GR, Assignment 7

Problem 1

The Schwarzschild metric is

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2.$$

where $$\Delta = 1 - \frac{2GM}{r}.$$

Since the proper time of an observer at infinity is t (there $\Delta \to 1$) we need to find $\frac{dr}{dt}$ for a radially infalling particles. We have that

$$(*) \qquad 1 = -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \Delta \left(\frac{dt}{d\tau}\right)^2 - \Delta^{-1} \left(\frac{dr}{d\tau}\right)^2$$

if we parametrize the geodesic with proper time (see Eq. 5.55 in the textbook).

Furthermore $K^\mu = (1,0,0,0)$ is a Killing vector for time translation which gives the conserved quantity

$$\frac{E}{m} = -K_\mu \frac{dx^\mu}{d\tau} = \Delta \frac{dt}{d\tau},$$

i.e. energy per mass. Thus

$$\frac{dt}{d\tau} = \frac{E/m}{\Delta}$$

Using (*) above we get that

$$\left(\frac{d\tau}{dt}\right)^2 = \Delta - \Delta^{-1}\left(\frac{dr}{dt}\right)^2$$

so $$\left(\frac{dr}{dt}\right)^2 = \Delta\left(\Delta - \left(\frac{d\tau}{dt}\right)^2\right)$$

$$= \Delta\left(\Delta - \frac{\Delta^2}{(E/m)^2}\right)$$

so $$\frac{dr}{dt} = -\Delta\sqrt{1 - \frac{\Delta^2}{(E/m)^2}}.$$

In the second half of the question we need to find the velocity measured by a static observer. We now that it measures the particle to have energy

$$E_0 = -p_\mu U^\mu$$

where p_μ is the particle's four-momentum and u^μ is the observer's four-velocity.

We know that $u^\mu = (a, 0, 0, 0)$ because the observer is static. To find a we note that

$$-1 = u_\mu u^\mu = g_{tt}\, a^2 = -\Delta a^2, \quad \text{i.e.}$$

$$u^\mu = (\Delta^{-1/2}, 0, 0, 0).$$

Thus

$$E_0 = -g_{\mu\nu}\, p^\mu u^\nu$$

$$= -g_{tt}\, m \frac{dt}{d\tau} \Delta^{-1/2} = \Delta^{1/2}\, m \frac{dt}{d\tau}$$

so

$$\frac{E_0}{m} = \frac{E/m}{\Delta^{1/2}}$$

Note the difference between E_0 and E. E comes from a Killing vector and thus includes gravitational energy (it is a conserved quantity). Conversely, E_0 is the rest mass + kinetic energy measured by some observer.

If v is the velocity measured by the observer then

$$E_0/m = \gamma = \frac{1}{\sqrt{1-v^2}}$$

so $$1-v^2 = \frac{1}{(E_0/m)^2} = \frac{\Delta}{(E/m)^2}$$

which gives $$\underline{\underline{v = \sqrt{1 - \frac{\Delta}{(E/m)^2}}}}$$

Problem 2/3

We can write the metric as

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2$$

just as in the case of an uncharged blackhole.

We can then directly use the results from last problem set to see that the Christoffel symbols are

$$\Gamma^t_{tr} = \partial_r \alpha, \qquad \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha, \qquad \Gamma^r_{rr} = \partial_r\beta$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r}, \qquad \Gamma^r_{\theta\theta} = -re^{-2\beta}, \qquad \Gamma^\phi_{r\phi} = \frac{1}{r},$$

$$\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

(see Eq. 5.12 in the textbook, I won't use the tetrad basis.)

Furthermore the Ricci tensor is

$$R_{tt} = e^{2(\alpha-\beta)} \left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha \right]$$

$$R_{rr} = - \partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \beta$$

$$R_{\theta\theta} = e^{-2\beta} \left[r (\partial_r \beta - \partial_r \alpha) - 1 \right] + 1$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}.$$

(see Eq. 5.14 in textbook).

We can write Einstein's equation as

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

where $T = T^{\mu}{}_{\mu}$.

I will assume the black hole has an electric charge but no magnetic charge $(P=0)$.

By spherical symmetry we know that

$$E_r = F_{rt} = F_{rt}(r)$$

and all other components vanish.

To find E_r we need to solve Maxwell's equation in curved spacetime. (It's not obvious that we will get the usual Coulomb force.)

We know that

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right)$$

(Note that the textbook is missing the factor $\frac{1}{4\pi}$ which must be there when we use Gaussian units.)

Then $$T = \frac{1}{4\pi}\left(F_{\mu\rho} F^{\mu\rho} - F_{\rho\sigma} F^{\rho\sigma} \right) = 0$$

so Einstein's equation reduces to

$$R_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Let's find the different components of $T_{\mu\nu}$.

It's easy to see that

$$T_{tt} = F_{tr} F_t{}^r - \frac{1}{4} g_{tt}\, 2\, F_{tr} F^{tr}$$

(to include both $F_{tr} F^{tr}$ and $F_{rt} F^{rt}$)

$$= \frac{1}{2} g_{tt} F_{tr} F^{tr}$$

$$T_{rr} = F_{rt} F_r{}^t - \frac{1}{4} g_{rr}\, 2 F_{tr} F^{tr}$$

$$= \frac{1}{2} g_{rr} F_{tr} F^{tr}$$

$$T_{\theta\theta} = -\frac{1}{2} g_{\theta\theta} F_{tr} F^{tr}$$

$$T_{\varphi\varphi} = -\frac{1}{2} g_{\varphi\varphi} F_{tr} F^{tr}.$$

Since $e^{2(\beta-\alpha)} T_{tt} + T_{rr}$

$$= e^{2\beta} \frac{1}{2} F_{tr} F^{tr} (1 - 1) = 0$$

we get that

$$0 = e^{2(\beta-\alpha)} R_{tt} + R_{rr}$$

$$= \frac{2}{r} \left[\partial_r \alpha + \partial_r \beta \right].$$

Thus $\alpha(r) = -\beta(r) + C$

where C is some constant.

By rescaling $t \to e^{-C} t$ we get

$$\underline{\underline{\alpha = -\beta}}.$$

Let's next look at Maxwell's equation

$$g^{\mu\nu} \nabla_\mu F_{\nu\sigma} = 0.$$

For $\sigma = t$ we get

$$0 = g^{tt} \nabla_t F_{tt} + g^{rr} \nabla_r F_{rt} + g^{\theta\theta} \nabla_\theta F_{\theta t} + g^{\varphi\varphi} \nabla_\varphi F_{\varphi t}$$

$$= -e^{-2\alpha} \left(-\Gamma^r_{tt} F_{rt} - \Gamma^r_{tt} F_{tr} \right)$$

$$+ e^{-2\beta} \left(\partial_r F_{rt} - \Gamma^r_{rr} F_{rt} - \Gamma^t_{rt} F_{rt} \right)$$

$$- r^{-2} \Gamma^r_{\theta\theta} F_{rt} - r^{-2} \sin^{-2}\theta \, \Gamma^r_{\varphi\varphi} F_{rt}.$$

Plugging in the Christoffel symbols and using that $F_{rt} = -F_{tr}$ this shows that

$$0 = e^{-2\beta}\left(\partial_r F_{rt} - \partial_r\beta\, F_{rt} - \partial_r\alpha\, F_{rt}\right)$$

$$+ \frac{1}{r} e^{-2\beta} F_{rt} + \frac{1}{r} e^{-2\beta} F_{rt}$$

or in other words

$$\partial_r F_{rt} + \frac{2}{r} F_{rt} = 0$$

where we used that $\alpha = -\beta$.

Thus $$F_{rt} = \frac{Q}{r^2}$$

where we get the constant Q from the case of flat spacetime in Gaussion units.

The only thing left is to find α.

We will use the equation

$$R_{\theta\theta} = 8\pi G T_{\theta\theta}.$$

This gives

$$-\left(e^{2\alpha}\left(2r\,\partial_r\alpha + 1\right) - 1\right) = - G\, r^2\, F_{tr} F^{tr}$$

where

$$F_{tr}F^{tr} = -e^{-2\alpha}e^{-2\beta}(F_{tr})^2 = -\frac{Q^2}{r^4}$$

so $$\partial_r(re^{2\alpha}) - 1 = -\frac{GQ^2}{r^2}.$$

It can be shown that

$$e^{2\alpha} = 1 - \frac{R_s}{r} + \frac{GQ^2}{r^2}$$

where $R_s = 2GM$ by comparing with the case where $Q = 0$.

In conclusion

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2$$

where $$\Delta = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}.$$

We have not yet used the other Maxwell equation $(\nabla_{[\mu}F_{\nu\rho]} = 0)$ or the remaining component of Einstein's equation (either $R_{tt} = 8\pi G T_{tt}$ or $R_{rr} = 8\pi G T_{rr}$) but they can be shown to be fulfilled by the solution we just found.

Problem 4.

This problem can be solved without evaluating any Christoffel symbols or using the geodesic equations. We will simply use the constants of motion.

The metric is

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

with $$\Delta = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}.$$

We assume that $GM^2 < Q^2$ which can easily be shown to give $\Delta > 0$ for all r.

Because of rotational invariance we can assume $\sin\theta = 1$. Furthermore there are two Killing vectors,

$$K^\mu = (1, 0, 0, 0) \quad \text{(symmetry under time-translation)}$$

and $$R^\mu = (0, 0, 0, 1) \quad \text{(rotational invariance)}$$

(see p. 207-208 in the textbook for further details)

The corresponding conserved quantities are energy (per mass)

$$E = -K_\mu \frac{dx^\mu}{d\tau} = \Delta \frac{dt}{d\tau}$$

and angular momentum (per mass unit)

$$L = R_\mu \frac{dx^\mu}{d\tau} = r^2 \frac{d\phi}{d\tau}.$$

We have another conserved quantity

$$\varepsilon = -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \qquad \text{(see Eq. 5.55)}$$

where we can parametrize the geodesic so that $\varepsilon = 1$ for massive particles and $\varepsilon = 0$ for massless particles. Then

$$\varepsilon = \Delta \left(\frac{dt}{d\tau}\right)^2 - \Delta^{-1} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2$$

$$= \frac{E^2}{\Delta} - \Delta^{-1} \left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{r^2},$$

i.e. $$\Delta \left(\varepsilon + \frac{L^2}{r^2}\right) = E^2 - \left(\frac{dr}{d\tau}\right)^2$$

or $$\frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{1}{2} E^2$$

where $V(r) = \frac{1}{2}\left(\varepsilon + \frac{L^2}{r^2}\right)\Delta$.

Using our knowledge from classical mechanics we see that we just need to show that $V(r)$ is a repulsive potential for sufficiendy small r.

This is clear from $V > 0$ (because $\Delta > 0$) and that $V \to \infty$ as $r \to 0$.

(The potential must then look something like,

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Problem 5

Let's write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\sigma\rho}\partial_\rho h^\mu{}_\sigma + \frac{1}{2}\eta^{\mu\nu}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} - \frac{1}{2}\eta^{\mu\nu}\partial_\mu h\,\partial_\nu h \right]$$

(I write the second term differently than the textbook.) Here $h = h^x{}_x$.

Since

$$\frac{\partial \mathcal{L}}{\partial h^{\alpha\beta}} = 0$$

the Euler-Lagrange equations reduce to

$$\partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x h^{\alpha\beta})} = 0.$$

As an example

$$\frac{\partial(\partial_\mu h^{\mu\nu}\partial_\nu h)}{\partial(\partial_x h^{\alpha\beta})} = \frac{\partial(\delta^\omega_\mu \delta^\xi_\nu \eta_{\xi\phi}\,\partial_\omega h^{\mu\nu}\,\partial_\xi h^{\xi\phi})}{\partial(\partial_x h^{\alpha\beta})}$$

$$= \delta^\omega_\mu \delta^\xi_\nu \eta_{\xi\phi}\left[\delta^x_\omega \delta^\mu_\alpha \delta^\nu_\beta (\partial_\xi h^{\xi\phi}) + (\partial_\omega h^{\mu\nu})\delta^x_\xi \delta^\xi_\alpha \delta^\phi_\beta\right]$$

$$= \partial_\beta h \, \delta^\chi_\alpha + \partial_\mu h^{\mu\chi} \eta_{\alpha\beta} .$$

Evaluating the other terms similarly we get that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\chi h^{\alpha\beta})} = \frac{1}{2} \Big[\delta^\chi_\alpha \partial_\beta h + \partial_\mu h^{\mu\chi} \eta_{\alpha\beta} - \partial_\alpha h^\chi{}_\beta - \partial_\alpha h^\chi{}_\beta + \partial^\chi h_{\alpha\beta} - \partial^\chi h \, \eta_{\alpha\beta} \Big]$$

Then $\partial_\chi \dfrac{\partial \mathcal{L}}{\partial(\partial_\chi h^{\alpha\beta})} = 0$ gives that

$$\frac{1}{2} \Big[\partial_\alpha \partial_\beta h + \partial_\mu \partial_\chi h^{\mu\chi} \eta_{\alpha\beta} - \partial_\chi \partial_\alpha h^\chi{}_\beta - \partial_\chi \partial_\beta h^\chi{}_\alpha + \Box h_{\alpha\beta} - \Box h \, \eta_{\alpha\beta} \Big] = 0$$

which is precisely $-G_{\alpha\beta} = 0$, i.e. $G_{\alpha\beta} = 0$.

Thus we get Einstein's equation in the weak field limit.

Problem 6

(a) The distance between the two masses is $2x$. Thus Newton's law of gravitation gives that

$$M\ddot{x} = -\frac{M^2}{(2x)^2}$$

where we have set $G = 1$ like in the textbook.

Let's simply show that the solution provided satisfies this equation with $x(t) \to +\infty$, at $t \to -\infty$.
(The argument for the other solution is similar.)
From

$$x(t) = \left(\frac{9Mt^2}{8}\right)^{1/3} = \left(\frac{9M}{8}\right)^{1/3} t^{2/3}$$

we get that

$$\dot{x} = \left(\frac{9M}{8}\right)^{1/3} \frac{2}{3}\, t^{-1/3}$$

and

$$\ddot{x} = \left(\frac{9M}{8}\right)^{1/3} \frac{2}{3} \left(\frac{-1}{3}\right) t^{-4/3}.$$

Thus

$$-\frac{M^2}{(2x)^2} = \frac{M^2}{4} \frac{t^{4/3}}{(9M/8)^{2/3}} = \frac{1}{9^{2/3}} M^{4/3}\, t^{-4/3}$$

$$= \frac{2}{9} M\left(\frac{9M}{8}\right)^{1/3} t^{-4/3} = -M\ddot{x}$$

which is what we wanted to show.

(b) To use non-relativistic mechanics we must have $v \ll c$, i.e $\dot{x} \ll 1$.

Omitting numerical factors this gives

$$\left(\frac{M}{t}\right)^{1/3} \ll 1,$$

i.e. $M \ll t$.

Thus $x \sim M^{1/3} t^{2/3} \gg M$,

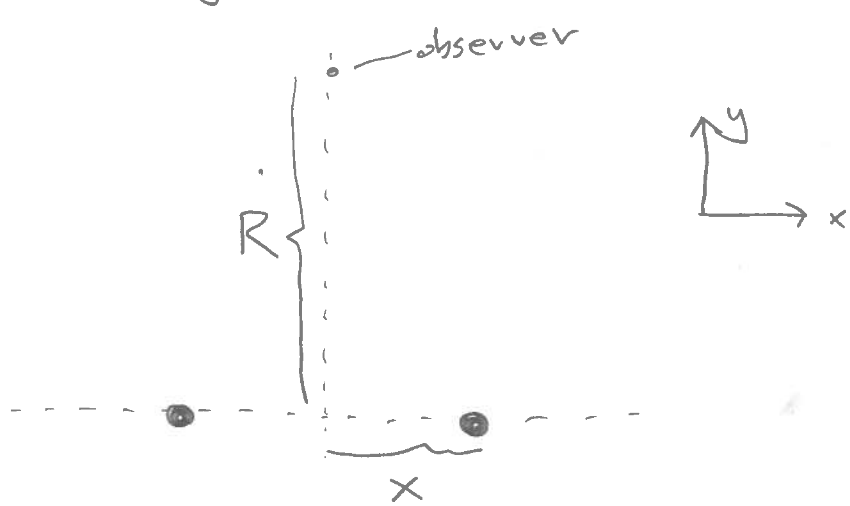
i.e. $\underline{\underline{x \gg M}}$.

We can also show this by noting that in the weak field limit

$$h_{00} \sim \frac{GM}{x}$$

so $|h| \ll 1$ gives $x \gg M$

(c)



We want to find $h^{TT}_{xx}(t)$ at $(0, R, 0)$.

Eq. 7.179 in the textbook tells us that in the transverse-traceless gauge

$$h^{TT}_{xx} = \frac{2}{r}\frac{d^2}{dt^2} I^{TT}_{xx}(t_r)$$

where r is the distance from the observer to the system and t_r is the retarded time (see below)

We can assume that $R \gg x$ because all our equations assume the observer is far from the source. Thus $r \simeq R$.

That also means that the unit vector pointing towards the observer is $n^i = (0, 1, 0)$.

Now Eq. 7.138 tells us that

$$I_{xx} = \int x^2 \, T^{00}(t, y) \, d^3y$$

where

$$T^{00} = \delta(y)\delta(z)\left[M\delta(x - x_-) + M\delta(x - x_+)\right]$$

is the energy density of the masses and we denote the solutions found in (a) by $x_\pm$. Thus

$$I_{xx} = M(x_-^2 + x_+^2) = 2M x_+^2.$$

To use the transverse-traceless gauge we need to evaluate

$$I^{TT}_{xx} = (P_x{}^k P_x{}^l - \frac{1}{2} P_{xx} P^{kl}) I_{kl}.$$

where $P_{ij} = \delta_{ij} - n_i n_j$

(see Eq. 7.178).

It's easy to see that I_{xx} is the only non-vanishing component.

Thus

$$I^{TT}_{xx} = (P_x{}^x P_x{}^x - \frac{1}{2} P_{xx} P^{xx}) I_{xx} = \frac{1}{2} I_{xx}.$$

In other words

$$I^{TT}_{xx} = M x_+^2 = M \left(\frac{9M}{8}\right)^{2/3} t^{4/3}$$

and

$$\ddot{I}^{TT}_{xx} = \frac{4M}{9} \left(\frac{9M}{8}\right)^{2/3} t^{-2/3}.$$

Putting everything together we see that

$$h^{TT}_{xx} = \frac{2}{R} \frac{4M}{9} \left(\frac{9M}{8}\right)^{2/3} t_r^{-2/3}.$$

The only thing left is to find the retarded time. If a wave front is emitted at time t_r and reaches the observer at time t then it has travelled distance $\sqrt{R^2+x^2} \simeq R$ in the meantime. Thus

$$t \simeq t_r + R$$

and we see that

$$\underline{\underline{h^{TT}_{xx} = \frac{2}{R} \frac{M^{5/3}}{9^{1/3}} (t-R)^{-2/3}}}$$