11. Classical Mechanics of Continuous Media

(I) Recall from point particle mechanics $\qquad q(t)$

$$\mathcal{L} = T - V \qquad T = \frac{1}{2} m \dot{q}^2$$

$$V = \frac{1}{2} k q^2 \qquad \text{spring - tension}$$

Ex A string

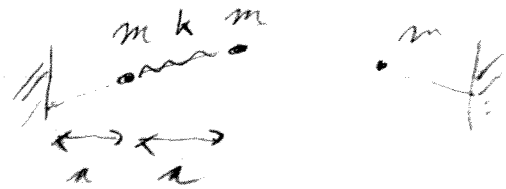
$q(x,t)$ $\qquad \rho(x)$ mass / length

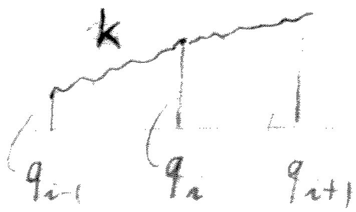
$$T = \int dx\, \rho(x)\, \dot{q}^2(x)\, \frac{1}{2}$$

$$V = \int dx\, \frac{1}{2}\, \tau \left(\frac{\partial}{\partial x} q\right)^2$$

$$\mathcal{L} = \frac{1}{2} \int dx \left(\rho\, \dot{q}^2 - \tau \left(\frac{\partial}{\partial x} q\right)^2\right)$$

Derivation:





$\tau = ka$

$$V_i = \frac{1}{2} k\, \Delta\ell^2 = \frac{1}{2} k \left(q_i - q_{i-1}\right)^2$$

$$V = \sum V_i = \frac{1}{2} k \sum_i \left(\frac{q_i - q_{i-1}}{a}\right)^2 a^2$$

$$= \frac{\tau}{2} \sum_i a \left(\frac{q_i - q_{i-1}}{a}\right)^2$$

$q_i \to q(x)$

$$\int dx \left(\frac{\partial}{\partial x} q\right)^2$$

$$T = \sum_i T_i = \frac{1}{2}\sum_i m_i \dot{q}_i^2 = \frac{1}{2}\sum_i a \, \frac{m_i}{a}\dot{q}_i^2$$

$$= \frac{1}{2}\int dx \, \rho \, \dot{q}(x)^2$$

$$V = \frac{1}{2}\int dx \, \tau \left(\frac{\partial}{\partial x} q\right)^2$$

$$L = \frac{1}{2}\int dx \underbrace{\left[\rho \dot{q}^2 - \tau\left(\frac{\partial q}{\partial x}\right)^2\right]}_{\text{Lagrange density}}$$

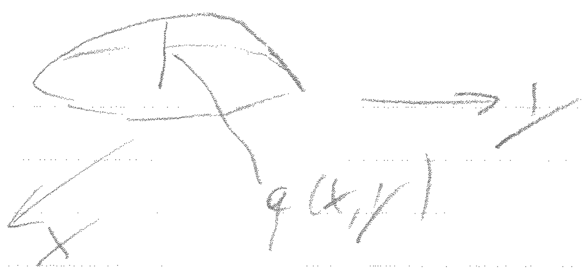
Lagrangian

Note: $\left.\begin{array}{l}\rho \to \rho(x) \\ \tau \to \tau(x)\end{array}\right\}$ generalization

Ex B Membrane

$$L = \frac{1}{2}\int dx\, dy \left[\rho(x,y)\dot{q}^2 - \tau(x,y)\left(\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right)\right]$$

sketch



Ex C Classical field

$\varphi(x,y,z)$

$$L = \frac{1}{2}\int dx\,dy\,dz \left[m\, \dot{\varphi}^2 - \tau\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi\right]$$

context: particle — field duality

(II) Derive EoM

$$\mathcal{L} = \frac{1}{2}\int dx \left[\rho(x)\, \dot{q}^2 - \tau(x) \left(\frac{\partial q}{\partial x}\right)^2 \right]$$

$$S = \int_{t_1}^{t_2} dt\ \mathcal{L}$$

$\delta S \overset{!}{=} 0$ fixed end points $q(x,t_1)$, $q(x,t_2)$

$$\delta S = \int_{t_1}^{t_2} dt \int dx \Big[\underbrace{\rho(x)\, \dot{q}\, \delta\dot{q}}_{\text{IBP in } t} - \underbrace{\tau \frac{\partial q}{\partial x}\, \delta\left(\frac{\partial q}{\partial x}\right)}_{\text{IBP in } x} \Big]$$

$$- \int_{t_1}^{t_2} \int dx\ \rho(x)\, \ddot{q}\, \delta q \qquad - \int_{t_1}^{t_2} \int dx\ \tau \frac{\partial^2 q}{\partial x^2}\, \delta q$$

$$+ \underbrace{\tau \frac{\partial q}{\partial x}\, \delta q \Big|_{x_1}^{x_2}}_{=0 \text{ if fixed b.c.}}$$

$q(x_1) = q(x_2) = 0$

$$= - \int_{t_1}^{t_2} dt \int dx \left(\rho(x)\, \ddot{q} - \tau \frac{\partial^2 q}{\partial x^2} \right) \delta q \overset{!}{=} 0$$

$$\Rightarrow \rho(x)\, \ddot{q} - \tau \frac{\partial^2 q}{\partial x^2} = 0 \qquad \rho(x) = \rho$$

$$\boxed{\ddot{q} - \frac{\tau}{\rho} \frac{\partial^2 q}{\partial x^2} = 0} \qquad \text{wave equation}$$

$$\frac{\tau}{\rho} = c^2$$

Solving Wave Equation

(WE) $\boxed{\ddot{q} - c^2 \frac{\partial^2 q}{\partial x^2} = 0}$ PDE → difficult

a) Wave ansatz $q(x,t) = f(t) \cos(kx+\varphi)$

(separation of variables) φ phase

$\ddot{f} + c^2 k^2 f = 0$ ODE → easy

ansatz: oscillatory $f(t) = A\cos(\omega t + \alpha)$

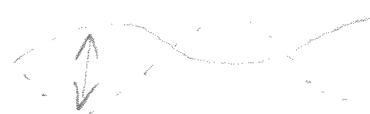
α phase

$\omega^2 - c^2 k^2 = 0 \Rightarrow \omega = \pm ck$

$q(x,t) = \left[A_1 \cos(ckt+\alpha) + A_2 \cos(-ckt+\alpha)\right] \cos(kx+\varphi)$

↑

standing wave solution



b) Wave ansatz (complex)

$q(x,t) = A e^{i(kx+\omega t+\alpha)}$

$\omega^2 - c^2 k^2 = 0$ $\omega = \pm ck$

$q(x,t) = A_1 e^{i(kx+ckt+\alpha_1)} + A_2 e^{i(kx-ckt+\alpha_2)}$

↑

travelling wave solution

crest: $x = -ct$ crest: $x = +ct$

left moving right moving

Role of Boundary Conditions

Ex: Infinite string : no spatial b.c.
$\downarrow$
any value of k allowed

Ex: finite string, Dirichlet b.c. $q(0) = q(L) = 0$

standing wave: $q(0) = 0 \Rightarrow \varphi = +\pi/2$
$q(L) = 0 \Rightarrow kL = n\pi$

$k_n = n \frac{\pi}{L}$ discrete set of k values $n = 0, 1, .$
infinite " " "

Note: for fixed amplitude of oscillation
$E_n \nearrow$ as $n \nearrow$

Ex: finite string, periodic b.c. $q(0) = q(L)$

$k_n L = 2\pi n \qquad k_n = n \frac{2\pi}{L} \qquad n = 0, 1, ..$

Connection with string theory

string theory elem. object strings
particle " " " point particles
each excitation of a string ~ particle
$n \nearrow \Rightarrow E(\text{particle}) \nearrow \Rightarrow m(\text{particle}) \nearrow$

$\Rightarrow \begin{cases} \text{unification} \\ \text{including gravity} \end{cases}$

quantization of classical string theory

$\downarrow$

quantum theory of gravity, E&M
particles

but: $D = 9 + 1$

need superstring ($\to$ predict supersymmetry)

Ex Neumann b.c. $\frac{\partial}{\partial x} q(0) = \frac{\partial}{\partial x} q(L) = 0$

standing wave: $\frac{\partial}{\partial x} q(0) = 0 \Rightarrow \varphi = 0$

$\frac{\partial}{\partial x} q(L) = 0 \Rightarrow kL = n\pi$

Ex Mixed b.c. e.g. $q(0) = 0$, $\frac{\partial}{\partial x} q(L) = 0$

$kL = \left(n - \frac{1}{2}\right)\pi$

General Solution

(WE) is linear:

Superposition principle: If q_1 & q_2 are solutions

$\Rightarrow A_1 q_1 + A_2 q_2$ is a solution

Ex finite string $q(0) = q(L) = 0$

$$q(x,t) = \sum_{n=-\infty}^{\infty} A_n(t) \sin(k_n x) \qquad (*)$$

Fourier series

superposition of standing waves

$A_n(t)$ Fourier coeffs.

Completeness theorem.

$\Rightarrow$ Any solution can be written as $(*)$

Ex infinite string

$(**)$ $q(x,t) = \int dk \cdot A_k(t)\, e^{ikx}$ Fourier integral

$A_k(t)$ Fourier transform

Completeness theorem

$\Rightarrow$ Any solution can be written as $(**)$