Phys 514 - Assignment 2 - EEP and Differential Geometry
Solutions

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1. Use the Einstein Equivalence Principle to calculate the vertical distance light has fallen (equivalently the distance from the Earth's surface) after travelling 1km if it is emitted horizontally in vacuum near the Earth's surface

Solution
The Einstein equivalence principle states that "in small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiment". To determine the physical distance light falls, consider a reference frame at the point where light is emitted to be in free-fall. Our approach is to determine the distance such a frame has 'fallen' over the time spent for light to traverse 1km.

The time taken for light to traverse 1km is simply

$$\Delta t = \Delta d/c = 10^3/3 \cdot 10^8 \quad s \approx 3 \cdot 10^{-6} \quad s \tag{1}$$

The frame is in free fall with respect to a frame fixed on the ground, falling with an acceleration of $9.8 \ m/s^2$, so the velocity of this frame after Δt is

$$v = a \cdot \Delta t = 3 \cdot 10^{-5} \ m/s \tag{2}$$

Now, to compute the vertical distance light has fallen, we should make use of Lorentz transforms. However, since $v/c = 10^{-13} <<< 1$, we can simplify our problem and make use of the simpler Galilean transforms. Performing this analysis yields a vertical fall of the emitted light frame, and subsequently the vertical fall of the emitted photon to be

$$\Delta y = \frac{1}{2} a \Delta t^2 \approx 5 \cdot 10^{-11} \ m \tag{3}$$

2. In class, the Eötvös experiment was discussed. Summarize the analysis and determine which latitude on earth is the best for performing the experiment. Explain your answer in both words and equations

Solution

The Eötvös experiment was designed to test if inertial mass (the mass appearing in $F = ma$) is equivalent to gravitational mass (the one appearing in $F = GmM/r^2$). This was done by making use of a torsion balance (a way to measure very weak forces by way of the twisting of a system). This torsion balance consisted of a rigid rod supporting two masses on either side (call the weights A and B) as well as a wire attached to the centre (to measure the torsion). See the figure for a breakdown of the scenario.

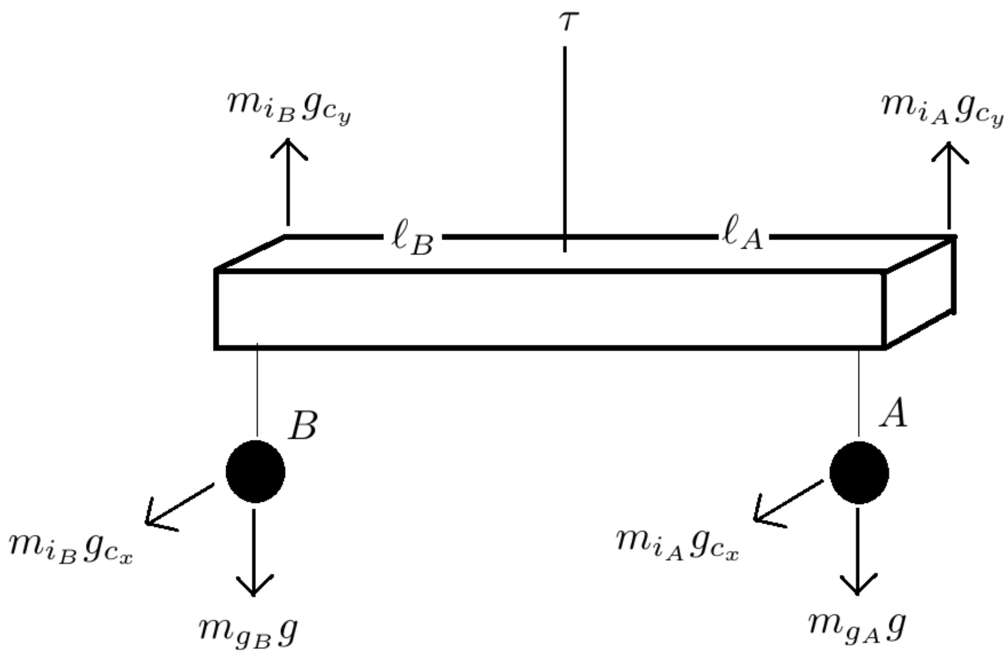

Figure 1: The torsion system for measuring changes between m_i and m_g. Figure inspiration from Jerome Quintin.

The idea of the experiment is to test the equivalence of the gravitational mass, m_g with the inertial mass m_i by utilizing an apparatus that takes both as inputs. The rod has attached to it two masses, A and B fixed at both ends, which will feel a gravitational force (in the vertical, or y direction) as well as a centrifugal force (in both the x and y directions). If the inertial and gravitational mass are one and the same, this system will not twist, and stay in static equilibrium. In our figure, g is the usual Newtonian constant, $9.8 m/s^2$, g_{c_x}, g_{c_y} correspond to the centrifugal accelerations in the x and y directions , and m_i, m_g correspond to the inertial and gravitational masses. Lets see how this works in equations.

The condition for no torque in the y (vertical) direction is simply given by

$$\ell_B(m_{g_B} g - m_{i_B} g_{c_y}) = \ell_A(m_{g_A} g - m_{i_A} g_{c_y}) \tag{4}$$

There is also the possibility that we will torque and twist around the wire suspending the rod. This torque is given by

$$\tau = \ell_A m_{i_A} g_{c_x} - \ell_B m_{i_B} g_{c_x} = \ell_A m_{i_A} g_{c_x} \left(1 - \frac{\ell_B m_{i_B}}{\ell_A m_{i_A}}\right) \tag{5}$$

We can write the ratio ℓ_B/ℓ_A by equation (4) as

$$\frac{\ell_B}{\ell_A} = \frac{m_{g_A} g - m_{i_A} g_{c_y}}{m_{g_B} g - m_{i_B} g_{c_y}} \tag{6}$$

Inserting this into the equation for τ yields

$$\tau = \ell_A m_{i_A} g_{c_x} \left(1 - \frac{m_{i_B}}{m_{i_A}} \left(\frac{m_{g_A} g - m_{i_A} g_{c_y}}{m_{g_B} g - m_{i_B} g_{c_y}}\right)\right) \tag{7}$$

$$= \ell_A m_{i_A} g_{c_x} \left(1 - \frac{\frac{m_{g_A}}{m_{i_A}} - \frac{g_{c_x}}{g}}{\frac{m_{g_B}}{m_{i_B}} - \frac{g_{c_x}}{g}}\right) \tag{8}$$

Now we note that the gravitational force of Earth is much larger than the centrifugal acceleration coming from the rotation of the planet, so we make the approximation $\frac{g_{c_x}}{g} \approx 0$ to simplify

$$\tau = \ell_A m_{i_A} g_{c_x} \left(1 - \frac{m_{g_A} m_{i_B}}{m_{i_A} m_{g_B}}\right) \tag{9}$$

$$= \ell_A m_{g_A} g_{c_x} \left(\frac{m_{i_A}}{m_{g_A}} - \frac{m_{i_B}}{m_{g_B}}\right) \tag{10}$$

And since there was no torque observed in the experiment, this implies that the inertial and gravitational masses are the same.

The best angle is the one that will maximize the torque. Since this is related by g_{c_x}, we can note that

$$g_{c_x} = g_c \cos\theta \tag{11}$$

Where θ is the angle from a vector originating at the centre of the earth to the north pole. The centrifugal acceleration is $g_c = \omega^2 R \sin\theta$ where ω is the constant angular speed of the earth, and R is the radius of the earth. Thus, to maximize the horizontal centrifugal acceleration, we must maximize

$$g_{c_x} = \omega^2 R \sin\theta \cos\theta \tag{12}$$

Which is maximized for $\theta = \pi/4$.

3. In class I did not discuss the **Hughes-Drever** experiment. This classic experiment is a test of one of the key aspects of the Einstein Equivalence Principle (EEP), namely local Lorentz invariance. Read up on this experiment and give a disucssion on how it works and why it is a test of the EEP

Solution

The Hughes-Drever experiment was originally designed to be a test of Mach's principle (stated that the inertia in our local frame should be influenced by the distribution of mass in the universe). Since by this logic, a nonuniform distribution of matter (such as the fact that there is much more matter in the direction of the Milky way bulge then any other direction within our galaxy) would introduce inertial anisotropies.

Hughes and Drever conducted experiments (separately) to test this hypothesis by examining the splitting of the ground state of the nucleus of a lithium-7 atom. Since this nucleus state has a spin of 3/2, its ground state splits into four evenly spaced magnetic energy levels (with spin quantum numbers $S_z = -3/2, -1/2, 1/2, 3/2$). The spatial distributions of energy levels are different, however, and so if the anisotropic mass distribution of the galaxy effected the inertial energy of these energy states, photons from transitions between these states would have different frequencies. Over the course of a 24 hour experiment (performed by Drever), the earth turned and the magnetic axis pointed at different regions of the sky. There was never any frequency change between photons emitted from transitions between the different energy levels. Thus, the sensitivity of the experiment allowed the maximum anisotropic effects to be limited to $10^{-25} GeV$, a statement strongly in line with Lorentz invariance, stating that there is no preferred frame.

4. Prove that the set of tangent vectors at a point p on a manifold $\mathcal{M}$ is a vector space

Solution

This can follow partially from section 2.3 of the text. The set of tangent vectors at a point p can be identified with the space of directional derivative operators along curves through the point p. The coordinate independent way to write a vector tangent to a point p is

$$v = v^\mu \partial_\mu = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu} \tag{13}$$

Where λ parametrizes the curve through p. The tangent space is then the set of directional derivative operators through the point p. We will write $v = \frac{d}{d\lambda}$ to specify a vector tangent to the curve $x^\mu(\lambda)$ through p in what follows. Also define u, and w as tangent vectors with different parametrizations through p, and f as some smooth test function. To prove that this set is indeed a vector space, we must satisfy the 8 axioms of vector spaces

(i) Associativity of addition

$$\begin{aligned}
(u + (v + w))[f] &= (u[f] + (v[f] + w[f])) \\
&= (\frac{dx^\mu}{d\eta}\frac{\partial f}{\partial x^\mu} + (\frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dx^\mu}{d\sigma}\frac{\partial f}{\partial x^\mu})) \\
&= ((\frac{dx^\mu}{d\eta}\frac{\partial f}{\partial x^\mu} + \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}) + \frac{dx^\mu}{d\sigma}\frac{\partial f}{\partial x^\mu}) \\
&= ((u + v) + w)[f]
\end{aligned}$$

(ii) Commutativity of addition

$$\begin{aligned}
(u + v)[f] &= (u[f] + v[f]) \\
&= \left(\frac{dx^\mu}{d\eta}\frac{\partial f}{\partial x^\mu} + \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}\right) \\
&= \left(\frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dx^\mu}{d\eta}\frac{\partial f}{\partial x^\mu}\right) \\
&= (v + u)[f]
\end{aligned}$$

(iii) Identity element of addition

Consider the curve c through p parametrized by ρ as $c(\rho) = p$. This provides our identity element (the 0 element in the vector space) since $\frac{\partial x^\mu}{\partial \rho} = 0$.

(iv) Inverse elements of addition

For every vector in the tangent space, $v = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$ we can define an inverse, $w = -\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$ by an appropriate choice of parametrization, such that $v + w = 0$.

(v) Compatibility of scalar multiplication with field multiplication

$$\begin{aligned}
a(b\mathbf{v})[f] &= a\left(b\frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}\right) \\
&= ab\left(\frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}\right) \\
&= (ab)\mathbf{v}[f]
\end{aligned}$$

(vi) Identity element of scalar multiplication

This one is trivial

$$\begin{aligned}
1v &= 1\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} \\
&= \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} = v
\end{aligned}$$

(vii) Distributivity of scalar multiplication with respect to vector addition

$$\begin{aligned} a(\mathbf{u}+\mathbf{v}) &= a\left(\frac{dx^\mu}{d\eta}\frac{\partial}{\partial x^\mu} + \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}\right) \\ &= a\frac{dx^\mu}{d\eta}\frac{\partial}{\partial x^\mu} + a\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} \\ &= a\mathbf{u} + a\mathbf{v} \end{aligned}$$

(viii) Distributivity of scalar multiplication with respect to field addition

$$\begin{aligned} (a+b)\mathbf{v} &= (a+b)\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} \\ &= a\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} + b\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} \\ &= a\mathbf{v} + b\mathbf{v} \end{aligned}$$

Since we have now shown our tangent space satisfies these properties, we can be confident in calling it a proper vector space.

5. Given two vector fields X and Y on M, find the local coordinate representation of the commutator $[X, Y]$

Solution

The local coordinate representation of the vector fields is $X = x^\mu \partial_\mu$ and $Y = y^\nu \partial_\nu$. The commutator is thus

$$\begin{aligned} [X,Y] = [x^\mu\partial_\mu, y^\nu\partial_\nu] &= x^\mu\partial_\mu(y^\nu\partial_\nu) - y^\nu\partial_\nu(x^\mu\partial_\mu) \\ &= x^\mu y^\nu \partial_\mu\partial_\nu + x^\mu\partial_\mu y^\nu\partial_\nu - y^\nu x^\mu\partial_\nu\partial_\mu - y^\nu\partial_\nu x^\mu\partial_\mu \\ &= x^\mu\partial_\mu y^\nu\partial_\nu - y^\nu\partial_\nu x^\mu\partial_\mu \end{aligned} \tag{14}$$

Now we can rename some of the indices since we contract along them to further simplify

$$[X,Y] = x^\mu\partial_\mu y^\nu\partial_\nu - y^\mu\partial_\mu x^\nu\partial_\nu = (x^\mu\partial_\mu y^\nu - y^\mu\partial_\mu x^\nu)\partial_\nu \tag{15}$$

Which is as good as we can get. Note that in the textbook, equation 2.23 states

$$[X,Y]^\mu = x^\lambda\partial_\lambda y^\mu - y^\lambda\partial_\lambda x^\mu \tag{16}$$

but it is important to remember that

$$[X,Y] = [X,Y]^\mu\partial_\mu \tag{17}$$

and our answer is consistent with that.

6. To gain familiarity with the abstract defintion of tangent vector, vector fields, and commutator, prove the following properties (some of them stated in class):
a) The tangent vector T_p (p a point on the manifold) operates linearly on functions, i.e.

$$T_p(af+bg) = aT_p(f) + bT_p(g) \tag{18}$$

where a, b are real numbers and f, g are functions on the manifold.
b) The tangent vector satisfies the Leibniz rule, i.e.

$$T_p(fg) = f(p)T(g) + g(p)T(f) \tag{19}$$

c) The commutator satisfies the Jacobi identity stated in class.
d) If X and Y are vector fields and f and g are functions on the manifold, then fX is a vector field and

$$[fX, gY] = fg[X,Y] + (fX(g))Y - (gY(f))X \tag{20}$$

Solutions
Recall from problem 4, that we had defined the notion of a tangent vector at a point p as $T_p = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$, evaluated at the point $x = p$. A tangent vector acting on a function f will simply be $T_p(f) = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} f$
a) The proof that it operates linearly on functions is straightforward,

$$\begin{aligned} T_p(af + bg) &= \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}(af) + \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}(bg) \\ &= a\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} f + b\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} g \\ &= aT_p(f) + bT_p(g) \end{aligned} \tag{21}$$

Thus it operates linearly

b) Next is the Leibniz rule

$$T_p(fg) = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}(fg) = f\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} g + g\frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} f = fT_p(g) + gT_p(f) \tag{22}$$

Therefore product rule is satisfied

c) The Jacobi identity states that for three vector fields, X, Y, Z the following relation is satisfied

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{23}$$

Lets start by looking at the first term

$$[X,[Y,Z]] = [X, YZ - ZY] = XYZ - XZY - YZX + ZYX \tag{24}$$

So the full identity can be written as

$$\begin{aligned} &[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = \\ &\quad XYZ - XZY - YZX + ZYX + \\ &\quad YZX - YXZ - ZXY + XZY + \\ &\quad ZXY - ZYX - XYZ + YXZ = 0 \end{aligned} \tag{25}$$

d) Let us start by filling in what we know

$$[fX, gY] = fx^\mu\partial_\mu(gy^\nu\partial_\nu) - gy^\nu\partial_\nu(fx^\mu\partial_\mu) \tag{26}$$

Performing product rule expansions yields

$$\begin{aligned}[fX,gY] = fx^\mu\partial_\mu(g)y^\nu\partial_\nu + fx^\mu g\partial_\mu(y^\nu)\partial_\nu + fx^\mu gy^\nu\partial_\mu(\partial_\nu)\\ -gy^\nu\partial_\nu(f)x^\mu\partial_\mu - gy^\nu f\partial_\nu(x^\mu)\partial_\mu - gy^\nu fx^\mu\partial_\nu(\partial_\mu)\end{aligned} \tag{27}$$

The terms with $\partial_\mu(\partial_\nu)$ and vice versa cancel out as partial derivatives commute, and so we are left with (after a bit of rearranging)

$$\begin{aligned}[fX,gY] &= fgx^\mu\partial_\mu y^\nu\partial_\nu - fgy^\nu\partial_\nu x^\mu\partial_\mu + fx^\mu\partial_\mu(g)y^\nu\partial_\nu - gy^\nu\partial_\nu(f)x^\mu\partial_\mu\\ &= fgXY - fgYX + (fX(g))Y - (gY(f))X\\ &= fg[X,Y] + (fX(g))Y - (gY(f))X\end{aligned} \tag{28}$$