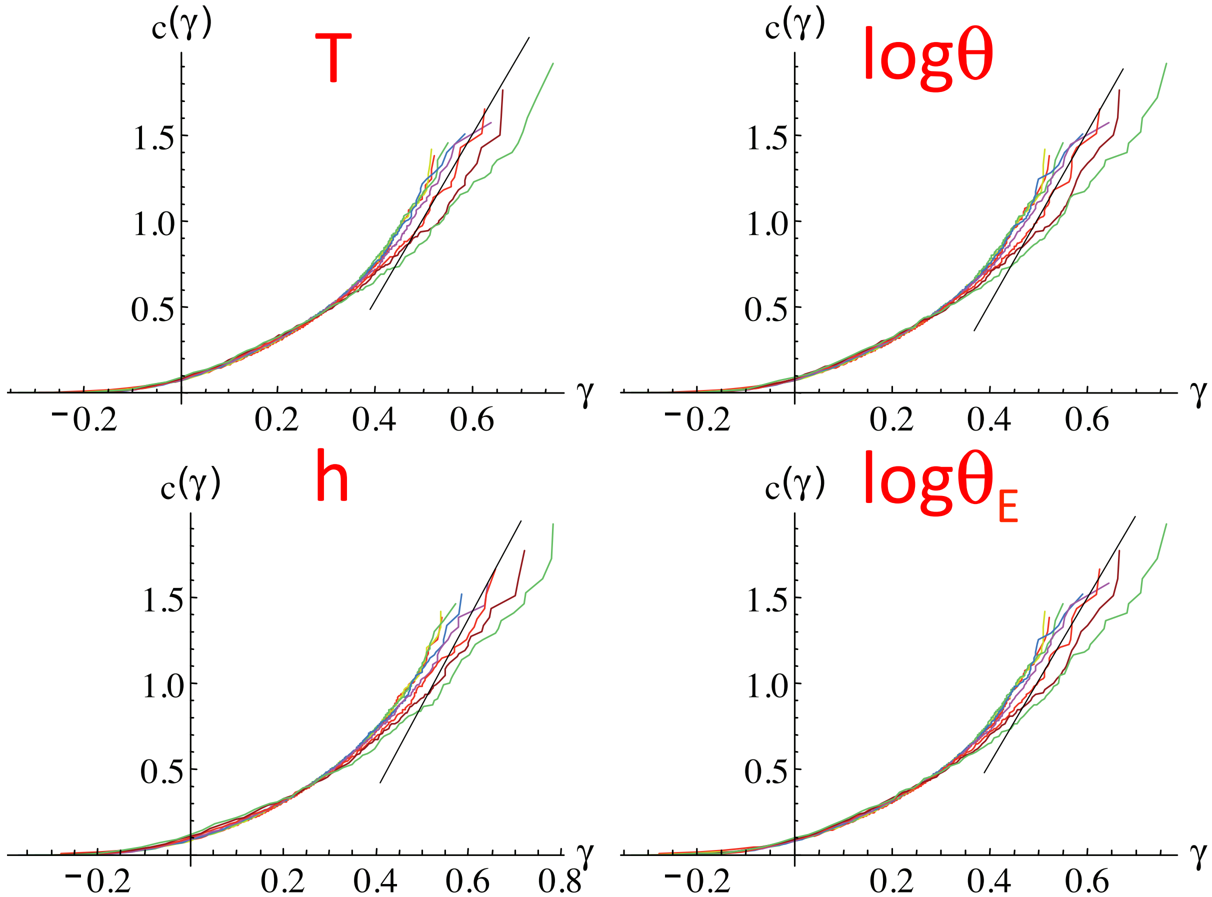
Aircraft at 200mb: 24 flight legs, each 4000 points long, 280 *m* resolution (i.e. 1120 *km*), dynamic variables



$c(\gamma) \approx -\log Pr/\log\lambda$

$c(\gamma)$ estimated from the PDMS method $c(\gamma) \approx -\log Pr/\log\lambda$ are shown for resolution degraded by factors of 2 from 280 *m* to ≈36 *km* (longest to shortest curves). For reference, lines of slope 3 (top row) and 5 (bottom row) are given corresponding to power law probability distributions with the given exponents.

Thermodynamic variables



The reference lines all have slopes of 5

Codimension and dimension multifractal formalisms

Codimension (stochastic)

Dimension (deterministic)

Singularities

$$\varepsilon_\lambda = \lambda^\gamma$$

density

$$l = \lambda^{-1} \qquad vol(B_\lambda) = \lambda^{-d}$$

$$\Pi_\lambda = P_\lambda$$

$$\varepsilon_\lambda = p_\lambda$$

$$\Pi_\lambda = \int_{B_\lambda} \varepsilon_\lambda d^d \underline{x} = \lambda^{-\alpha_d}$$

integral

$$\Pi_\lambda = \varepsilon_\lambda vol(B_\lambda) = \lambda^{\gamma - d}$$

$$\alpha_d = d - \gamma$$

Probabilities

$$\Pr(\varepsilon_\lambda = \lambda^\gamma) \approx \lambda^{-c(\gamma)}$$

$$Number\left(\Pi_\lambda = \lambda^{-\alpha_d}\right) = \lambda^{-f_d(\alpha_d)}$$

$$Number = \lambda^d \Pr$$

$$f_d(\alpha_d) = d - c(\gamma)$$

Statistical Moments

$$\langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)}$$

$$\sum_{i=1}^{\lambda^d} \Pi_{\lambda,i}^q = \lambda^{-\tau_d(q)}$$

$$\sum_{i=1}^{\lambda^d} \Pi_{\lambda,i}^q = \left\langle \sum_{i=1}^{\lambda^d} \left(\lambda^{-d}\varepsilon_\lambda\right)^q \right\rangle = \lambda^{d(q-1)} \langle \varepsilon_\lambda^q \rangle = \lambda^{K(q)-d(q-1)}$$

$$\tau_d(\alpha_d) = d(q-1) - K(q)$$

$$c(\gamma) \overset{L.T.}{\leftrightarrow} K(q); \quad f_d(\alpha_d) \overset{L.T.}{\leftrightarrow} \tau(q)$$