1-point torus blocks in 2d

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The computation of the conformal blocks for the thermal one-point function in a 2d CFT that should reproduce the results from [1].

Useful results

In our notation, Δ denotes a holomorphic conformal dimension.

The following relations will be used a lot in the calculations:

$$[L_1, L_{-1}] = 2L_0$$

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}$$

Using these we derive the value of $\left[L_1, L_{-1}^N\right]$. Let's expand the commutator once by using the usual rule:

$$\begin{aligned}\left[L_1, L_{-1}^N\right] &= L_{-1}\left[L_1, L_{-1}^{N-1}\right] + [L_1, L_{-1}]L_{-1}^{N-1} = L_{-1}\left[L_1, L_{-1}^{N-1}\right] + 2L_0 L_{-1}^{N-1} \\ &= L_{-1}\left[L_1, L_{-1}^{N-1}\right] + 2L_{-1}^{N-1}L_0 + 2(N-1)L_{-1}^{N-1}\end{aligned}$$

From this we can guess something of the form $\left[L_1, L_{-1}^N\right] = L_{-1}^{N-1} p_N$ and we obtain the recursion relation

$$p_N = p_{N-1} + 2(L_0 + N - 1)$$

which is solved by $p_N = 2NL_0 + N^2 - N$. Together these give

$$\boxed{\left[L_1, L_{-1}^N\right] = NL_{-1}^{N-1}(2L_0 + N - 1)} \tag{1}$$

We can so the same thing to compute $\left[L_{-1}, L_1^M\right]$ and we find that it can be written as $L_1^{M-1} q_M$ with the following recursion relation

$$q_M = q_{M-1} - 2L_0 + 2(M-1)$$

This is solved by $q_M = -2ML_0 + M(M-1)$ and we find the relation

$$\boxed{\left[L_{-1}, L_1^M\right] = -ML_1^{M-1}(2L_0 - M + 1)} \tag{2}$$

We will also need the following relation for a primary field

$$[L_n, \phi(z)] = \Delta_\phi (n+1) z^n \phi(z) + z^{n+1} \partial\phi(z)$$

and especially the cases $n = \pm 1$

$$[L_1, \phi] = 2\Delta_\phi z\phi + z^2 \partial\phi$$

$$[L_{-1}, \phi] = \partial\phi$$

Setup of the calculation

For a given primary state $|\Delta\rangle$ the descendant at level N is simply $|\Delta, N\rangle = L_{-1}^N |\Delta\rangle$ and the levels are orthogonal. The first thing we need to compute the blocks is the Gram matrix B_{MN}, which is diagonal with entries $\langle\Delta, N|\Delta, N\rangle$. It's easy to compute it by using the relation (1) and the fact that L_1 kills the primary state.

$$\begin{aligned}\langle\Delta, N|\Delta, N\rangle &= \langle\Delta| L_1^N L_{-1}^N |\Delta\rangle = \left\langle L_1^{N-1}\left[L_1, L_{-1}^N\right]\right\rangle_\Delta = \left\langle L_1^{N-1} N L_{-1}^{N-1}(2L_0 + N - 1)\right\rangle_\Delta \\ &= N(2\Delta + N - 1)\left\langle L_1^{N-1} L_{-1}^{N-1}\right\rangle_\Delta = N(2\Delta + N - 1)\cdot(N-1)(2\Delta + N - 2)\left\langle L_1^{N-2} L_{-1}^{N-2}\right\rangle_\Delta \\ &= \cdots = N!(2\Delta + N - 1)(2\Delta + N - 2)\cdots(2\Delta)\langle\Delta|\Delta\rangle = \boxed{N!(2\Delta)_N}\end{aligned} \tag{3}$$

where $(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. This can also be derived by using two different ways of writing the two-point function, as can be seen in equations (113)-(117) of [2].

The next thing that we need is the three-point function

$$A_{MN}(z) \equiv \left\langle L_1^M \phi(z) L_{-1}^N\right\rangle_\Delta$$

where $\phi(z)$ is the primary operator with dimension Δ_ϕ of which we compute the one-point function. We will derive a recursion relation for the functions $A_{MN}(z)$ by taking one of the L_1 all the way to the right.

$$\begin{aligned}A_{MN}(z) &= \left\langle L_1^{M-1}[L_1, \phi(z)]L_{-1}^N\right\rangle_\Delta + \left\langle L_1^{M-1}\phi(z)L_1 L_{-1}^N\right\rangle_\Delta \\ &= \left\langle L_1^{M-1}(2\Delta_\phi z\phi(z) + z^2\partial\phi(z))L_{-1}^N\right\rangle_\Delta + \left\langle L_1^{M-1}\phi(z)\left[L_1, L_{-1}^N\right]\right\rangle_\Delta \\ &= 2\Delta_\phi z A_{M-1,N}(z) + z^2\partial_z A_{M-1,N}(z) + N(2\Delta + N - 1)A_{M-1,N-1}(z)\end{aligned} \tag{4}$$

We can derive another recursion relation by taking one L_{-1} and putting it all the way to the left.

$$\begin{aligned}A_{MN}(z) &= \left\langle L_1^M[\phi(z), L_{-1}]L_{-1}^{N-1}\right\rangle_\Delta + \left\langle L_1^M L_{-1}\phi(z)L_{-1}^{N-1}\right\rangle_\Delta \\ &= \left\langle L_1^M(-\partial\phi(z))L_{-1}^{N-1}\right\rangle_\Delta + \left\langle\left[L_1^M, L_{-1}\right]\phi(z)L_{-1}^{N-1}\right\rangle_\Delta \\ &= -\partial_z A_{M,N-1}(z) + M(2\Delta - M + 1)A_{M-1,N-1}(z)\end{aligned} \tag{5}$$

Now remember that conformal symmetry says that $A_{00}(z) = \langle\phi(z)\rangle_\Delta = \langle\phi(1)\rangle_\Delta z^{-\Delta_\phi}$. From now on we set $\langle\phi(1)\rangle_\Delta = 1 = A_{00}(1)$ because it will appear in each term and cancel in the recursion relations. By looking at the recursion relations (4) and (5) we can see that the z dependance must be

$$A_{MN}(z) = A_{MN}(1)z^{-\Delta_\phi + M - N}$$

Plugging this in the recursion relations leads directly to the following recursions for $A_{MN}(1) \equiv A_{MN}$:

$$\boxed{A_{MN} = N(2\Delta + N - 1)A_{M-1,N-1} + (\Delta_\phi + M - N - 1)A_{M-1,N}} \tag{6}$$

$$\boxed{A_{MN} = M(2\Delta - M + 1)A_{M-1,N-1} + (\Delta_\phi + N - M - 1)A_{M,N-1}} \tag{7}$$

Note that A_{MN} is symmetric. Also at the end everything we care about is $A_{MN}(1)$ because the one-point function should not depend on the point.

Solving the recursion relations

In order to solve the recursion relations (6) and (7) we will view A_{MN} as a symmetric matrix. The first thing that we will compute is the values of the first line A_{0M} and first column A_{M0}, which are the same since the matrix is symmetric. We can use the recursion relations to find (the terms with negative indices vanish)

$$A_{0M} = A_{M0} = (\Delta_\phi + M - 1)A_{M-1,0} = (\Delta_\phi + M - 1)(\Delta_\phi + M - 2)A_{M-2,0} = \cdots = \boxed{(\Delta_\phi)_M}$$

To compute the blocks we actually only need the diagonal elements of the matrix, A_{NN}. Using the recursion relation (6) we see that each of them depend first of all only on $A_{N-1,N-1}$ and $A_{N-1,N}$, which are respectively the elements on the line above that are directly to the left and directly on top of the matrix element that we want to compute. Each of these can be related to the line above them by the recursion relation and at the end each A_{NN} will depend only on the first N elements of the first line. (Note that we could do the same thing with (7) to use the first column.) We can then write

$$A_{NN} = \sum_{k=0}^{N} G(N,k) A_{0k} \tag{8}$$

However the coefficients $G(N,k)$ could be hard to get since they contain a contribution from every possible path in the matrix leading from A_{NN} to A_{0k} by going either directly up or on a diagonal to the left at each step up. In fact it turns out that when we look at the recursion relations this coefficient doesn't depend on the path but just on the endpoints so that it is simpler to get (still not clear why). The number of paths is easy to determine since to reach the top we need to do N steps but to reach A_{0k} we need to go to the left $N-k$ times so the number of possible ways to do this is $\binom{N}{N-k} = \binom{N}{k}$. With this we can write

$$G(N,k) = \binom{N}{k} g(N,k)$$

where $g(N,k)$ doesn't depend on the path. Because of this nice property we can pick the easiest path and compute its contribution directly. Let's choose the one where we go on a diagonal $N-k$ times then we go up k times. For the first $N-k$ terms we pick the first part of (6) from N to $N-k$, which will give a contribution of $N(N-1)\cdots(N-k+1)(2\Delta+N-1)(2\Delta+N-2)\cdots(2\Delta+N-k)$. For the rest we pick the second contribution and since the column index is fixed to $N-k$ at this point and only M varies from $N-k$ to 0 we get $(\Delta_\phi - 1)(\Delta_\phi - 2)\cdots(\Delta_\phi - k)$. Putting all of this together and rewriting a little bit gives

$$g(N,k) = \frac{N!(2\Delta+N-1)!(\Delta_\phi - 1)!}{k!(2\Delta+k-1)!(\Delta_\phi - k - 1)!}$$

Finally putting everything we gathered in (8) we find

$$\boxed{A_{NN} = \sum_{k=0}^{N} \binom{N}{k} \frac{N!}{k!} \frac{(2\Delta+N-1)!}{(2\Delta+k-1)!} \frac{(\Delta_\phi + N - 1)!}{(\Delta_\phi - k - 1)!}} \tag{9}$$

which is the right result when compared to the equation in [1].

I guess it would not be too much harder to find the full matrix A_{MN} but it's not necessary for our purposes.

References

[1] L. Hadasz, Z. Jaskolski, and P. Suchanek, "Recursive representation of the torus 1-point conformal block," *Jhep*, vol. 01, p. 063, 2010. DOI: `10.1007/JHEP01(2010)063`. arXiv: `0911.2353 [hep-th]`.

[2] D. Simmons-Duffin, "The Conformal Bootstrap," in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015*, 2017, pp. 1–74. DOI: `10.1142/9789813149441_0001`. arXiv: `1602.07982 [hep-th]`. [Online]. Available: `http://inspirehep.net/record/1424282/files/arXiv:1602.07982.pdf`.