Dimension of Lie groups

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We show how to find the dimension of the most common Lie groups (number of free real parameters in a generic matrix in the group) and we discuss the agreement with their algebras.

1 Orthogonal groups

1.1 $O(n)$ and $SO(n)$

The group $O(n)$ is composed of $n \times n$ real matrices that are orthogonal, so that satisfy $O^T O = I$. In general a $n \times n$ matrix has n^2 elements, but the constraint of orthogonality adds some relation between them and decreases the number of independent elements. To find exactly by how much the number of elements is reduced, we have to note that the constraint equation satisfies $(O^T O)^T = O^T O$ so that it is symmetric and has only $\frac{n(n+1)}{2}$ independent components (n on the diagonal and $\frac{n(n-1)}{2}$ on the upper triangle). Equating this matrix to the identity then creates $\frac{n(n+1)}{2}$ constraints that involve the elements of O and it reduces the number of independent elements to

$$dim[O(n)] = n^2 - \frac{n(n+1)}{2} = \boxed{\frac{n(n-1)}{2}}.$$

To consider the group $SO(n)$, we add the requirement that the matrix has a determinant of 1. This seems to be an additional constraint, but in fact taking the determinant of the orthogonality constraint tells us that $det(O)^2 = 1 \Rightarrow det(O) = \pm 1$. Asking for unit determinant is then just some choice of sign and doesn't add an extra constraint, which leads to

$$dim[SO(n)] = dim[O(n)] = \boxed{\frac{n(n-1)}{2}}.$$

As an example, consider the 2×2 matrix $O = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The orthogonality constraint is written as

$$O^T O = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$1 = a^2 + c^2$$
$$\Rightarrow 0 = ab + cd.$$
$$1 = b^2 + d^2$$

There are four unknowns and 3 equations so there is one free variable, which agrees with the formula for the dimension of $O(2)$. If we solve this system and compute $det(O) = ad - bc$, we will get that it is either $+1$

or -1. Working with $SO(2)$ then just means to ignore the solutions which have a determinant of -1, so it doesn't reduce the number of free elements. Actually a matrix in $O(2)$ can be written as

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ \mp\sin\theta & \pm\cos\theta \end{pmatrix}$$

and to get an element of $SO(2)$ we pick the upper sign.

1.2 $o(n)$ and $so(n)$

When we talk about Lie algebras, we ask for matrices that, when exponentiated, give an element of a Lie group. For the Lie algebra $o(n)$, we are looking for matrices A such that e^A (defined by the series expansion of the exponential) is orthogonal. Note that the orthogonality condition can be rewritten as $O^{-1} = O^T$. It's easy to see from the expansion $e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!} = 1 + A + \frac{A^2}{2} + ...$ that the inverse of e^A is e^{-A} and its transpose is e^{A^T}. This means that for O to be orthogonal we need A to be anti-symmetric $A^T = -A$.

Let's check the number of independent parameters in an anti-symmetric $n \times n$ matrix to make sure that it agrees with the dimension of $O(n)$. The first condition coming from the anti-symmetry is that the diagonal elements must vanish because $A_{aa} = -A_{aa} \Rightarrow A_{aa} = 0$. Next, the elements in the lower triangle are just the negative of the ones in the upper triangle, so we are left with only $\frac{n^2-n}{2}$ independent elements, meaning that

$$dim[o(n)] = \boxed{\frac{n(n-1)}{2}},$$

the same as for $O(n)$.

To study the algebra of $SO(n)$, we need to remember (or discover) the identity

$$det(e^A) = e^{Tr\,A}.$$

This is easy to see for a diagonal matrix since $e^{diag(a_1,...,a_n)} = diag(e^{a_1},...,e^{a_n})$. The case of a general matrix follows since we can just find a basis in which it is diagonal. Using this, the requirement that the matrix O has determinant 1 is equivalent to asking that the matrix A has a trace of zero. This should be a new requirement for a matrix in $so(n)$, but actually an anti-symmetric matrix is already traceless so there is no additional constraint and

$$dim[so(n)] = dim[o(n)] = \boxed{\frac{n(n-1)}{2}}.$$

It's interesting to note that we don't see the difference between $SO(n)$ and $O(n)$ at the level of the algebra. This can be understood from the fact that the part of $O(n)$ with a negative determinant can't be expressed as an exponential because e^A must have a positive determinant, so the only part that is present in the algebra $o(n)$ is exactly the one that corresponds to $so(n)$.

2 Unitary groups

2.1 $U(n)$ and $SU(n)$

To consider the group $U(n)$, we look at complex $n \times n$ matrices that are unitary, so they satisfy $U^\dagger U = I$ (remember that the hermitian conjugate of a matrix is the transpose of its complex conjugate). Originally a complex matrix has n^2 complex elements, which means that it has $2n^2$ free real parameters. The constraints are coming from the unitarity condition, which is in fact hermitian since $(U^\dagger U)^\dagger = U^\dagger U$. This means that we need to know the number of real parameters in a hermitian matrix and substract that from $2n^2$. The diagonal of a hermitian matrix must be real since it must be equal to its complex conjugate. This gives n parameters. Hermiticity fixes the elements on the lower triangle in terms of the ones in the upper triangle

since they must be complex conjugates of each other. This leaves us with $\frac{n^2-n}{2}$ complex parameters on the upper triangle, which means $n(n-1)$ real ones. Overall this sums to $n + n(n-1) = n^2$ real parameters in a hermitian matrix, which is then the number of constraints on the matrix U coming from the unitarity constraint. The number of real parameters in a general unitary matrix is then found to be

$$dim[U(n)] = 2n^2 - n^2 = \boxed{n^2}.$$

When we talk about the group $SU(n)$, we ask that the determinant of the matrix be 1. Taking the determinant of the hermiticity condition gives $|det(U)|^2 = 1 \Rightarrow det(U) = e^{i\theta}$. There is still a continuum of possible values for the determinant of a unitary matrix, so fixing it adds a new constraint. Basically we ask for $\theta = 0$, which can be expressed in terms of the elements of U if we want. The result is then

$$dim[SU(n)] = dim[U(n)] - 1 = \boxed{n^2 - 1}.$$

For an example,

2.2 $u(n)$ and $su(n)$

Like before we want to write an element of the Lie group as e^H. The unitarity requirement can be rewritten as $U^{-1} = U^\dagger$ and it's easy to see again that $(e^H)^{-1} = e^{-H}$ and $(e^H)^\dagger = e^{H^\dagger}$ so that unitarity asks for the elements of the Lie algebra to be anti-hermitian matrices satisfying $H^\dagger = -H$. To see how many free real parameters there are in such a matrix, first note that the elements of the diagonal must be imaginary because they are equal to minus their complex conjugate. This gives n parameters. The elements of the lower triangle are fixed as minus the complex conjugate of the elements of the upper triangle so there are only $\frac{n^2-n}{2}$ other complex parameters, which gives $n(n-1)$ real ones. The total is then

$$dim[u(n)] = n + n(n-1) = \boxed{n^2},$$

agreeing with $U(n)$.

For $su(n)$, the formula involving the determinant shown above requires that $Tr\, H = 0$. This time it adds an additinal constraint because an anti-hermitian matrix is only required to have an imaginary trace. This means that we have

$$dim[su(n)] = dim[u(n)] - 1 = \boxed{n^2 - 1}.$$

3 Linear groups

3.1 $SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$

The groups of the $SL(n, .)$ family are simply general non-degenerate matrices (non-zero determinant) that have unit determinant. For $SL(n, \mathbb{R})$ we consider real matrices, which start with n^2 parameters. The condition that the determinant is 1 is the only constraints so that we get

$$dim[SL(n, \mathbb{R})] = \boxed{n^2 - 1}.$$

For $SL(n, \mathbb{C})$ we care about complex matrices so we start with $2n^2$ real parameters. The unit determinant condition actually gives 2 constraints since the determinant could be any complex number and we fix its real and imaginary parts. The result is

$$dim[SL(n, \mathbb{C})] = \boxed{2n^2 - 2}.$$

3.2 $sl(n,\mathbb{R})$ and $sl(n,\mathbb{C})$

4 Summary

Name of the group	Properties	Dimension	Algebra
$O(n)$	$O^T O = I$	$\frac{n(n-1)}{2}$	anti-symmetric matrices
$SO(n)$	$O^T O = I$ and $det(O) = 1$	$\frac{n(n-1)}{2}$	anti-symmetric matrices
$U(n)$	$U^\dagger U = I$	n^2	anti-hermitian matrices
$SU(n)$	$U^\dagger U = I$ and $det(U) = 1$	$n^2 - 1$	traceless anti-hermitian matrices
$SL(n,\mathbb{R})$	$det(L) = 1$	$n^2 - 1$	traceless real matrices
$SL(n,\mathbb{C})$	$det(L) = 1$	$2(n^2 - 1)$	traceless complex matrices